ARCHIVUM MATHEMATICUM (BRNO)
Tomus 59 (2023), 155–162

A PRIORI BOUNDS FOR POSITIVE RADIAL SOLUTIONS OF QUASILINEAR EQUATIONS OF LANE–EMDEN TYPE

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Abstract. We consider the quasilinear equation $\Delta_p u + K(|x|)u^q = 0$, and present the proof of the local existence of positive radial solutions near 0 under suitable conditions on K. Moreover, we provide a priori estimates of positive radial solutions near ∞ when $r^{-\ell}K(r)$ for $\ell \geq -p$ is bounded near ∞.

1. Introduction

We consider the equation

$$\Delta_p u + K(|x|)u^q = 0\,, \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $n > p > 1$ and $q > p-1$. Let $r = |x|$ and $\frac{d}{dr}u(r) = u_r(r)$. Then, the radial version of (1.1) is

$$r^{1-n}(r^{n-1}|u_r|^{p-2}u_r)_r + K(r)u^q = 0\,. \tag{1.2}$$

For $p = 2$, the basic assumption of K for local solutions is (K):

(i) $K(r) \geq 0, \not\equiv 0$; $K(r)$ is continuous on $(0,\infty)$;

(ii) $\int_0 rK(r)\,dr < \infty$, i.e., $rK(r)$ is integrable near 0.

Under condition (K), (1.2) with $p = 2$ and $u(0) = \alpha > 0$, has a unique positive solution $u_\alpha \in C^2(0,\varepsilon) \cap C[0,\varepsilon)$ for small $\varepsilon > 0$. In order to obtain local solutions (1.2) near 0, we assume (KP): (i) of (K), and for $r > 0$ small,

$$\int_0^r t^{\frac{1-n}{p-1}}\Big(\int_0^t s^{n-1}K(s)ds\Big)^{\frac{1}{p-1}}dt < \infty\,.$$

For $p = 2$, this integrability is (ii) of (K). If $K(r) = r^l$, then it is easy to see that (KP) holds for $l > -p$. As a typical example, the equation

$$\Delta_p u + |x|^l u^q = 0 \tag{1.3}$$

possesses a local radial solution $\overline{u}_\alpha$ with $\overline{u}_\alpha(0) = \alpha$ for each $\alpha > 0$, and has the scaling invariance:

$$\overline{u}_\alpha(r) = \alpha\overline{u}_1(\alpha^{\frac{1}{m}}r) \tag{1.4}$$

2020 *Mathematics Subject Classification*: primary 35J92; secondary 35B45.

Key words and phrases: quasilinear equation, positive solution, a priori bound.
This research was supported by the research fund of Hanbat National University in 2018.
Received August 26, 2022, accepted December 16, 2022. Editor Š. Nečasová.
DOI: 10.5817/AM2023-2-155

with $m = \frac{p+l}{q-(p-1)}$. Moreover, (1.3) has a singular solution which is invariant under the scaling in (1.4), the so-called self-similar solution. That is,

$$U(x) = L|x|^{-m},$$

where L is defined by

$$L = L(n,p,q,l) = [m^{p-1}(n-1-(m+1)(p-1))]^{\frac{1}{q-(p-1)}}\,. \tag{1.5}$$

This singular solution can be defined for $l > -p$ and $q > \frac{(p-1)(n+l)}{n-p}$ because $n-1-(m+1)(p-1) > 0$. Then, we observe the asymptotic self-similar behavior.

Theorem 1.1. *Let $n > p > 1$ and $q > \frac{(p-1)(n+l)}{n-p}$ with $l > -p$. If $r^{-l}K(r) \to 1$ as $r \to \infty$, then any positive solution u of* (1.2) *near ∞ satisfies one of the two asymptotic behavior: either*

$$\liminf_{r\to\infty} r^m u(r) \le L \le \limsup_{r\to\infty} r^m u(r) < \infty \tag{1.6}$$

with $L = L(n,p,q,l)$ given by (1.5) *or $r^{(n-p)/(p-1)}u(r) \to C > 0$ as $r \to \infty$.*

Moreover, (1.6) can be the asymptotic self-similarity

$$\lim_{r\to\infty} r^m u(r) = L\,.$$

In a forthcoming paper, we study entire solutions of (1.2) with this asymptotic behavior in a supercritical range.

