ARCHIVUM MATHEMATICUM (BRNO)
Tomus 59 (2023), 133–140

EQUIVALENCE OF ILL-POSED DYNAMICAL SYSTEMS

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Abstract. The problem of topological classification is fundamental in the study of dynamical systems. However, when we consider systems without well-posedness, it is unclear how to generalize the notion of equivalence. For example, when a system has trajectories distinguished only by parametrization, we cannot apply the usual definition of equivalence based on the phase space, which presupposes the uniqueness of trajectories.

In this study, we formulate a notion of "topological equivalence" using the axiomatic theory of topological dynamics proposed by Yorke [7], where dynamical systems are considered to be shift-invariant subsets of a space of partial maps. In particular, we study how the type of problems can be regarded as invariants under the morphisms between systems and how the usual definition of topological equivalence can be generalized.

This article is intended to also serve as a brief introduction to the axiomatic theory of ordinary differential equations (or topological dynamics) based on the formalism presented in [6].

1. Introduction

The purpose of the present article is to explain what an axiomatic theory of ordinary differential equations is and how it enables us to classify "flows" without well-posedness assumptions.

In the first place, it is natural to ask why we need an axiomatic theory of ordinary differential equations here. A short answer is that the usual criteria of classification require too much to be applicable to those without well-posedness.

In the study of dynamical systems of flows, we classify systems according to the notion of topological equivalence, which is defined as follows [5].

Definition 1.1 (Topological equivalence)**.** Let X and Y be topological spaces. Two flows $\Phi\colon \mathbb{R}\times X \to X$ and $\Psi\colon \mathbb{R}\times Y \to Y$ are *topologically equivalent* if there exists a homeomorphism $h\colon X \to Y$ such that each orbit of Φ is mapped to an orbit of Ψ preserving the orientation of the orbit.

2020 *Mathematics Subject Classification*: primary 37B02; secondary 37B55, 34A06, 34A34.

Key words and phrases: dynamical systems, topological dynamics, topological equivalence, axiomatic theory of ordinary differential equations.

This research was supported by a Grant-in-Aid for JSPS Fellows (20J01101).

Received August 21, 2022, accepted October 4, 2022. Editor R. Šimon Hilscher.

DOI: 10.5817/AM2023-1-133

However, it poses an inherent difficulty to generalize it to the systems without well-posedness in the sense of Hadamard, in particular, uniqueness. Let us illustrate this point with examples.

Example 1.2. If the uniqueness of orbits is not assumed, topological equivalence as defined above does not define an equivalence relation. For example, let us consider two "flows" defined on $\mathbb{R}$ by

(1) $\dot{x} = 1$

(2) $\dot{x} = 3x^{\frac{2}{3}}$.

The identity map is a homeomorphism that sends each orbit of (1) to an orbit of (2), preserving the orientation. However, the number of equilibria is clearly different between these two systems.

The problem of the last example can be amended straightforwardly by requiring that the inverse of homeomorphisms also preserve the orbits. However, even if we require so, problems remain.

Example 1.3. The following systems on $\mathbb{R}$ are indistinguishable if we use the same criteria as in Definition 1.1:

(1) $\dot{x} = 1$.

(2) $\dot{x} \in \{1/2, 1\}$.

(3) $\dot{x} \in [1/2, 1]$.

Here, systems (2) and (3) are differential inclusions (the definition and details can be found, for example, in [1]). Even if we require that the inverse of homeomorphisms also preserve the orbits, we still cannot distinguish them. This is because there exists only one orbit if we ignore the parametrization.

Thus, in the classification of systems without well-posedness, it is necessary to consider a kind of "topological equivalence", which does not entirely ignore the parametrization. One of the valuable properties of an axiomatic theory of ODE is that we may consider the space of solutions without mentioning problems. This enables us to construct a general framework to treat such classification problems.

In general, an axiomatic theory of ODE consists of two ingredients. One is a space of partial maps, later regarded as a space of "solutions". Another is a set of axioms to be satisfied by such "solutions". Depending on the selection of the above two elements, possibly we obtain many different theories. However, there are mainly two formalisms of the axiomatic theory of ODE. In J.A. Yorke's formalism, partial maps with open domains are considered [7]. On the other hand, V.V. Filippov's theory is based on partial maps with closed domains [3, 4]. This difference in the choice of the class of partial maps results in a significant difference in the treatment. Here we consider a generalization of Yorke's formalism since it is easier to consider the generalization of flows and the problem of their classification within this framework, although V.V. Filippov's theory is much more developed (actually, the details of J.A. Yorke's formalism have not been published except for a small portion).

