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RICCATI MATRIX DIFFERENTIAL EQUATION AND THE DISCRETE ORDER PRESERVING PROPERTY

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Abstract. In this paper we recall discrete order preserving property related to the discrete Riccati matrix equation. We present results obtained by applying this property to the solutions of the Riccati matrix differential equation.

1. Notation

In the whole paper we denote by $\mathbf{S}$ the set of real symmetric $n \times n$ matrices. For any two symmetric matrices Q, $\hat{Q} \in \mathbf{S}$, by the inequality $Q \le \hat{Q}$ we mean that the symmetric matrix $\hat{Q} - Q$ is non-negative definite. For any two $n \times n$ symmetric matrix functions $Q(t)$, $\hat{Q}(t)$, by the inequality $Q(t) \le \hat{Q}(t)$, $t \in M$ we mean that both functions are defined for all $t \in M$ and that $\hat{Q}(t) - Q(t)$ is non-negative definite on M.

2. Discrete order preserving property

In this section we recall the order preserving property of the discrete Riccati matrix equation and its modifications. By the *discrete Riccati matrix equation* we mean the difference equation

$$R[Q]_k := Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k) = 0\,, \tag{2.1}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$, and Q_k are real $n \times n$ matrices, Q_k are symmetric and the $2n \times 2n$ matrices $\mathcal{S}_k$ with block entries $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ are symplectic. This means that

$$\mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \quad \mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

If a matrix $S = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ is symplectic, then its inverse exists, $S^{-1} = \left(\begin{smallmatrix} \mathcal{D}^T & -\mathcal{B}^T \\ -\mathcal{C}^T & \mathcal{A}^T \end{smallmatrix}\right)$, it is also symplectic matrix and the identities

$$\begin{aligned} &\mathcal{A}^T \mathcal{C} = \mathcal{C}^T \mathcal{A}\,, \quad \mathcal{B}^T \mathcal{D} = \mathcal{D}^T \mathcal{B}\,, \quad \mathcal{A}^T \mathcal{D} - \mathcal{C}^T \mathcal{B} = I\,, \\ &\mathcal{A}\mathcal{B}^T = \mathcal{B}\mathcal{A}^T\,, \quad \mathcal{C}\mathcal{D}^T = \mathcal{D}\mathcal{C}^T\,, \quad \mathcal{A}\mathcal{D}^T - \mathcal{B}\mathcal{C}^T = I \end{aligned} \tag{2.2}$$

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hold. Other known properties of symplectic matrices are formulated in the next lemma.

Lemma 2.1. *Let $S = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ be a $2n \times 2n$ symplectic matrix and $Q_0 \in \mathbf{S}$ such that $\mathcal{A} + \mathcal{B}Q_0$ is invertible and denote $Q_1 := (\mathcal{C} + \mathcal{D}Q_0)(\mathcal{A} + \mathcal{B}Q_0)^{-1}$. Then Q_1 is symmetric, $(\mathcal{A} + \mathcal{B}Q_0)^{-1} = \mathcal{D}^T - \mathcal{B}^T Q_1$ and $Q_0 = (-\mathcal{C}^T + \mathcal{A}^T Q_1)(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1}$. Further, $(\mathcal{A} + \mathcal{B}Q_0)^{-1}\mathcal{B} \geq 0$ iff $(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1}\mathcal{B}^T \geq 0$.*

Proof. All can be done by direct calculation, using the properties (2.2). □

The following result about order preserving property of the Riccati equation (2.1) is from [3].

Proposition 2.2 (Proposition 2.4 from [3])**.** *Assume that Q and $\hat{Q}$ are symmetric solutions of the Riccati equation (2.1) on $[0, N]_{\mathbb{Z}} := [0, N] \cap \mathbb{Z}$ such that $(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}\mathcal{B}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$. If $Q_0 \leq \hat{Q}_0$ ($Q_0 < \hat{Q}_0$), then $Q_k \leq \hat{Q}_k$ ($Q_k < \hat{Q}_k$) on $[0, N+1]_{\mathbb{Z}}$. Moreover, in this case $(\mathcal{A}_k + \mathcal{B}_k \hat{Q}_k)^{-1}\mathcal{B}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$ as well.*

Without the assumption $(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}\mathcal{B}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$, the conclusion of Proposition 2.2 does not hold in general. See example in [3, Remark 2.5]. It can be shown that this assumption is necessary.

