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SOLUTIONS OF AN ADVANCE-DELAY DIFFERENTIAL EQUATION AND THEIR ASYMPTOTIC BEHAVIOUR

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Abstract. The paper considers a scalar differential equation of an advance-delay type

$$\dot{y}(t) = -\left(a_0 + \frac{a_1}{t}\right) y(t-\tau) + \left(b_0 + \frac{b_1}{t}\right) y(t+\sigma)\,,$$

where constants a_0, b_0, τ and σ are positive, and a_1 and b_1 are arbitrary. The behavior of its solutions for $t \to \infty$ is analyzed provided that the transcendental equation

$$\lambda = -a_0 \mathrm{e}^{-\lambda\tau} + b_0 \mathrm{e}^{\lambda\sigma}$$

has a positive real root. An exponential-type function approximating the solution is searched for to be used in proving the existence of a semi-global solution. Moreover, the lower and upper estimates are given for such a solution.

1. Preliminaries

In [6, Section 2.1] a general scalar linear equation

$$\dot{y}(t) = -c(t)y(t-\tau(t)) + d(t)y(t+\sigma(t)) \tag{1.1}$$

is considered with Lipschitz continuous $c, d\colon [t_0,\infty) \to [0,\infty)$, $\tau\colon [t_0,\infty) \to [0,r_1]$ and $\sigma\colon [t_0,\infty) \to [0,r_2]$, where $r_i > 0$, $i = 1,2$, and the existence of right semi-global solutions to (1.1) is proved . The right semi-global solution is defined as follows. A continuous function $y\colon [t_0 - r_1,\infty) \to \mathbb{R}$ is a right semi-global solution to (1.1) on $[t_0 - r_1,\infty)$ if it is continuously differentiable on $[t_0,\infty)$ and satisfies (1.1) on $[t_0,\infty)$.

The present paper considers a particular case of equation (1.1), specifying c and d as

$$c(t) = a_0 + \frac{a_1}{t}\,,\quad d(t) = b_0 + \frac{b_1}{t}\,,$$

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where a_0, b_0 are positive constants, a_1 and b_1 are arbitrary reals, $\tau > 0$ and $\sigma > 0$. Then (1.1) can be written as

$$\dot{y}(t) = -\Big(a_0 + \frac{a_1}{t}\Big)y(t-\tau) + \Big(b_0 + \frac{b_1}{t}\Big)y(t+\sigma)\,. \tag{1.2}$$

Equation (1.2) is now analyzed. An approximate solution of (1.2) is found in the form

$$y_{as}(t) = \mathrm{e}^{\lambda t} t^{-r}\Big(1 - \frac{A}{t}\Big) \tag{1.3}$$

where λ is a positive real root of the transcendental equation

$$\lambda = -a_0\mathrm{e}^{-\lambda\tau} + b_0\mathrm{e}^{\lambda\sigma} \tag{1.4}$$

and the coefficients A and r will be specified dependending on a_i, b_i, $i = 0, 1$, τ, σ and λ. Then, it is proved that there exists a right semi-global solution to (1.2) which can be estimated from below and from above by functions having a form of the suggested approximate solution y_{as}.

2. Auxiliary lemma

Being a part of Theorem 2 in [6], the following auxiliary result will be used in the paper to prove the existence of a solution with properties described above in Section 1.

Lemma 2.1. *Consider bounded continuous functions $\mathcal{L}, \mathcal{R}\colon [t_0 - r_1, \infty) \to \mathbb{R}$, $\mathcal{L}(t) \leqslant \mathcal{R}(t)$, $t \in [t_0 - r_1, \infty)$ and a Lipschitz continuous function $\varphi\colon [t_0 - r_1, t_0] \to \mathbb{R}$ satisfying $\varphi(t_0) = 0$. Moreover, let*

$$\mathcal{L}(t) \leqslant -c(t)\exp\left(\int_t^{t-\tau(t)} \mathcal{L}(s)\,\mathrm{ds}\right) + d(t)\exp\left(\int_t^{t+\sigma(t)} \mathcal{L}(s)\,\mathrm{ds}\right), \tag{2.1}$$