In this article, we consider the problem of classification of general dynamical systems based on Yorke's formalism of axiomatic theory of ODE. While the theory given here is based on [6], we use an improved formulation in this article, and new results on the description of dynamics are also presented.

In what follows, we assume that X is a second-countable metric space and G is a locally compact second-countable metrizable topological group, e.g., $\mathbb{R}$.

2. Yorke's formalism

First, let us define the notion of partial maps as used here.

Definition 2.1 (Partial maps). A continuous map $\phi\colon D \to X$ is a *partial map from G to X* if $D \subset G$ is a nonempty open set.

The set of all partial maps is denoted by $C_p(G, X)$. For each $\phi\colon D \to X$, we set $\operatorname{dom}\phi := D$.

A partial map $\phi \in C_p(G, X)$ with a connected domain is *maximally defined* if, for all $\psi \in C_p(G, X)$ with a connected domain, the condition $\operatorname{dom}\phi \subset \operatorname{dom}\psi$ and $\phi = \psi$ on $\operatorname{dom}\phi$ implies $\phi = \psi$.

The set of all maximally defined partial maps is denoted by $C_s(G, X)$.

Remark 2.2. It is convenient to define the inverse image of a subset $A \subset X$ under a partial map $\phi\colon G \to X$ by

$$\phi^{-1}(A) := \{g \in G \mid g \in \operatorname{dom}\phi \text{ and } \phi(g) \in A\}\,.$$

In particular, we have $\operatorname{dom}\phi = \phi^{-1}(X)$. By the continuity on the domain, the inverse image of an open set under a partial map is always open.

We topologize $C_s(G, X)$ by introducing the topology of compact convergence (with modifications). That is, we define $\phi_n \to \phi$ as $n \to \infty$ in $C_s(G, X)$ if and only if, for all compact subsets $K \subset \operatorname{dom}\phi$, we have $K \subset \operatorname{dom}\phi_n$ for sufficiently large n and $\sup_{t\in K} d(\phi_n(t), \phi(t)) \to 0$ as $n \to \infty$.

This topology can also be described using the compact-open topology (Lemma 2.3 in [6]). In this description, subbases are the sets of the form

$$W(K, V) := \{\phi \in C_s(G, X) \mid K \subset \operatorname{dom}\phi \text{ and } \phi(K) \subset V\}\,,$$

where $K \subset G$ is compact and $V \subset X$ is open.

The problem here is that $C_s(G, X)$ need not be Hausdorff in this topology.

Example 2.3 (Yorke [7]). Consider a sequence of maps $\{\phi_n\}_{n\in\mathbb{N}} \subset C_s(\mathbb{R}, \mathbb{R})$ given by

$$\phi_n(t) := \frac{1}{t^2 + \frac{1}{n}}\,,$$

and partial maps $\phi^{\pm}$ defined by $\phi^{\pm}(t) = \frac{1}{t^2}$, $\operatorname{dom}\phi^+ = (0, \infty)$ and $\operatorname{dom}\phi^- = (-\infty, 0)$. Then the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ converges to both ϕ^+ and ϕ^- in $C_s(\mathbb{R}, \mathbb{R})$. Consequently, $C_s(\mathbb{R}, \mathbb{R})$ is not Hausdorff.

It is worth noting that $C_s(G, X)$ satisfies separation axioms weaker than Hausdorff.

Proposition 2.4. *The space $C_s(G,X)$ is T_1.*

Proof. As X is a second-countable metric space, it is in particular T_1. Let ϕ and ψ be two distinct partial maps in $C_s(G,X)$. Since ϕ and ψ are maximally defined, either $\operatorname{dom}\phi\cap\operatorname{dom}\psi$ is empty or there exists $g\in\operatorname{dom}\phi\cap\operatorname{dom}\psi$ with $\phi(g)\neq\psi(g)$. In the former case, we may take any $g'\in\operatorname{dom}\psi$ to obtain $\phi\notin W(\{g'\},X)$. In the latter case, there exists an open neighborhood V of $\psi(g)$ such that $\phi(g)\notin V$. Therefore, we have $\phi\notin W(\{g\},V)$. □

Remark 2.5. If X is discrete, $C_s(G,X)$ is Hausdorff.

To justify the use of sequences in the analysis, we consider the following construction originally due to Yorke.