1.1. **Lower bound.** The p-Laplace equation has the radial form

$$(|u_r|^{p-2}u_r)_r + \frac{n-1}{r}|u_r|^{p-2}u_r = 0\,, \tag{1.7}$$

where $n > p > 1$. Then, (1.7) possesses a solution $|x|^{-\theta}$ with $\theta = \frac{n-p}{p-1}$. Let u be a positive radial solution satisfying the quasilinear inequality

$$r^{1-n}(r^{n-1}|u_r|^{p-2}u_r)_r = (|u_r|^{p-2}u_r)_r + \frac{n-1}{r}|u_r|^{p-2}u_r \le 0\,. \tag{1.8}$$

If $u_r(r_0) \le 0$ for some $r_0 > 0$, then $u_r(r) \le 0$ for $r > r_0$. Hence, u is monotone near ∞. Assume $u_r \le 0$ for $r \ge r_0$ with some $r_0 > 1$. Setting $V(t) = r^\theta u(r)$ for $t = \log r \ge t_0 = \log r_0$, we see that $g(t) = \theta V(t) - V'(t) = r^{\theta+1}(-u_r(r)) = r^{\frac{n-1}{p-1}}(-u_r(r))$ satisfies

$$\frac{d}{dt}(g^{p-1}(t)) = (n-1)g^{p-1}(t) + r^n[(-u_r)^{p-1}]_r \ge 0$$

for $t \ge t_0$. Hence, g is increasing for $t \ge t_0$. Then, V satisfies that for $t > T \ge t_0$,

$$V'(t) - \theta V(t) \le V'(T) - \theta V(T)\,.$$

Suppose $V'(T) < 0$. Setting $c = \theta V(T) - V'(T)$, we have $(e^{-\theta t}V(t))' \le -ce^{-\theta t}$ and

$$V(t) \le e^{\theta(t-T)}(V(T) - \frac{c}{\theta}) + \frac{c}{\theta} = e^{\theta(t-T)}\frac{V'(T)}{\theta} + \frac{c}{\theta}\,.$$

Hence, V has a finite zero. Therefore, in order for u to be positive near ∞, V must be increasing and $(r^\theta u(r))_r \ge 0$ near ∞. This is true obviously in the other case that $u_r > 0$ near ∞.

Lemma 1.2. *Let $n > p > 1$. If u is a positive radial solution satisfying* (1.8) *near* ∞*, then $r^{\frac{n-p}{p-1}} u(r)$ is increasing.*

Now, we classify positive solutions of (1.8) near ∞ into two groups according to their behaviors. If $r^{\frac{n-p}{p-1}} u$ converges to a positive constant at ∞, then we call u a fast decaying solution. Otherwise, u is a slowly decaying solution if $r^{\frac{n-p}{p-1}} u(r) \to \infty$ as $r \to \infty$.

1.2. **Known results.** One of Liouville's theorems related to p-Laplace equation is the nonexistence of nontrivial nonnegative solutions in $W^{1,p}_{\mathrm{loc}}(\mathbf{R}^n) \cap C(\mathbf{R}^n)$ to the following quasilinear inequality

$$-\Delta_p u \geq c|x|^l u^q$$

with $c > 0$ and $l > -p$, when $n > p > 1$ and

$$q \leq \frac{(p-1)(n+l)}{n-p}.$$

See [1, Theorem 3.3 (iii)]. For the existence of nontrivial solutions to

$$\Delta_p u + u^q = 0\,,$$

on $\mathbf{R}^n$ with $n > p > 1$ and $q > p-1$, it is necessary and sufficient that $q \geq \frac{n(p-1)+p}{n-p}$ [6]. On the other hand, (1.3) with $q = q_s := \frac{n(p-1)+p+pl}{n-p}$ admits the one-parameter family of positive solutions given by

$$\overline{u}_\alpha(x) = \frac{\alpha}{(1 + \xi(\alpha^{\frac{p}{n-p}}|x|)^{\frac{p+l}{p-1}})^{\frac{n-p}{p+l}}}$$

with $\xi = \xi_{p,n} = \frac{p-1}{(n-p)(n+l)^{1/(p-1)}}$ and $\overline{u}_\alpha(0) = \alpha > 0$. A radial solution $u(x) = u(|x|)$ of (1.3) satisfies the equation

$$(|u_r|^{p-2}u_r)_r + \frac{n-1}{r}|u_r|^{p-2}u_r + r^l u^q = 0\,. \tag{1.9}$$

For $l > -p$, (1.9) with $u(0) = \alpha > 0$, has a unique positive solution $u \in C^1(0,\epsilon) \cap C[0,\epsilon)$ for small $\epsilon > 0$ such that $|u_r|^{p-2}u_r \in C^1[0,\epsilon)$. If $q < q_s$, then every local solution of (1.9) has a finite zero [2, 5]. In the opposite case $q > q_s$, every local solution of (1.9) is to be a slowly decaying solution [2, 3, 5].