The following result is a generalization of the order preserving property from Proposition 2.2. It contains equivalence instead of implication and it is formulated in a more general way that omits the notion of Riccati equation.

Theorem 2.3. *Let $S = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ be a $2n \times 2n$ symplectic matrix and $Q \in \mathbf{S}$ be such that the inverse $(\mathcal{A} + \mathcal{B}Q)^{-1}$ exists. The following statements are equivalent:*

(i) $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0\,,$

(ii) $\forall \hat{Q} \in \mathbf{S} : Q \leq \hat{Q} \implies (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \leq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}$,

(iii) $\forall \hat{Q} \in \mathbf{S} : (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \geq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1} \implies Q \geq \hat{Q}\,.$

Further, the following statements are equivalent:

(iv) $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \leq 0\,,$

(v) $\forall \hat{Q} \in \mathbf{S} : (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \leq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1} \implies Q \leq \hat{Q}\,,$

(vi) $\forall \hat{Q} \in \mathbf{S} : Q \geq \hat{Q} \implies (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \geq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}$.

Before the proof of Theorem 2.3 we present the following two lemmas, which are used in the proof of this theorem.

Lemma 2.4. *Let $\mathcal{A}$, $\mathcal{B}$ be $n \times n$ matrices, $\mathcal{A}\mathcal{B}^T$ symmetric, and $Q \in \mathbf{S}$ be such that the inverse $(\mathcal{A} + \mathcal{B}Q)^{-1}$ exists and $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0$. If $\hat{Q} \in \mathbf{S}$ and $Q \leq \hat{Q}$, then the inverse $(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}$ exists as well and $(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \geq 0$.*

Proof. First notice that $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0$ is equivalent with $\mathcal{B}(\mathcal{A} + \mathcal{B}Q)^T \geq 0$ and further $Q \leq \hat{Q}$ implies $B(\mathcal{A} + \mathcal{B}\hat{Q})^T \geq 0$. Now let v be an n-vector such that $(\mathcal{A} + \mathcal{B}\hat{Q})^T v = 0$. The inequality $Q \leq \hat{Q}$ implies $0 \leq v^T \mathcal{B}(\hat{Q} - Q)\mathcal{B}^T v$ and from this we further get

$$
\begin{aligned}
0 &\leq v^T\mathcal{B}(\hat{Q}-Q)\mathcal{B}^Tv = v^T(\mathcal{B}\mathcal{A}^T+\mathcal{B}\hat{Q}\mathcal{B}^T-\mathcal{B}\mathcal{A}^T-\mathcal{B}Q\mathcal{B}^T)v \\
&= v^T\mathcal{B}(\mathcal{A}+\mathcal{B}\hat{Q})^Tv - v^T\mathcal{B}(\mathcal{A}+\mathcal{B}Q)^Tv = -v^T\mathcal{B}(\mathcal{A}+\mathcal{B}Q)^Tv\,.
\end{aligned}
$$

From positive semidefinity of $\mathcal{B}(\mathcal{A}+\mathcal{B}Q)^T$ we have that $v^T\mathcal{B}(\hat{Q}-Q)\mathcal{B}^Tv=0$ and from positive semidefinity of $\hat{Q}-Q$ we further have that $(\hat{Q}-Q)\mathcal{B}^Tv=0$. From this relationship together with $(\mathcal{A}+\mathcal{B}\hat{Q})^Tv=0$ we get that $(-\mathcal{A}^T-Q\mathcal{B}^T)v=0$ and hence $v=0$. This proves that the inverse $(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}$ exists. Now, inequality $B(\mathcal{A}+\mathcal{B}\hat{Q})^T\geq 0$ implies that $(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}\mathcal{B}\geq 0$. □

Lemma 2.5. *Let $S=\left(\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right)$ be a $2n\times 2n$ symplectic matrix. For any matrices $Q,\hat{Q}\in\mathbf{S}$, such that the inverses $(\mathcal{A}+\mathcal{B}Q)^{-1}$ and $(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}$ exist, we have the identity*

$$
\begin{aligned}
&(\mathcal{C}+\mathcal{D}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}-(\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1} \\
&= (\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q+(\hat{Q}-Q)(\mathcal{A}+\mathcal{B}Q)^{-1}\mathcal{B}(\hat{Q}-Q)\right](\mathcal{A}+\mathcal{B}\hat{Q})^{-1}.
\end{aligned}
\tag{2.3}
$$