Definition 2.6. For a subset $S\subset C_s(G,X)$, we define a partial map $e_S\colon G\times S\to G\times X$ by

$$e_S(g,\phi):=\big(g,\phi(g)\big)\,.$$

For each subset $W\subset G\times X$, we define

$$S^*W:=e_S^{-1}(W)=\big\{(g,\phi)\mid \phi\in S, g\in\operatorname{dom}\phi,\big(g,\phi(g)\big)\in W\big\}\,.$$

We call S^*W the *star-construction* defined by S and W.

Lemma 2.7. *For a nonempty subset $S\subset C_s(G,X)$, the partial map $e_S\colon G\times S\to G\times X$ is well-defined, that is, it is continuous on the domain, which is nonempty and open.*

Proof. It is sufficient to show that $e_S^{-1}(W)$ is open if $W\subset G\times X$ is open. Let W be open and $(g,\phi)\in e_S^{-1}(W)$. Then, we can find an open neighborhood U_0 of g and V of $\phi(g)$ with $U_0\times V\subset W$. Since $U_0\cap\phi^{-1}(V)$ is an open neighborhood of g and G is locally compact, there exists another open neighborhood U of g such that $U\subset\bar U\subset U_0\cap\phi^{-1}(V)$ and $\bar U$ is compact. Then $U\times W(\bar U,V)$ is an open neighborhood of (g,ϕ) contained in $e_S^{-1}(W)$. □

The map e_S can be seen as an extended evaluation map, and consequently, the star-construction S^*W is an abstraction of the initial value problem on W with solutions in S. We can show that the space S^*W is Hausdorff and second-countable under our assumptions on X and G (Theorem 2.8 in [6]).

In the next definition, we introduce the main additional axioms, which are an abstraction of the conditions for well-posedness.

Definition 2.8. Let $S\subset C_s(G,X)$.

(1) The subspace S satisfies the *compactness axiom* if e_S is a proper map.

(2) The subspace S satisfies the *existence axiom* on W if $e_S\colon e_S^{-1}(W)\to W$ is surjective.

(3) The subspace S satisfies the *uniqueness axiom* on W if $e_S\colon e_S^{-1}(W)\to W$ is injective.

(4) The subspace S has a *domain* D if e_S is defined on $D\times S$.

The interpretation of the existence and uniqueness axioms is straightforward. The compactness axiom is an abstraction of the continuous dependence on the initial conditions (see Theorem 2.3 in [6]). If a space of solutions S has a domain D, we may regard S to be globally defined on D.

Remark 2.9. The formulation of the theory given here is somewhat different from that in [6] or [7], which does not involve the extended evaluation map e_S. However, it is easily observed to be equivalent.

The apparatuses introduced so far enable us to describe an initial value problem and the corresponding space of solutions. Based on this framework, the dynamics are described using the shift map.

Theorem 2.10 (Shift map, Theorem 3.1 in [6])**.** *The shift map* $\sigma\colon G\times C_p(G,X)\to C_p(G,X)$*, which is defined by*

$$\sigma(g,\phi)(x) := \phi(xg)$$

for $x\in \mathrm{dom}(\phi)g^{-1}$*, is continuous and satisfies the following conditions:*

(1) *For each* $\phi\in C_p(G,X)$ *we have* $\sigma(e,\phi)=\phi$*.*

(2) *For all* g*,* $h\in G$ *and* $\phi\in C_p(G,X)$*, we have* $\sigma(g,\sigma(h,\phi))=\sigma(gh,\phi)$*.*

That is, σ *is a left* G*-action.*

The correspondence with the usual theory of dynamical systems is given by the following theorem, which claims that flows can be identified with well-behaved subspaces of $C_s(G,X)$. We may regard this to be one of the fundamental theorems of Yorke's theory.

Theorem 2.11 (Theorem 2.3 in [7] and Theorem 3.3 in [6])**.** *Let* X *be locally compact. Then a* σ*-invariant subset* $S\subset C_s(G,X)$ *satisfies the compactness, existence, and uniqueness axioms and has domain* G *if and only if it is given by a left* G*-action* $\pi_S : G\times X\to X$ *on* X *via*

$$S := \{\pi_S(\cdot,x)\mid x\in X\}\,. \tag{2.1}$$

Thus, our theory subsumes that of flows, and in this sense, it is a generalization of the theory of topological dynamics.

3. Concatenation of solutions and conditional evolution of trajectories

For the description of dynamics in the case $G=\mathbb{R}$, an interesting question is when the concatenation of solutions is admissible. In Yorke's formalism, this property is formulated as follows.