$$\mathcal{R}(t) \geqslant -c(t)\exp\left(\int_t^{t-\tau(t)} \mathcal{R}(s)\,\mathrm{ds}\right) + d(t)\exp\left(\int_t^{t+\sigma(t)} \mathcal{R}(s)\,\mathrm{ds}\right) \tag{2.2}$$

on $[t_0, \infty)$ and

$$\mathcal{L}(t) \leqslant -c(t_0)\exp\left(\int_{t_0}^{t_0-\tau(t_0)} \mathcal{L}(s)\,\mathrm{ds}\right) + d(t_0)\exp\left(\int_{t_0}^{t_0+\sigma(t_0)} \mathcal{L}(s)\,\mathrm{ds}\right) + \varphi(t), \tag{2.3}$$

$$\mathcal{R}(t) \geqslant -c(t_0)\exp\left(\int_{t_0}^{t_0-\tau(t_0)} \mathcal{R}(s)\,\mathrm{ds}\right) + d(t_0)\exp\left(\int_{t_0}^{t_0+\sigma(t_0)} \mathcal{R}(s)\,\mathrm{ds}\right) + \varphi(t) \tag{2.4}$$

on $[t_0 - r_1, t_0]$. Then, there exists a right semi-global solution $y(t)$ of (1.1) on $[t_0 - r_1, \infty)$ such that $y(t_0 - r_1) = 1$ and

$$\exp\left(\int_{t_0-r_1}^{t} \mathcal{L}(s)\,\mathrm{ds}\right) \leqslant\ y(t) \leqslant \exp\left(\int_{t_0-r_1}^{t} \mathcal{R}(s)\,\mathrm{ds}\right), \quad t \in [t_0 - r_1, \infty)\,. \tag{2.5}$$

Remark 2.2. In applications of Lemma 2.1, a crucial role is played by a proper choice of functions $\mathcal{L}$ and $\mathcal{R}$ because this is often not an easy task. From this point of view, an important contribution of the paper is, among others, in the construction of such functions.

3. Existence of approximate solutions

In this section we will look for approximate solutions of the equation (1.2) in the form (1.3).

Consider transcendental equation (1.4). In the rest of the paper, a typical assumption is that this equation, which we rewrite in the form

$$f(\lambda) := \lambda + a_0 \mathrm{e}^{-\lambda\tau} - b_0 \mathrm{e}^{\lambda\sigma} = 0\,,$$

has a real root $\lambda = \lambda^*$ such that $f'(\lambda^*) \neq 0$ or $f'(\lambda^*) > 0$ where

$$f'(\lambda) = 1 - a_0 \tau \mathrm{e}^{-\lambda\tau} - b_0 \sigma \mathrm{e}^{\lambda\sigma}\,.$$

The lemma below gives sufficient conditions for the existence of such a real root.

Lemma 3.1. *Let μ, ν be positive numbers such that $\mu < \nu$, $f(\mu) < 0$ and $f(\nu) > 0$. If, moreover,*

$$V(\mu, \nu) := 1 - a_0 \tau \mathrm{e}^{-\mu\tau} - b_0 \sigma \mathrm{e}^{\nu\sigma} > 0\,,$$

then there exists a positive root λ^* of equation (1.4) such that $f'(\lambda^*) > 0$.