2. Local existence

Let $n \geq p > 1$, $l > -p$ and $q \geq p-1$. First, in order to prove the local existence of positive radial solutions of (1.3), we consider the integral equation

$$u(r) = \alpha - \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l} u^q(s) ds)^{\frac{1}{p-1}}\, dt$$

with $\alpha > 0$.

2.1. **Integral representation.** On a space

$$S = \{u \in C[0,\varepsilon] \mid 0 \le u \le \alpha\}$$

we study a nonlinear operator T from S to $C[0,\varepsilon]$ by

$$T(u)(r) = \alpha - T_1(u)(r),$$

where

$$T_1(u)(r) = \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l} u^q(s) ds)^{\frac{1}{p-1}}\, dt\,.$$

For $\varepsilon > 0$ small enough, T_1 satisfies that

$$0 \le T_1 \le \alpha^{\frac{q}{p-1}} \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l} ds)^{\frac{1}{p-1}} dt \le (\frac{\alpha^q}{n+l})^{\frac{1}{p-1}} \frac{p-1}{p+l} \varepsilon^{\frac{p+l}{p-1}} \le \alpha\,.$$

Hence, $T(S) \subset S$. Minkowski's inequality for $p \ge 2$ shows that for $u_1, u_2 \in S$,

$$\begin{aligned}
\|T(u_2) - T(u_1)\| &\le \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l} \big|u_2^{\frac{q}{p-1}} - u_1^{\frac{q}{p-1}}\big|^{p-1} ds)^{\frac{1}{p-1}}\, dt \\
&\le \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l}\, ds)^{\frac{1}{p-1}}\, dt\, \|u_2 - u_1\| \\
&= \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} (\frac{1}{n+l})^{\frac{1}{p-1}} \frac{p-1}{p+l} \varepsilon^{\frac{p+l}{p-1}} \|u_2 - u_1\|\,.
\end{aligned}$$

For $1 < p < 2$, we observe that for $u_1, u_2 \in S$,

$$\begin{aligned}
\|T(u_2) - T(u_1)\| &\le \int_0^r t^{\frac{1-n}{p-1}} \frac{\alpha^{\frac{q(2-p)}{p-1}}}{p-1} (\int_0^t s^{n-1+l}\, ds)^{\frac{2-p}{p-1}} (\int_0^t s^{n-1+l} |u_2^q - u_1^q|\, ds)\, dt \\
&\le \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1+l}\, ds)^{\frac{1}{p-1}}\, dt\, \|u_2 - u_1\| \\
&= \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} (\frac{1}{n+l})^{\frac{1}{p-1}} \frac{p-1}{p+l} \varepsilon^{\frac{p+l}{p-1}} \|u_2 - u_1\|\,.
\end{aligned}$$

Now, we assume that

$$\frac{p-1}{p+l} \max\{(\frac{\alpha^q}{n+l})^{\frac{1}{p-1}}, \frac{q}{p-1} \alpha^{\frac{q-(p-1)}{p-1}} (\frac{1}{n+l})^{\frac{1}{p-1}}\} \varepsilon^{\frac{p+l}{p-1}} < \min\{\alpha, 1\}.$$

Then, T is a contraction mapping in S and thus T has a unique fixed point $\bar{u}_\alpha$.

Generally, we consider the integral equation under condition (KP),

$$u(r) = \alpha - \int_0^r t^{\frac{1-n}{p-1}} (\int_0^t s^{n-1} K(s) u^q(s) ds)^{\frac{1}{p-1}}\, dt\,.$$

Then, the integrability of (KP) shows in the same way the local existence of a positive solution u_α with $u_\alpha(0) = \alpha > 0$ to (1.2). Then, it is easy to see that there exists a sequence $\{r_j\}$ going to 0 such that

$$\lim_{j\to\infty} r_j^{n-1} |u_r(r_j)|^{p-2} u_r(r_j) = 0\,, \tag{2.1}$$

and $u_\alpha(r)$ is decreasing as long as u remains positive. Moreover, u_α is strictly decreasing after K becomes positive.