Proof. We use the identities (2.2) and Lemma 2.1 and we get the identity by direct calculation:

$$
\begin{aligned}
&(\mathcal{C}+\mathcal{D}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}-(\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1} \\
&\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}(\mathcal{C}^T+\hat{Q}\mathcal{D}^T)-(\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1} \\
&\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1} \\
&\qquad\times\left[(\mathcal{C}^T+\hat{Q}\mathcal{D}^T)(\mathcal{A}+\mathcal{B}Q)-(\mathcal{A}^T+\hat{Q}\mathcal{B}^T)(\mathcal{C}+\mathcal{D}Q)\right](\mathcal{A}+\mathcal{B}Q)^{-1} \\
&\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q\right](\mathcal{A}+\mathcal{B}Q)^{-1} \\
&\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q\right](\mathcal{A}+\mathcal{B}Q)^{-1}(\mathcal{A}+\mathcal{B}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1} \\
&\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q\right](\mathcal{A}+\mathcal{B}Q)^{-1}\left[\mathcal{A}+\mathcal{B}Q+\mathcal{B}(\hat{Q}-Q)\right](\mathcal{A}+\mathcal{B}\hat{Q})^{-1} \\
&\quad=(\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[\hat{Q}-Q+(\hat{Q}-Q)(\mathcal{A}+\mathcal{B}Q)^{-1}\mathcal{B}(\hat{Q}-Q)\right](\mathcal{A}+\mathcal{B}\hat{Q})^{-1}.
\end{aligned}
$$

□

Proof of Theorem 2.3. The implication (i) $\Longrightarrow$ (ii) and the implication (i) $\Longrightarrow$ (iii) follows immediately from the identity (2.3) and Lemma 2.4.

Now we prove the implication (ii) $\Longrightarrow$ (i). Let's suppose (i) does not hold, that is, there exists such n-vector v that $v^T(\mathcal{A}+\mathcal{B}Q)^{-1}\mathcal{B}v<0$. We now show that then also (ii) does not hold. We take $\hat{Q}=Q+tI$, where t is a positive real number such that the inverse $(\mathcal{A}+\mathcal{B}Q+t\mathcal{B})^{-1}$ exists. Then $Q\leq\hat{Q}$. Further we get from the identity (2.3) that

$$
\begin{aligned}
&(\mathcal{C}+\mathcal{D}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1}-(\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1} \\
&\qquad\qquad= (\mathcal{A}+\mathcal{B}\hat{Q})^{T-1}\left[tI+t^2(\mathcal{A}+\mathcal{B}Q)^{-1}\mathcal{B}\right](\mathcal{A}+\mathcal{B}\hat{Q})^{-1}.
\end{aligned}
$$

Now we take vector $u = (\mathcal{A} + \mathcal{B}\hat{Q})v$ and we get

$$u^T\big[(\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1} - (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1}\big]u \\ = tv^T v + t^2 v^T(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B}v\,,$$

which is negative for sufficiently large t. Hence, there exists $\hat{Q} = Q + tI$ such that the matrix $(\mathcal{C}+\mathcal{D}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1} - (\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1}$ is not positive semi-definite.

In the proof of the implication (iii) $\implies$ (i) we use Lemma 2.1. Let $Q_1 := (\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1}$. Then $\mathcal{D}^T - \mathcal{B}^T Q_1$ is invertible and $Q = (-\mathcal{C}^T + \mathcal{A}^T Q_1)(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1}$. Now again we suppose that (i) does not hold. Then also the inequality $(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1}\mathcal{B}^T \geq 0$ does not hold, that is, there exists such n-vector v that $v^T(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1}\mathcal{B}^T v < 0$. We now show that then also (iii) does not hold. We take $\hat{Q}_1 = Q_1 - tI$, where t is a positive real number such that the inverse $(\mathcal{D}^T - \mathcal{B}^T Q_1 + t\mathcal{B}^T)^{-1}$ exists. Then $Q_1 \geq \hat{Q}_1$. Denote $\hat{Q} := (-\mathcal{C}^T + \mathcal{A}^T\hat{Q}_1)(\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{-1}$. Now we get from the identity (2.3), applied on the symplectic matrix $\left(\begin{smallmatrix} \mathcal{D}^T & -\mathcal{B}^T \\ -\mathcal{C}^T & \mathcal{A}^T \end{smallmatrix}\right)$, that

$$\begin{aligned}\hat{Q} - Q &= (-\mathcal{C}^T + \mathcal{A}^T\hat{Q}_1)(\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{-1} - (-\mathcal{C}^T + \mathcal{A}^T Q_1)(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1} \\ &= (\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{T-1}\left[-tI - t^2(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1}\mathcal{B}^T\right](\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{-1}.\end{aligned}$$