Proof. It may be seen that there exists a root $\lambda = \lambda^*$ of equation $f(\lambda) = 0$ such that $\lambda^* \in (\mu, \nu)$. Since, for $\lambda \in (\mu, \nu)$, $f'(\lambda) > V(\mu, \nu) > 0$, the root is the only one in the interval (μ, ν) and $f'(\lambda^*) > 0$. □

Example 3.2. Let $a_0 = 1$, $b_0 = 2$, $\sigma = 1/10$ and $\tau = 1$. Then,

$$f(\lambda) := \lambda + \mathrm{e}^{-\lambda} - 2\mathrm{e}^{0.1\lambda}.$$

All the hypotheses of Lemma 3.1, where $\mu = 2$, $\nu = 3$, are satisfied since

$$f(2) = 2 + \mathrm{e}^{-2} - 2\mathrm{e}^{0.2} \doteq -0.307 < 0, \quad f(3) := 3 + \mathrm{e}^{-3} - 2\mathrm{e}^{0.3} \doteq 0.350 > 0$$

and

$$V(2, 3) := 1 - \mathrm{e}^{-2} - 2 \cdot 10^{-1} \mathrm{e}^{0.3} \doteq 0.595 > 0\,.$$

Therefore, there exists a positive root $\lambda = \lambda^* \in (2, 3)$ of the equation $f(\lambda) = 0$ such that $f'(\lambda^*) > 0$. We refer to Figure 1, where the positive root $\lambda^* \doteq 2.479$, shown in red, has the property $f'(\lambda^*) > 0$. The remaining real roots are $\lambda^{**} \doteq -1.047$ and $\lambda^{***} \doteq 25.426$.

The formula, used below is a consequence of the binomial one: For $t \to \infty$ and $\alpha, \beta \in \mathbb{R}$, the asymptotic representation

$$(t - \alpha)^\beta = t^\beta \left[1 - \frac{\alpha\beta}{t} + \frac{\beta(\beta - 1)\alpha^2}{2t^2} + o\left(\frac{1}{t^2}\right) \right] \tag{3.1}$$

holds.

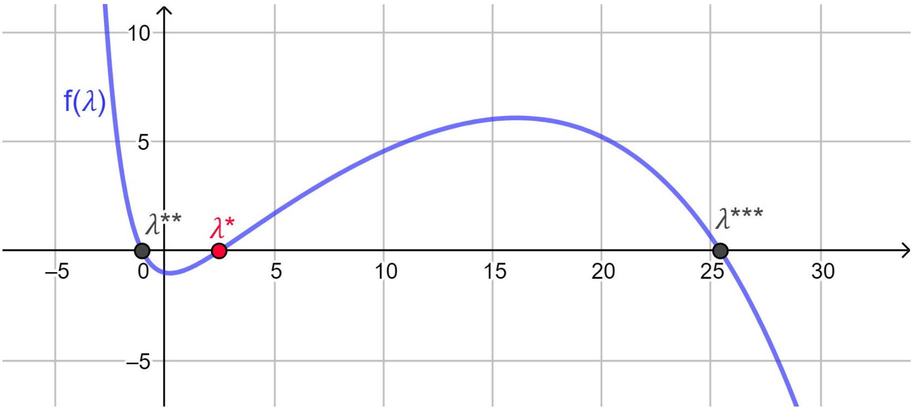

FIG. 1: To Example 3.2: Function $f(\lambda)$ and its real roots

Let positive constants a_0, b_0, τ and σ be given. Suppose that the equation (1.4) has a real root λ^* such that $f'(\lambda^*) \neq 0$. The calculation below indicates that taking $\lambda = \lambda^*$ the parameters r and A in formula (1.3) for y_{as} should be chosen as

$$r = \frac{a_1 \mathrm{e}^{-\lambda^* \tau} - b_1 \mathrm{e}^{\lambda^* \sigma}}{f'(\lambda^*)}\,, \tag{3.2}$$

$$A = \frac{r\left(-a_1 \tau \mathrm{e}^{-\lambda^* \tau} - b_1 \sigma \mathrm{e}^{\lambda^* \sigma}\right) + 0.5 r(-r-1)\left(a_0 \tau^2 \mathrm{e}^{-\lambda^* \tau} - b_0 \sigma^2 \mathrm{e}^{\lambda^* \sigma}\right)}{f'(\lambda^*)}\,. \tag{3.3}$$