2.2. **Fowler transform.** Let $n > p > 1$ and $q > \frac{(n+l)(p-1)}{n-p}$ with $l > -p$. Set $m = \frac{p+l}{q-(p-1)}$. Fowler transform $V(t) = r^m u(r)$, $t = \log r$, of a positive solution to (1.2) satisfies

$$(p-1)(mV - V')^{p-2}(V'' - mV') - \xi(mV - V')^{p-1} + k(t)V^q = 0\,, \tag{2.2}$$

where $\xi = n-1-(m+1)(p-1) = \frac{L^{q-(p-1)}}{m^{p-1}}$ with L given by (1.5), and $k(t) = r^{-l}K(r)$. Furthermore, if $-r^{m+1}u_r(r) = mV - V' > 0$, then (2.2) can be rewritten as

$$(p-1)(V'' - mV') - \xi(mV - V') = -\frac{k(t)V^q}{(mV - V')^{p-2}}$$

and

$$(p-1)V'' + aV' - \xi mV = -\frac{k(t)V^q}{(mV - V')^{p-2}},$$

where $a = n - 1 - (2m+1)(p-1)$. Setting $b = \xi m = \frac{L^{q-(p-1)}}{m^{p-2}}$, we have

$$(p-1)V'' + aV' - (b - \frac{k(t)V^{q-1}}{(mV - V')^{p-2}})V = 0\,.$$

That is,

$$(p-1)V'' + aV' - \frac{1}{m^{p-2}}L^{q-(p-1)}V + \frac{k(t)}{(mV - V')^{p-2}}V^q = 0\,, \tag{2.3}$$

which holds as long as the local solution remains positive.

3. A priori estimates

In order to obtain upper bounds, we argue similarly as in Lemma 2.16, Lemma 2.20, Theorem 2.25 in [4].

3.1. **Upper bound.** Let $n > p \geq -\ell$. If u is a positive solution satisfying the inequality

$$(r^{n-1}|u_r|^{p-2}u_r)_r \leq -cr^{n-1+\ell}u^q \tag{3.1}$$

near ∞ for some $c > 0$, then

$$r^{n-1}|u_r|^{p-2}u_r \leq r_0^{n-1}|u_r(r_0)|^{p-2}u_r(r_0) - c\int_{r_0}^{r} s^{n-1+\ell}u^q(s)\,ds \tag{3.2}$$

for $r > r_0$, if r_0 is sufficiently large. Then, we may assume that $u_r(r_0) \leq 0$. Indeed, if $u_r(r_0) > 0$, then

$$r^{n-1}|u_r|^{p-2}u_r \leq r_0^{n-1}|u_r(r_0)|^{p-2}u_r(r_0) - cu^q(r_0)\frac{1}{n+\ell}(r^{n+\ell} - r_0^{n+\ell})$$

as long as u_r is positive. Hence, u_r is eventually negative. Therefore, (3.2) gives

$$r^{n-1}|u_r|^{p-2}u_r \leq -cu^q(r)\frac{1}{n+\ell}(r^{n+\ell} - r_0^{n+\ell})$$

and thus,

$$\frac{u_r}{u^{q/(p-1)}} \leq -c_1 r^{\frac{1+\ell}{p-1}}$$

for some $c_1 > 0$. Hence, we obtain

$$u(r) \le \begin{cases} Cr^{-\frac{p+\ell}{q-(p-1)}} & \text{if} \quad \ell > -p\,, \\ C(\log r)^{-\frac{p-1}{q-(p-1)}} & \text{if} \quad \ell = -p \end{cases}$$

for some $C > 0$. Combining the a priori estimates and Lemma 1.2, we have the following assertion.

Theorem 3.1. *Let $n > p \ge -\ell$ and $q > \frac{(p-1)(n+\ell)}{n-p}$. Then, every positive solution to* (3.1) *near ∞ satisfies that*

$$C_1 r^{-\frac{p+\ell}{q-(p-1)}} \ge u(r) \ge C_2 r^{-\frac{n-p}{p-1}}$$

for $\ell > -p$ and

$$C_1 (\log r)^{-\frac{p-1}{q-(p-1)}} \ge u(r) \ge C_2 r^{-\frac{n-p}{p-1}}$$

for $\ell = -p$.