Definition 3.1. A subspace $S\subset C_s(\mathbb{R},X)$ satisfies the *switching axiom* if S contains the map defined by

$$\psi(t)=\begin{cases}\phi_1(t) & (t\le\tau)\\ \phi_2(t) & (t\ge\tau)\end{cases}$$

whenever ϕ_1, $\phi_2 \in S$ satisfy

$$\phi_1(\tau) = \phi_2(\tau)$$

for some $\tau \in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2$.

Remark 3.2. Compared to other axiomatic theories of ODE or semiflows, such as Filippov's theory or Ball's theory of generalized semiflows [2], it is one of the characteristics of Yorke's formalism that it does not require the concatenation property by default.

The rules of time evolution for a system $S \subset C_s(\mathbb{R}, X)$ are described in terms of the conditional evolution of trajectories. For example, an autonomous ODE can be regarded to describe how a trajectory may be extended given the present position in the phase space. Therefore, we introduce the following notion of conditional solution spaces, which represent the rules of time evolution as inferred from the past data.

Definition 3.3. Let $S \subset C_s(\mathbb{R}, X)$. For $\phi \in S$ and $\tau \in \operatorname{dom}\phi$, we define

$$S(\phi|_{(-\infty,\tau]}) := \{\psi|_{[\tau,\infty)} \mid \psi \in S \text{ and } \psi(t) = \phi(t) \ \text{ for } t \le \tau\}$$

$$S(\tau, x) := \{\psi|_{[\tau,\infty)} \mid \psi \in S \text{ and } \psi(\tau) = x\}$$

The following result makes it clear that the switching axiom is actually an axiom restricting the rules of time evolution. In short, knowing all the past makes no difference if and only if the switching axiom holds.

Proposition 3.4. *A subspace $S \subset C_s(\mathbb{R}, X)$ satisfies the switching axiom if and only if*

$$S\left(\phi|_{(-\infty,\tau]}\right) = S\big(\tau, \phi(\tau)\big)$$

for all $\phi \in S$ and $\tau \in \operatorname{dom}\phi$.

Proof. Let S satisfy the switching axiom, and fix $\phi \in S$ and $\tau \in \operatorname{dom}\phi$. By definition, we have

$$S\left(\phi|_{(-\infty,\tau]}\right) \subset S\big(\tau, \phi(\tau)\big).$$

If $\psi|_{[\tau,\infty)} \in S(\tau, \phi(\tau))$, we have $\phi(\tau) = \psi(\tau)$ and therefore we may apply the switching axiom to deduce that $\psi|_{[\tau,\infty)} \in S\left(\phi|_{(-\infty,\tau]}\right)$. Therefore $S\left(\phi|_{(-\infty,\tau]}\right) = S(\tau, \phi(\tau))$.

Conversely, let

$$S\left(\phi|_{(-\infty,\tau]}\right) = S\big(\tau, \phi(\tau)\big)$$

for all $\phi \in S$ and $\tau \in \operatorname{dom}\phi$ and fix ϕ_1, $\phi_2 \in S$ with $\phi_1(\tau) = \phi_2(\tau)$ for some $\tau \in \operatorname{dom}\phi_1 \cap \operatorname{dom}\phi_2$. Then we have

$$\phi_2|_{[\tau,\infty)} \in S\big(\tau, \phi_2(\tau)\big) = S\big(\tau, \phi_1(\tau)\big) = S\big(\phi_1|_{(-\infty,\tau]}\big).$$

Therefore there exists $\psi \in S$ with $\psi|_{[\tau,\infty)} = \phi_2|_{[\tau,\infty)}$ and $\psi|_{(-\infty,\tau]} = \phi_1|_{(-\infty,\tau]}$. Consequently, S satisfies the switching axiom. □

The next result is obvious.

Corollary 3.5. *If a subspace $S \subset C_s(\mathbb{R}, X)$ satisfies the uniqueness axiom on $\mathbb{R} \times X$, S satisfies the switching axiom.*

Thus, the concatenation property can be seen as an analog for the Markov property, and we may assume it if the state of the time evolution is completely determined by the position in the phase space. Also, it follows that if the rule of the time evolution involves other state variables, such as the history of the trajectory, then we cannot expect the concatenation property to hold.

4. Generalizations of topological equivalence

So far, we have considered individual systems. At this point, we may ask how the relationship between them is described under this framework. In general, to consider the relationship between mathematical objects, it is necessary to introduce the notion of morphisms. For the star-constructions, we may define it as follows.