Substituting the assumed form of the solution (1.3) into equation (1.2) we obtain an approximate expression

$$\begin{aligned}
&\lambda \mathrm{e}^{\lambda t} t^{-r} - (r + A\lambda) \mathrm{e}^{\lambda t} t^{-r-1} + A(r+1) \mathrm{e}^{\lambda t} t^{-r-2} \propto \\
&-\left(a_0 + \frac{a_1}{t}\right) \mathrm{e}^{\lambda(t-\tau)} (t-\tau)^{-r} \left(1 - \frac{A}{t-\tau}\right) + \left(b_0 + \frac{b_1}{t}\right) \mathrm{e}^{\lambda(t+\sigma)} (t+\sigma)^{-r} \left(1 - \frac{A}{t+\sigma}\right).
\end{aligned}$$

Using formula (3.1), we have

$$\begin{aligned}
&\lambda \mathrm{e}^{\lambda t} t^{-r} - (r + A\lambda) \mathrm{e}^{\lambda t} t^{-r-1} + A(r+1) \mathrm{e}^{\lambda t} t^{-r-2} \propto \\
&\qquad - \left(a_0 + \frac{a_1}{t}\right) \mathrm{e}^{\lambda(t-\tau)} t^{-r} \left[1 + \frac{r\tau}{t} - \frac{r(-r-1)\tau^2}{2t^2}\right] \left(1 - \frac{A}{t} - \frac{A\tau}{t^2}\right) \\
&\qquad\quad + \left(b_0 + \frac{b_1}{t}\right) \mathrm{e}^{\lambda(t+\sigma)} t^{-r} \left[1 - \frac{r\sigma}{t} - \frac{r(-r-1)\sigma^2}{2t^2}\right] \left(1 - \frac{A}{t} + \frac{A\sigma}{t^2}\right)
\end{aligned}$$

and, consequently,

$$
\begin{aligned}
\lambda - (r+A\lambda)t^{-1} + A(r+1)t^{-2} \propto \\
-\left(a_0+\frac{a_1}{t}\right)\mathrm{e}^{-\lambda\tau}\left[1+\frac{r\tau}{t}-\frac{r(-r-1)\tau^2}{2t^2}\right]\left(1-\frac{A}{t}-\frac{A\tau}{t^2}\right) \\
+\left(b_0+\frac{b_1}{t}\right)\mathrm{e}^{\lambda\sigma}\left[1-\frac{r\sigma}{t}-\frac{r(-r-1)\sigma^2}{2t^2}\right]\left(1-\frac{A}{t}+\frac{A\sigma}{t^2}\right).
\end{aligned}
$$

Matching the coefficients corresponding at the powers of t, we derive

$$
\begin{aligned}
t^0 &: & \lambda &= -a_0\mathrm{e}^{-\lambda\tau}+b_0\mathrm{e}^{\lambda\sigma},\\
t^{-1} &: & -(r+A\lambda) &= a_0A\mathrm{e}^{-\lambda\tau}-a_0\mathrm{e}^{-\lambda\tau}r\tau-a_1\mathrm{e}^{-\lambda\tau}-b_0A\mathrm{e}^{\lambda\sigma}-b_0r\sigma\mathrm{e}^{\lambda\sigma}+b_1\mathrm{e}^{\lambda\sigma},\\
t^{-2} &: & A(r+1) &= a_0A\tau\mathrm{e}^{-\lambda\tau}+a_1A\mathrm{e}^{-\lambda\tau}+b_0A\sigma\mathrm{e}^{\lambda\sigma}-b_1A\mathrm{e}^{\lambda\sigma}\\
&&&\quad + r\left(a_0A\tau\mathrm{e}^{-\lambda\tau}-a_1\tau\mathrm{e}^{-\lambda\tau}+b_0A\sigma\mathrm{e}^{\lambda\sigma}-b_1\sigma\mathrm{e}^{\lambda\sigma}\right)\\
&&&\quad -\frac{r(r+1)}{2}\left(a_0\tau^2\mathrm{e}^{-\lambda\tau}-b_0\sigma^2\mathrm{e}^{\lambda\sigma}\right).
\end{aligned}
$$