In Theorem 3.1, we use the notation ℓ instead of l to consider the case of $\ell = -p$. It is interesting to study the existence of positive entire solutions of (1.1) with the logarithmic asymptotic behavior at ∞.

Lemma 3.2. *Let $q > \frac{(p-1)(n+l)}{n-p}$. Assume $K(r) = O(r^l)$ at ∞ for some $l > -p$. If u is a positive solution to* (3.1) *near ∞ and $u(r) = O(r^{-m-\varepsilon})$ with some $\varepsilon > 0$ at ∞, then $u(r) = O(r^{\frac{p-n}{p-1}})$ at ∞.*

Proof. Integrating (1.2) over $[r, \infty)$, we obtain

$$u(r) = \int_r^\infty t^{\frac{1-n}{p-1}} \Big(\int_0^t K(s) u^q(s) s^{n-1}\, ds\Big)^{\frac{1}{p-1}}\, dt\,.$$

On the other hand, we have

$$\begin{aligned} \int_0^t K(s) u^q(s) s^{n-1}\, ds &\le C + C \int_1^t s^{n-1+l-q(m+\varepsilon)}\, ds \\ &= \begin{cases} C + Ct^{n+l-q(m+\varepsilon)} & \text{if} \quad n+l \ne q(m+\varepsilon)\,, \\ C + C\log t & \text{if} \quad n+l = q(m+\varepsilon)\,. \end{cases} \end{aligned}$$

If $n+l < q(m+\varepsilon)$, we are done. If $n+l \ge q(m+\varepsilon)$, then

$$u(r) \le \begin{cases} Cr^{\frac{p-n}{p-1}} + Cr^{\frac{p-n}{p-1}} (\log r)^{\frac{1}{p-1}} & \text{if} \quad n+l = q(m+\varepsilon)\,, \\ Cr^{\frac{p-n}{p-1}} + Cr^{\frac{p+l}{p-1} - \frac{q(m+\varepsilon)}{p-1}} & \text{if} \quad p+l < q(m+\varepsilon) < n+l\,. \end{cases}$$

In case $n+l = q(m+\varepsilon)$, we replace ε by $\frac{n-p}{p-1} - m - \delta$ in the above arguments, where $\delta > 0$ is so small that $\delta < \frac{n-p}{p-1} - m$. Note that $m < \frac{n-p}{p-1}$ iff $q > \frac{(p-1)(n+l)}{n-p}$.

$$u(r) \le \begin{cases} Cr^{\frac{p-n}{p-1}} & \text{if} \quad n+l = q(m+\varepsilon)\,, \\ Cr^{\frac{p-n}{p-1}} + Cr^{\frac{p+l}{p-1} + \frac{q(p+l)}{(p-1)^2} - \frac{q^2(m+\varepsilon)}{(p-1)^2}} & \text{if} \quad p+l < q(m+\varepsilon) < n+l\,. \end{cases}$$

In case $q(m+\varepsilon) < n+l$, we iterate this process to obtain

$$\begin{aligned} u(r) &\le Cr^{\frac{p-n}{p-1}} + Cr^{\frac{p+l}{p-1}\sum_{i=0}^{j-1}(\frac{q}{p-1})i - \frac{q^j(m+\varepsilon)}{(p-1)^j}} \\ &= Cr^{\frac{p-n}{p-1}} + Cr^{-m-\varepsilon(\frac{q}{p-1})^j} \end{aligned}$$

for any positive integer j. Since $q > p-1$, we reach the conclusion after a finite number of iterations. □

Lemma 3.3. *Let $q > \frac{(p-1)(n+l)}{n-p}$. Assume $K(r) = O(r^l)$ at ∞ for some $l > -p$. If $u(r) = o(r^{-m})$ at ∞, then $(r^m u(r))_r < 0$ near ∞.*

Proof. Let $V(t) = r^m u(r)$, $t = \log r$. Then, V satisfies (2.3). Suppose $V'(T) = 0$ for some T near ∞ and $k(t)V^{q-(p-1)}(t) < m^{p-2}b$ for $t \in [T,\infty)$. Then, $V''(T) > 0$ and $V(t)$ is strictly increasing near T but for $t > T$. Since $V \to 0$ at ∞, there exists $T_1 > T$ such that $V'(T_1) = 0$ and

$$V''(T_1) = \frac{1}{p-1}(b - \frac{1}{m^{p-2}}k(T_1)V^{q-(p-1)}(T_1))V(T_1) \le 0\,,$$

a contradiction. □

Theorem 3.4. *Let $q > \frac{(p-1)(n+l)}{n-p}$. Assume $K(r) = O(r^l)$ at ∞ for some $l > -p$. If $u(r) = o(r^{-m})$ at ∞, then $u(r) = O(r^{\frac{p-n}{p-1}})$ at ∞.*