Now we take vector $u = (\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)v$ and we get

$$u^T\left[\hat{Q} - Q\right]u = -tv^T v - t^2 v^T(\mathcal{D}^T - \mathcal{B}^T Q_1)^{-1}\mathcal{B}^T v\,,$$

which is positive for sufficiently large t. Hence, there exists $\hat{Q} \in \mathbf{S}$ such that the matrix $(\mathcal{C}+\mathcal{D}\hat{Q})(\mathcal{A}+\mathcal{B}\hat{Q})^{-1} - (\mathcal{C}+\mathcal{D}Q)(\mathcal{A}+\mathcal{B}Q)^{-1} = \hat{Q}_1 - Q_1 = -tI$ is negative semidefinite but $\hat{Q} - Q$ is not negative semidefinite.

The proof of the equivalence of (iv)–(vi) is analogous. □

Remark 2.6. In Theorem 2.3 in the statements (ii), (iii), (v) and (vi), we can replace the set $\mathbf{S}$ with the set $\mathbf{M} = \{Q \in \mathbf{S} : \mathcal{A} + \mathcal{B}Q$ is invertible and $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0\}$. In the proof of the implication (ii) $\implies$ (i) there exists also a sufficiently large t such that $(\mathcal{A} + \mathcal{B}Q)^{-1}$ exists and $B(\mathcal{A}^T + Q\mathcal{B}^T + t\mathcal{B}^T) \geq 0$, so $\hat{Q} = Q + tI \in \mathbf{M}$, and in the proof of the implication (iii) $\implies$ (i) there exists also a sufficiently large t such that $(\mathcal{D}^T - \mathcal{B}^T\hat{Q}_1)^{-1}$ exists and $\mathcal{B}^T(\mathcal{D} - Q_1\mathcal{B} + t\mathcal{B}) \geq 0$, which is equivalent with $(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \geq 0$.

The following corollary we get directly from Theorem 2.3.

Corollary 2.7. *Let $S = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ be a $2n \times 2n$ symplectic matrix and $Q, \hat{Q} \in \mathbf{S}$ be such that both inverses $(\mathcal{A} + \mathcal{B}Q)^{-1}$ and $(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}$ exist and both inequalities $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \leq 0, (\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \leq 0$ hold, or both inequalities $(\mathcal{A} + \mathcal{B}Q)^{-1}\mathcal{B} \geq 0, (\mathcal{A} + \mathcal{B}\hat{Q})^{-1}\mathcal{B} \geq 0$ hold. Then*

$$Q \leq \hat{Q} \quad \Leftrightarrow \quad (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1} \leq (\mathcal{C} + \mathcal{D}\hat{Q})(\mathcal{A} + \mathcal{B}\hat{Q})^{-1}.$$

3. Riccati matrix differential equation

In this section we present application of the discrete order preserving property to the continuous case, that is to the Riccati matrix differential equation. By the *Riccati matrix differential equation* we mean the equation

$$Q'(t) + A^T(t)Q(t) + Q(t)A(t) + Q(t)B(t)Q(t) - C(t) = 0\,, \tag{3.1}$$

where $A(t)$, $B(t)$, $C(t)$ and $Q(t)$ are real $n \times n$ matrix functions of t and $B(t)$, $C(t)$, $Q(t)$ are symmetric.

At first we introduce the classic version of the order preserving property of the Riccati matrix differential equation.