From the second equation (after substituting the first one), we get

$$
r=\frac{a_1\mathrm{e}^{-\lambda\tau}-b_1\mathrm{e}^{\lambda\sigma}}{1-a_0\tau\mathrm{e}^{-\lambda\tau}-b_0\sigma\mathrm{e}^{\lambda\sigma}}=\frac{a_1\mathrm{e}^{-\lambda\tau}-b_1\mathrm{e}^{\lambda\sigma}}{f'(\lambda)} \tag{3.4}
$$

and, from the third one,

$$
A=\frac{1}{f'(\lambda)}\left(r\left(-a_1\tau\mathrm{e}^{-\lambda\tau}-b_1\sigma\mathrm{e}^{\lambda\sigma}\right)+\frac{r(-r-1)}{2}\left(a_0\tau^2\mathrm{e}^{-\lambda\tau}-b_0\sigma^2\mathrm{e}^{\lambda\sigma}\right)\right)
$$

which corresponds to (3.2) and (3.3) for $\lambda=\lambda^*$.

4. Existence of semi-global solutions

In this section we formulate and prove the main result of the paper. As mentioned in Section 2, its proof is based on Lemma 2.1.

Theorem 4.1. *Let the transcendental equation* (1.4) *have a positive real root* λ^* *such that* $f'(\lambda^*)>0$. *Then, for every fixed* $\varepsilon\in(0,1)$ *and* $t\geqslant t_0-\tau$, *there exist a* $t_0\in\mathbb{R}$ *and a right semi-global positive solution* $y(t)$ *of equation* (1.2) *on* $[t_0-\tau,\infty)$ *satisfying the inequalities*

$$
K_1\mathrm{e}^{\lambda^*t}t^{-r}\left(1-\frac{A-\varepsilon}{t}\right)\leqslant y(t)\leqslant K_2\mathrm{e}^{\lambda^*t}t^{-r}\left(1-\frac{A+\varepsilon}{t}\right),\quad t\geqslant t_0-\tau \tag{4.1}
$$

with the coefficients A, r *defined by formulas* (3.2), (3.3) *and*

$$
K_1=(t_0-\tau)^r\exp\left(-\lambda^*(t_0-\tau)+\frac{A-\varepsilon}{t_0-\tau}\right),
$$

$$
K_2=(t_0-\tau)^r\exp\left(-\lambda^*(t_0-\tau)+\frac{A+\varepsilon}{t_0-\tau}\right).
$$

Proof. Let $\varepsilon\in(0,1)$ be fixed and let t_0 be large enough for the asymptotic relations discussed below to hold. Let us take

$$
\mathcal{L}(t):=\lambda^*-\frac{r}{t}+\frac{A-\varepsilon}{t^2}, \tag{4.2}
$$

$$\begin{aligned}\mathcal{R}(t) &:= \lambda^* - \frac{r}{t} + \frac{A+\varepsilon}{t^2}\,,\\ \varphi(t) &:= \mathcal{L}(t) - \mathcal{L}(t_0)\,.\end{aligned} \tag{4.3}$$

We will verify all the hypotheses of Lemma 2.1 and prove that there exists a solution of (1.2) satisfying inequalities (4.6). Obviously, $\mathcal{L}(t) \leqslant \mathcal{R}(t)$ on $[t_0-\tau,\infty)$. We need to show, that (2.1) holds, i.e., that

$$\begin{aligned}\lambda^* - \frac{r}{t} + \frac{A-\varepsilon}{t^2} \leqslant &-\left(a_0 + \frac{a_1}{t}\right)\exp\left(\int_t^{t-\tau}\left(\lambda^* - \frac{r}{s} + \frac{A-\varepsilon}{s^2}\right)\mathrm{d}s\right)\\ &+\left(b_0 + \frac{b_1}{t}\right)\exp\left(\int_t^{t+\sigma}\left(\lambda^* - \frac{r}{s} + \frac{A-\varepsilon}{s^2}\right)\mathrm{d}s\right).\end{aligned} \tag{4.4}$$