Proof. Let $\varphi(r) = r^m u(r)$. Then, φ satisfies

$$\varphi_{rr} + (1 + \frac{a}{p-1})\frac{1}{r}\varphi_r - \frac{b}{(p-1)r^2}\varphi + \frac{k}{(p-1)(m\varphi - r\varphi_r)^{p-2}r^2}\varphi^q = 0\,.$$

For $\varepsilon > 0$, define the elliptic operator

$$\mathcal{L}_\varepsilon\varphi = \Delta\varphi - [2m + (n-1)\frac{p-2}{p-1}]\frac{x\cdot\nabla\varphi}{|x|^2} - m(\frac{L^{q-(p-1)}}{m^{p-1}} - \varepsilon)\frac{\varphi}{|x|^2}\,,$$

where $\frac{L^{q-(p-1)}}{m^{p-1}} = n-1-(m+1)(p-1)$. It follows from Lemma 3.3 that for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\mathcal{L}_\varepsilon\varphi = m\varepsilon\frac{\varphi}{r^2} - \frac{k\varphi^q}{(p-1)r^2(m\varphi - r\varphi_r)^{p-2}} \ge (m\varepsilon - \frac{k\varphi^{q-(p-1)}}{(p-1)m^{p-2}})\frac{\varphi}{r^2} \ge 0$$

in $\mathbf{R}^n\backslash B_{R_\varepsilon}(0)$. For $0 < \varepsilon < n-1-(m+1)(p-1)$, let $\eta_\varepsilon(x) = |x|^{\sigma_\varepsilon}$ with σ_ε being the negative root of $\sigma(\sigma-1) + (n-1-2m-(n-1)\frac{p-2}{p-1})\sigma - m(\frac{L^{q-(p-1)}}{m^{p-1}} - \varepsilon) = 0$, i.e.,

$$\sigma_\varepsilon = \frac{1}{2}\left[-(n-2-2m-(n-1)\frac{p-2}{p-1}) - \sqrt{D}\right]\,,$$

where $D = (n-1-2m-(n-1)\frac{p-2}{p-1})^2 + 4m(\frac{L^{q-(p-1)}}{m^{p-1}} - \varepsilon)$. Setting $C_\varepsilon = \varphi(R_\varepsilon)R_\varepsilon^{-\sigma_\varepsilon}$, we see that $\mathcal{L}_\varepsilon(\varphi - C_\varepsilon\eta_\varepsilon) \ge 0$ in $\mathbf{R}^n\backslash B_{R_\varepsilon}(0)$ and $\varphi(R_\varepsilon) = C_\varepsilon\eta_\varepsilon(R_\varepsilon)$, $\varphi - C_\varepsilon\eta_\varepsilon \to 0$ as $r \to \infty$. Then, the maximum principle implies that $\varphi - C_\varepsilon\eta_\varepsilon \le 0$ in $\mathbf{R}^n\backslash B_{R_\varepsilon}(0)$. Hence, $\varphi(r) \le C_\varepsilon\eta_\varepsilon(r)$ at ∞. Then, Lemma 3.2 implies the conclusion. □

Proof of Theorem 1.1. When $k(t) = r^{-l}K(r) \to 1$ as $t = \log r \to +\infty$, it follows from Theorem 3.1 and (2.3) that slowly decaying solutions satisfy

$$\liminf_{r\to\infty} r^m u(r) \le L \le \limsup_{r\to\infty} r^m u(r) < \infty\,.$$

Indeed, at every local minimum (maximum) point of $V(t) = r^m u(r)$, V satisfies

$$\frac{1}{m^{p-2}} L^{q-(p-1)} V \ge (\le) \frac{k(t)}{(mV)^{p-2}} V^q\,.$$

If V is monotonically increasing near $+\infty$, then it is easy to see that $V \to L$ as $t \to +\infty$ by (2.3). If V is monotonically decreasing and $V \to 0$, then it follows from Lemma 1.2 and Theorem 3.4 that $r^{\frac{n-p}{p-1}} u(r) \to C$ for some $C > 0$. $\square$

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