Definition 4.1 (Morphisms of the star-construction)**.** Let $S \subset C_s(G, X)$, $S' \subset C_s(G', X')$, $W \subset G \times X$ and $W' \subset G' \times X'$. A *morphism* between the star-constructions S^*W and $(S')^*W'$ is a triplet of continuous maps

$$H\colon S^*W \to (S')^*W' , \qquad k\colon W \to W' , \qquad \eta\colon S \to S'$$

such that following diagrams commute:

$$\begin{array}{ccc} S^*W & \xrightarrow{H} & (S')^*(W') \\ {\scriptstyle e_S}\big\downarrow & & \big\downarrow{\scriptstyle e_{S'}} \\ W & \xrightarrow{k} & W' \end{array} \qquad \begin{array}{ccc} S^*W & \xrightarrow{H} & (S')^*(W') \\ {\scriptstyle p_S}\big\downarrow & & \big\downarrow{\scriptstyle p_{S'}} \\ S & \xrightarrow{\eta} & S' \end{array}$$

where p_S and $p_{S'}$ are projections to the map component. We denote a morphism by $\langle H, k, \eta\rangle\colon S^*W \to (S')^*W'$.

If there exists a morphism such that H, k and η are homeomorphisms, then $\langle H, k, \eta\rangle$ is an *isomorphism* and S^*W and $(S')^*W'$ are *isomorphic*.

The axioms listed in Definition 2.8 are preserved by isomorphisms.

Example 4.2. It can be shown that the three systems in Example 1.3 are not isomorphic. Indeed, system (1) satisfies the uniqueness axiom and the compactness axiom. The other systems lack uniqueness. While system (3) satisfies the compactness axiom, system (2) does not.

The equivalence class of subsets of $C_s(G, X)$ under the isomorphism relation is rather large. For example, continuous flows are identified:

Theorem 4.3 (Theorem 4.5 in [6])**.** *Let X be locally compact, and G be connected. If a σ-invariant subset $S \subset C_s(G, X)$ satisfies the compactness, existence, and uniqueness axioms on $G \times X$ and has domain G, $S^*(G \times X)$ is isomorphic to $S_0^*(G \times X)$, where*

$$S_0 := \{\psi_x \in C_s(G, X) \mid x \in X \quad \textit{and} \quad \psi_x(g) = x \quad \textit{for all} \quad g \in G\} .$$

As Yorke's axioms are abstraction of the well-posedness properties of the initial value problems, the classification induced by the isomorphism notion can be seen as that of the types of problems based on how well-posed they are. Considering this point, a more useful notion is defined as follows.

Definition 4.4 (Phase space preserving morphism)**.** Let $S \subset C_s(G, X)$, $S' \subset C_s(G', X')$, $W \subset G \times X$ and $W' \subset G' \times X'$. A morphism $\langle H, k, \eta\rangle$ between the star-constructions S^*W and $(S')^*W'$ *preserves phase space* if k has a form $k = (\tau, h)$, where $\tau\colon W \to G'$ and $h\colon W \to X'$ are continuous and $h(g_1, x) = h(g_2, x)$ for all $(g_1, x), (g_2, x) \in W$.

S^*W and $(S')^*W'$ are *isomorphic via phase space preserving isomorphisms* if there exists a phase space preserving isomorphism $\langle H, k, \eta\rangle$ between S^*W and $(S')^*W'$ such that $\langle H^{-1}, k^{-1}, \eta^{-1}\rangle$ also preserves phase space.

The notion of being isomorphic via phase space preserving isomorphisms respects basic dynamical properties, although the direction of time may be reversed.

Theorem 4.5 (Theorem 4.14 in [6])**.** *Let the star-constructions $S^*(G \times X)$ and $(S')^*(G' \times X')$ be isomorphic via a phase space-preserving isomorphism $\langle H, k, \eta\rangle$: $S^*(G \times X) \to (S')^*(G' \times X')$. Then we have*

$$\hat{h}\left(\mathcal{O}(\phi)\right) = \mathcal{O}\left(\eta(\phi)\right)$$

for all $\phi \in S$, where $k = (\tau, h)$. Here an orbit $\mathcal{O}(\phi)$ of $S^*(G \times X)$ is the set of the form $\mathcal{O}(\phi) := \{\phi(g) \mid g \in \operatorname{dom}\phi\}$, where $\phi \in S$, and the map $\hat{h}$ is defined by $\hat{h}(x) := h(g, x)$ for some $g \in W_x := \{g \in G \mid (g, x) \in W\}$.

The notion of isomorphisms can be improved if we require an additional isotopy condition. Then we obtain another, more stringent generalization of the usual topological equivalence.

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