Let us denote the right-hand side of the inequality (4.4) by $T\mathcal{L}(t)$. Then, after integration, we get

$$\begin{aligned}T\mathcal{L}(t) =& -\left(a_0 + \frac{a_1}{t}\right)\cdot\exp\left(-\lambda^*\tau - r\ln(t-\tau) + r\ln(t) - (A-\varepsilon)\left(\frac{1}{t-\tau} - \frac{1}{t}\right)\right)\\ &+\left(b_0 + \frac{b_1}{t}\right)\cdot\exp\left(\lambda^*\sigma - r\ln(t+\sigma) + r\ln(t) - (A-\varepsilon)\left(\frac{1}{t+\sigma} - \frac{1}{t}\right)\right)\\ =& -\left(a_0 + \frac{a_1}{t}\right)\cdot\exp\left(-\lambda^*\tau + \ln\left(\frac{t-\tau}{t}\right)^{-r} - \frac{(A-\varepsilon)\tau}{(t-\tau)t}\right)\\ &+\left(b_0 + \frac{b_1}{t}\right)\cdot\exp\left(\lambda^*\sigma + \ln\left(\frac{t+\sigma}{t}\right)^{-r} + \frac{(A-\varepsilon)\sigma}{(t+\sigma)t}\right)\\ =& -\left(a_0 + \frac{a_1}{t}\right)\mathrm{e}^{-\lambda^*\tau}\cdot\left(\frac{t-\tau}{t}\right)^{-r}\cdot\exp\left(-\frac{(A-\varepsilon)\tau}{(t-\tau)t}\right)\\ &+\left(b_0 + \frac{b_1}{t}\right)\mathrm{e}^{\lambda^*\sigma}\cdot\left(\frac{t+\sigma}{t}\right)^{-r}\cdot\exp\left(\frac{(A-\varepsilon)\sigma}{(t+\sigma)t}\right).\end{aligned}$$

Using formula (3.1) and the Maclaurin series for e^x, we obtain

$$\begin{aligned}\mathcal{E}_\tau(t) := \exp\left(-\frac{(A-\varepsilon)\tau}{(t-\tau)t}\right) &= \exp\left(-(A-\varepsilon)\tau\frac{1}{t^2}\left(1 + \frac{\tau}{t} + o\left(\frac{1}{t^2}\right)\right)\right)\\ &= 1 - \frac{(A-\varepsilon)\tau}{t^2} + o\left(\frac{1}{t^2}\right),\\ \mathcal{E}_\sigma(t) := \exp\left(\frac{(A-\varepsilon)\sigma}{(t+\sigma)t}\right) &= \exp\left((A-\varepsilon)\sigma\frac{1}{t^2}\left(1 - \frac{\sigma}{t} + o\left(\frac{1}{t^2}\right)\right)\right)\\ &= 1 + \frac{(A-\varepsilon)\sigma}{t^2} + o\left(\frac{1}{t^2}\right).\end{aligned}$$

By formula (3.1),

$$T\mathcal{L}(t) = -\left(a_0 + \frac{a_1}{t}\right)\mathrm{e}^{-\lambda^*\tau}\cdot\left[1 + \frac{r\tau}{t} - \frac{r(-r-1)\tau^2}{2t^2} + o\left(\frac{1}{t^2}\right)\right]\cdot\mathcal{E}_\tau$$

$$
\begin{aligned}
&+\left(b_0+\frac{b_1}{t}\right)\mathrm{e}^{\lambda^*\sigma}\cdot\left[1-\frac{r\sigma}{t}-\frac{r(-r-1)\sigma^2}{2t^2}+o\left(\frac{1}{t^2}\right)\right]\cdot\mathcal{E}_\sigma\\
=&-\left(a_0+\frac{a_1}{t}\right)\mathrm{e}^{-\lambda^*\tau}\cdot\left(1+\frac{r\tau}{t}-\frac{r(-r-1)\tau^2}{2t^2}-\frac{(A-\varepsilon)\tau}{t^2}+o\left(\frac{1}{t^2}\right)\right)\\
&+\left(b_0+\frac{b_1}{t}\right)\mathrm{e}^{\lambda^*\sigma}\cdot\left(1-\frac{r\sigma}{t}-\frac{r(-r-1)\sigma^2}{2t^2}+\frac{(A-\varepsilon)\sigma}{t^2}+o\left(\frac{1}{t^2}\right)\right).
\end{aligned}
$$

