ARCHIVUM MATHEMATICUM (BRNO)
Tomus 59 (2023), 191–200

UNIFORM ATTRACTORS IN SUP-NORM FOR SEMI LINEAR PARABOLIC PROBLEM AND APPLICATION TO THE ROBUST STABILITY THEORY

Oleksiy Kapustyan, Olena Kapustian, Oleksandr Stanzytskyi, and Ihor Korol

Abstract. In this paper we establish the existence of the uniform attractor for a semi linear parabolic problem with bounded non autonomous disturbances in the phase space of continuous functions. We applied obtained results to prove the asymptotic gain property with respect to the global attractor of the undisturbed system.

1. Introduction

Stability property of stationary points plays an important role in robust control theory. The notion of input-to-state stability, firstly appeared in [23] now is widely used to nonlinear systems of different nature [24]. Other approaches in the control theory for nonlinear systems can be found in [2]–[11]. In recent years there have appeared many papers devoted to adaptation of input-to-state stability theory to infinite dimensional case [7]–[13]. One of the central object in the qualitative theory of dissipative infinite-dimensional systems is a global attractor [19], [22]. Stability properties of global attractors, including impulsive perturbations, can be found in [1]–[5], [9]. Recently in [6], [21] there have been obtained results about input-to-state stability and asymptotic gain properties with respect to global attractors of semi linear heat and wave equations in L^2 space. This results requires that the corresponding non autonomous problem generated semi process family with uniform attractor [3] which tends to the global attractor of undisturbed system. In the present paper we apply this scheme to the case of the phase space $\mathbb{C}_0$ of continuous functions supplied with sup-norm. Similar results for other type of perturbations were studied in [25], [26]. The work consists of two parts. In the first part we set the problem, provide necessary definitions and auxiliary results,

2020 *Mathematics Subject Classification*: primary 26A12; secondary 34C10.
Key words and phrases: parabolic equations, attractor, stability.
Oleksiy Kapustyan and Oleksandr Stanzytskyi was partially funded by the Ministry of Education and Science of Ukraine (Grant No. 21BF038-01 from 24.02.2021), Olena Kapustian was partially funded by the Ministry of Education and Science of Ukraine (Grant No. M/27-2022 from 23.05.2022).
Received September 2, 2022, accepted December 1, 2022. Editor M. Kružík.
DOI: 10.5817/AM2023-2-191

and prove that under suitable assumptions mild solutions of the perturbed system generate a semi process family on $\mathbb{C}_0$ which has a uniform attractor. In the second part we use this result to establish the asymptotic gain properties with respect to the global attractor of the unperturbed system.

2. Setting of the problem and uniform attractors

We consider the following problem

$$\left\{\begin{array}{l} \frac{\partial u}{\partial t} = Au + f(u) + h(t,x)\,, \quad (t,x) \in (0,\infty)\times\Omega\,, \\ u|_{\partial\Omega} = 0\,, \\ u(0,x) = u_0(x)\,, \end{array}\right. \tag{2.1}$$

where $u(t,x)$ is an unknown function, $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary,

$$Au = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u\,.$$

Assume that

(2.2) $-A$ is a strongly elliptic self adjoint operator with bounded sufficiently smooth coefficients,

$f\colon \mathbb{R}\to\mathbb{R}$ is locally Lipschitz, $f(0)=0$ and

$$\exists C>0 \quad \text{such that } \forall\,|s|\geq C \quad s\cdot f(s)\leq 0\,. \tag{2.3}$$

Assume that $h\in L^\infty(0,+\infty;X)$, where

$$X = C_0(\Omega) = \left\{v\in\mathbb{C}(\overline{\Omega})|v=0 \text{ on } \partial\Omega\right\}$$

supplied with the sup-norm $\|v\|_X = \sup_{x\in\Omega}|v(x)|$. In the future we will use the spaces $H^1 = W^{1,2}(\Omega)$, $H_0^1 = \left\{v\in H^1, v|_{\partial\Omega}=0\right\}$, $H^2 = W^{2,2}(\Omega)$, $L^2 = L^2(\Omega)$. We will study qualitative behaviour of mild solutions of (2.1) in the phase space X.