Proposition 3.1 (Proposition 6, Chapter 2 in [1])**.** *Let $Q(t)$, $\hat{Q}(t)$ be symmetric solutions of the Riccati matrix differential equation* (3.1) *on an interval $\mathcal{I}$. If, for some a in $\mathcal{I}$, $Q(a) \le \hat{Q}(a)$, then $Q(t) \le \hat{Q}(t)$ for all t in $\mathcal{I}$. If $Q(a) < \hat{Q}(a)$, then $Q(t) < \hat{Q}(t)$ for all t in $\mathcal{I}$.*

The relation between the Riccati matrix differential equation and the discrete Riccati matrix equation can be seen from the form of the solution of the differential equation, which is shown in the next lemma.

Lemma 3.2. *Let*

$$S(t) = \begin{pmatrix} \tilde{X}(t) & \bar{X}(t) \\ \tilde{U}(t) & \bar{U}(t) \end{pmatrix}$$

be the solution of linear Hamiltonian system corresponding to the equation (3.1)*,*

$$S'(t) = \begin{pmatrix} A(t) & B(t) \\ C(t) & -A^T(t) \end{pmatrix} \cdot S(t),$$

with the initial condition $S(0) = I$ on an interval $\mathcal{I}_0$ and let $\tilde{X}(t) + \bar{X}(t)Q_0$ be invertible on $\mathcal{I}_0$. Then

$$Q(t) = (\tilde{U}(t) + \bar{U}(t)Q_0)(\tilde{X}(t) + \bar{X}(t)Q_0)^{-1} \tag{3.2}$$

is solution of the Riccati equation (3.1) *with the initial condition*

$$Q(0) = Q_0 \tag{3.3}$$

on $\mathcal{I}_0$. Moreover, the matrix $S(t)$ is symplectic.

Proof. We can directly substitute the right side of (3.2) into the equation (3.1) and verify the result. See also the proof of [4, Lemma 2]. □

Now follows the main result, a modification of the order preserving property of the Riccati matrix differential equation.

Theorem 3.3. *Let $Q(t)$ be unique symmetric solution of the Riccati matrix differential equation* (3.1) *with the initial condition $Q(0) = Q_0$ on an interval $\mathcal{I}_0$. Let $S(t) = \left(\begin{smallmatrix} \tilde{X}(t) & \bar{X}(t) \\ \tilde{U}(t) & \bar{U}(t) \end{smallmatrix}\right)$ be the solution of the corresponding Hamiltonian system as in*

Lemma 3.2. The following statements are equivalent:

$$\text{(i)}\quad (\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}\bar{X}(t) \geq 0\,,\ t\in M\,,$$
$$\text{(ii)}\quad \forall \hat{Q}_0 \in \mathbf{S}:\ Q_0 \leq \hat{Q}_0 \implies Q(t) \leq \hat{Q}(t)\,,\ t\in M\,,$$
$$\text{(iii)}\quad \forall \hat{Q}_0 \in \mathbf{S}:\ Q(t) \geq \hat{Q}(t)\,,\ t\in M \implies Q_0 \geq \hat{Q}_0\,,$$

where M is any subset of $\mathcal{I}_0$ and $\hat{Q}(t)$ is unique symmetric solution of the Riccati matrix differential equation (3.1) *with the initial condition $\hat{Q}(0)=\hat{Q}_0$.*

Further, the following statements are equivalent:

$$\text{(iv)}\quad (\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}\bar{X}(t) \leq 0\,,\ t\in M\,,$$
$$\text{(v)}\quad \forall \hat{Q}_0 \in \mathbf{S}:\ Q(t) \leq \hat{Q}(t)\,,\ t\in M \implies Q_0 \leq \hat{Q}_0\,,$$
$$\text{(vi)}\quad \forall \hat{Q}_0 \in \mathbf{S}:\ Q_0 \geq \hat{Q}_0 \implies Q(t) \geq \hat{Q}(t)\,,\ t\in M\,.$$

Proof. By Lemma 3.2 and the assumption that the solutions $Q(t)$ and $\hat{Q}(t)$ are unique, we have that $Q(t)=(\tilde{U}(t)+\bar{U}(t)Q_0)(\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}$, $\hat{Q}(t)=(\tilde{U}(t)+\bar{U}(t)\hat{Q}_0)(\tilde{X}(t)+\bar{X}(t)\hat{Q}_0)^{-1}$ and the matrix $S(t)$ is symplectic on $\mathcal{I}_0$. Thus, the statements (i)–(iii) and the statements (iv)–(vi) are equivalent for any fixed $t\in M$ because of Theorem 2.3. □