Matching the coefficients corresponding at the powers of t, we have

$$
\begin{aligned}
t^0 \ &: \ \lambda^* = -a_0\mathrm{e}^{-\lambda^*\tau}+b_0\mathrm{e}^{\lambda^*\sigma},\\
t^{-1} &: -r \ = -a_0\mathrm{e}^{-\lambda^*\tau}r\tau-a_1\mathrm{e}^{-\lambda^*\tau}-b_0r\sigma\mathrm{e}^{\lambda^*\sigma}+b_1\mathrm{e}^{\lambda^*\sigma}.
\end{aligned}
$$

The first equation holds due to (1.4). The second one, after simplication, is equivalent to (3.4). For the validity of $\mathcal{L}(t)\leqslant T\mathcal{L}(t)$ on $[t_0-\tau,\infty)$ with t_0 sufficiently large, we need that

$$
\begin{aligned}
t^{-2}: A-\varepsilon\leqslant -a_0\mathrm{e}^{-\lambda^*\tau}&\left(-\frac{r(-r-1)\tau^2}{2}-(A-\varepsilon)\tau\right)-a_1\mathrm{e}^{-\lambda^*\tau}r\tau\\
&+b_0\mathrm{e}^{\lambda^*\sigma}\left(-\frac{r(-r-1)\sigma^2}{2}-(A-\varepsilon)\sigma\right)-b_1\mathrm{e}^{\lambda^*\sigma}r\sigma.
\end{aligned}
$$

This may be rewritten as

$$
A-\varepsilon\leqslant (A-\varepsilon)(1-f'(\lambda^*))+Af'(\lambda^*),
$$

which holds because $0<\varepsilon f'(\lambda^*)$. Therefore, $\mathcal{L}(t)\leqslant T\mathcal{L}(t)$ on $[t_0-\tau,\infty)$.

Since inequalities (2.2), (2.3) and (2.4) may be proved in much the same way, the computations are omitted. The estimates (4.1) follow from (2.5) with $r_1=\tau$ and $\mathcal{L}$, $\mathcal{R}$ defined by (4.2) and (4.3). □

Example 4.2. Consider a particular case of equation (1.2) where $a_0 = \mathrm{e}/2$, $b_0 = 3\mathrm{e}^{-0.1}/2$, $a_1=\mathrm{e}$, $b_1=\mathrm{e}^{-0.1}$, $\tau=1$ and $\sigma=0.1$, i.e.,

$$
\dot{y}(t)=\mathrm{e}\left(\frac{1}{2}+\frac{1}{t}\right)y(t-1)+\mathrm{e}^{-0.1}\left(\frac{3}{2}+\frac{1}{t}\right)y(t+0.1). \tag{4.5}
$$

Then

$$
f(\lambda):=\lambda+\frac{1}{2}\mathrm{e}^{1-\lambda}-\frac{3}{2}\mathrm{e}^{0.1(-1+\lambda)},
$$

equation $f(\lambda)=0$ has a positive root $\lambda=1$ and

$$
f'(\lambda)|_{\lambda=1}=\left(1-\frac{1}{2}\mathrm{e}^{1-\lambda}-\frac{0.3}{2}\mathrm{e}^{0.1(-1+\lambda)}\right)\Bigg|_{\lambda=1}=0.35>0.
$$

Let $\varepsilon\in(0,1)$ be fixed and let t_0 be sufficiently large. By formulas (3.2), (3.3) we compute $r=A=0$. Then

$$
K_1=\exp\left(-(t_0-1)-\frac{\varepsilon}{t_0-1}\right),\quad K_2=\exp\left(-(t_0-1)+\frac{\varepsilon}{t_0-1}\right).
$$

By Theorem 4.1, there exist a right semi-global positive solution $y(t)$ of equation (4.5) on $[t_0 - \tau, \infty)$ satisfying the inequalities

$$K_1 e^t \left(1 + \frac{\varepsilon}{t}\right) \leqslant y(t) \leqslant K_2 e^t \left(1 - \frac{\varepsilon}{t}\right), \qquad t \geqslant t_0 - \tau. \tag{4.6}$$

Note that equation (4.5) has a family of exact solutions $y(t) = c \exp t$ where c is an arbitrary constant. If $K_1 < c < K_2$, then these solutions satisfy inequalities (4.6) for $t \to \infty$.

5. Concluding remarks

The paper proves the existence of right semi-global solutions to equation (1.2) deriving their upper and lower estimates, suggested by the form of approximate solutions. The research was motivated by investigations [2, 3, 4] and [6]. The auxiliary Lemma 2.1 is a particular case of Theorem 2 in [6] where this result was proved by the method of monotone iterative sequences, we refer, e.g., to [11]. The investigation carried out is close to [9] dealing with asymptotic properties of solutions of similar classes of equations. In [5] asymptotic properties of solutions are studied for the so-called p-type retarded functional differential equations. For the class of p-type advanced-delayed differential equations a further study may be envisaged of applying (generalizing) the results achieved to such equations. Referring to [10], let us also remark that the subject of the paper is closely related to the Hartman-Wintner theorem for retarded functional differential equations that deals with L_2-perturbations of autonomous delay equations. For rudiments of delayed, advanced, and some classes of advanced-delayed equations, we refer to [1, 7, 8].

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References

[1] Agarwal, R.P., Berezansky, L., Braverman, E., Domoshnitsky, A., *Nonoscillation Theory of Functional Differential Equations with Applications*, Springer, 2012.

[2] Diblík, J., Kúdelčíková, M., *Nonoscillating solutions of the equation* $\dot{x}(t) = -(a+b/t)x(t-\tau)$, Stud. Univ. Žilina Math. Ser. **15** (1) (2002), 11–24.

[3] Diblík, J., Kúdelčíková, M., *Inequalities for positive solutions of the equation* $\dot{y}(t) = -(a_0 + a_1/t)x(t-\tau_1) - (b_0 + b_1/t)x(t-\tau_2)$, Stud. Univ. Žilina Math. Ser. **17** (1) (2003), 27–46.

[4] Diblík, J., Kúdelčíková, M., *Inequalities for the positive solutions of the equation* $\dot{y}(t) = -\sum_{i=1}^{n}(a_i + b_i/t)y(t-\tau_i)$, Differential and Difference Equations and Applications (2006), 341–350, Hindawi Publ. Corp., New York.

[5] Diblík, J., Svoboda, Z., *Positive solutions of p-type retarded functional differential equations*, Nonlinear Anal. **64** (8) (2006), 1831–1848.

[6] Diblík, J., Vážanová, G., *Lower and upper estimates of semi-global and global solutions to mixed-type functional differential equations*, Adv. Nonlinear Anal. **11** (1) (2022), 757–784.

[7] Györi, I., Ladas, G., *Oscillation Theory of Delay Differential Equations*, Clarendon Press, Oxford, 1991.

[8] Hale, J.K., Lunel, S.M.V., *Introduction to Functional Differential Equations*, Springer-Verlag, 1993.

[9] Pinelas, S., *Asymptotic behavior of solutions to mixed type differential equations*, Electron. J. Differential Equations **2014** (210) (2014), 1–9.

[10] Pituk, M., *The Hartman-Wintner theorem for functional-differential equations*, J. Differential Equations **155** (1) (1999), 1–16.

[11] Zeidler, E., *Nonlinear Functional Analysis and its Application, Part I, Fixed-Point Theorems*, Springer-Verlag, 1985.

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