Corollary 3.4. *Let $Q(t),\hat{Q}(t)$ be unique symmetric solutions of the Riccati matrix differential equation* (3.1) *with the initial conditions $Q(0)=Q_0$ and $\hat{Q}(0)=\hat{Q}_0$ on an interval $\mathcal{I}_0$. Let $\left(\begin{smallmatrix}\tilde{X}(t) & \bar{X}(t)\\ \tilde{U}(t) & \bar{U}(t)\end{smallmatrix}\right)$ be the solution of the corresponding Hamiltonian system as in Lemma 3.2 and let M be a subset of $\mathcal{I}_0$. If inequalities*

$$(\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}\bar{X}(t) \leq 0\,,\ (\tilde{X}(t)+\bar{X}(t)\hat{Q}_0)^{-1}\bar{X}(t) \leq 0\,,\ t\in M$$

hold, or inequalities

$$(\tilde{X}(t)+\bar{X}(t)Q_0)^{-1}\bar{X}(t) \geq 0\,,\ (\tilde{X}(t)+\bar{X}(t)\hat{Q}_0)^{-1}\bar{X}(t) \geq 0\,,\ t\in M$$

hold, then

$$Q_0 \leq \hat{Q}_0 \quad \Leftrightarrow \quad Q(t) \leq \hat{Q}(t)\,,\ t\in M\,.$$

Proof. The proof is analogous to the proof of Theorem 3.3, we use Corollary 2.7. □

In the last part of this section we present a simple example to illustrate the difference between the order preserving property from Proposition 3.1 and Theorem 3.3.

Example 3.5. Let us have the Riccati equation

$$Q'(t)-Q^2(t)-I=0\,. \tag{3.4}$$

The corresponding linear Hamiltonian system is

$$S'(t)=\begin{pmatrix}0 & -I\\ I & 0\end{pmatrix}\cdot S(t)$$

and its unique solution with the initial condition $S(0) = I$ is

$$S(t) = \begin{pmatrix} I\cos t & -I\sin t \\ I\sin t & I\cos t \end{pmatrix} .$$

The solution of the equation (3.4) with $Q(0) = Q_0$ is

$$Q(t) = (I\sin t + Q_0\cos t)(I\cos t - Q_0\sin t)^{-1} = \tan(It + \arctan Q_0)$$

and it is defined on the interval $\mathcal{I}_0 = \big(-\frac{\pi}{2} - \arctan\lambda_{\min}, \frac{\pi}{2} - \arctan\lambda_{\max}\big)$, where $\lambda_{\min}$ is the smallest and $\lambda_{\max}$ is the largest eigenvalue of Q_0. The inequality in the statement (i) from Theorem 3.3 is $(I\cos t - Q_0\sin t)^{-1}\sin t \le 0$ and the largest subset of $\mathcal{I}_0$ where it is true is the interval $M = \big(-\frac{\pi}{2} - \arctan\lambda_{\min}, 0\big]$.

Now, from Theorem 3.3, we get, that for all $\hat{Q}_0 \in \mathbf{S}$: $Q_0 \le \hat{Q}_0$ implies that the solution $\hat{Q}(t) = \tan(It + \arctan\hat{Q}_0)$ is defined for all $t \in M$ and that the inequality $Q(t) \le \hat{Q}(t)$ holds for all $t \in M$. Moreover, M is the largest subset of $\mathcal{I}_0$ with this property.

For a comparison, from Proposition 3.1 we get, that for all $\hat{Q}_0 \in \mathbf{S}$: $Q_0 \le \hat{Q}_0$ implies $Q(t) \le \hat{Q}(t)$ on any subset of $\mathcal{I}_0$ such that the solution $\hat{Q}(t)$ is defined there.

From Theorem 3.3 we may further get other results regarding this equation (3.4), analogous to the one presented above.

The converse statement to that one in Proposition 3.1 was proven in [2] and it says that if a symmetric matrix equation has the order preserving property and the matrix dimension is $n \ge 2$, then it is the Riccati equation. Therefore it is possible, that also converse statements to those in Theorem 3.3 or in Corollary 3.4 can be proven, as well as converse statements to Theorem 2.3 and Corollary 2.7, which represent the discrete case. This remains an open problem. There is only the result published in [4] that deals with the continuous version of the order preserving property and the discrete Riccati equation.

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