Definition 2.1. The function $u\in\mathbb{C}([0,T];X)$ is a mild solution of (2.1) with initial data $u_0\in X$ if for all $t\in[0,T]$ we have

$$u(t) = T(t)u_0 + \int_0^t T(t-s)F(u(s))ds + \int_0^t T(t-s)h(s)\,ds\,, \tag{2.4}$$

where $F\colon X\to X$, $F(u)(x) = f(u(x))$, $T(t)$ is a C_0 semigroup of bounded operators, generated by A in X.

We prove that for all initial condition $u_0\in X$ there exists a unique global mild solution of (2.1) with $u(0)=u_0$, which will be denoted by $u(t) = S_h(t,0,u_0)$.

Taking the set $\Sigma(h)$ of all time shifts of h we show that the semiprocess family $\{S_\sigma\}_{\sigma\in\Sigma(h)}$ (see definition below) has uniform attractor $\Theta_{\Sigma(h)}$ in the phase space

X and, moreover, for the global attractor Θ of the unperturbed system ($h \equiv 0$) we have:

$$\operatorname{dist}_X(\Theta_{\Sigma(h)}, \Theta) \to 0 \quad \text{as} \quad h \to 0\,, \tag{2.5}$$

where

$$\operatorname{dist}_X(A, B) = \sup_{a\in A} \inf_{b\in B} \|a - b\|_X\,.$$

Limit equality (2.5) allow us to get the following result concerning robust stability: there exists a continuous strictly increasing function γ, vanishing at the origin, such that $\forall u_0 \in X$

$$\overline{\lim_{t\to\infty}} \|S_h(t, 0, u_0)\|_\Theta \le \gamma(\|h\|_\infty)\,, \tag{2.6}$$

where

$$\|u\|_\Theta := \inf_{\xi\in\Theta} \|\xi - u\|_X, \quad \|h\|_\infty = \sup_{t\ge 0} \|h(t)\|_X\,.$$

To prove (2.5), (2.6) we need some auxiliary results. First let us assume that $h \in L^2_{loc}(0, +\infty; X)$. Then, using Lipschitz continuity of f, we can use the classical result [17] (see Th. 1.4, Ch. 6) and claim that for every $u_0 \in X$ there exists $T = T(u_0, h) > 0$ such that there exists a unique mild solution of (2.1), $u \in \mathbb{C}([0, T]; X)$ with $u(0) = u_0$. Moreover, condition (2.3) allow us to use well-known comparison principle [12] and deduce the following estimate holds

$$\|u(t)\|_X \le Me^{-\lambda t}\|u_0\|_X + \frac{MC_1}{\lambda} + \int_0^t Me^{-\lambda(t-s)}\|h(s)\|_X\, ds\,, \tag{2.7}$$

where constant $C_1 > 0$ depends on f and positive constants M, λ are taken from the inequality

$$\|T(t)\| \le Me^{-\lambda t} \quad \forall\, t \ge 0\,. \tag{2.8}$$

This estimate shows that every mild solution is global, i.e., defined on $[0, +\infty)$.

In the sequel we will use the following facts. It is known that A is the infinitesimal generator of an analytic semigroup (still denoted by $T(t)$) in $L^p(\Omega)$, $p \ge 2$ [17]. Both in $L^p(\Omega)$, $p \ge 2$ and in X, we have the following estimates [3], [10]: there exist $c > 0$, $\alpha \in (0, 1)$, $\delta \in (\frac{1}{2}, 1)$ such that

$$\forall\, u_0 \in L^2(\Omega) \quad \|T(t)u_0\|_{H^2} \le \frac{c}{t}\|u_0\|_{L^2}\,, \tag{2.9}$$

$$]3pt]\forall\, u_0 \in X \qquad \|T(t)u_0\|_{C^{1+\alpha}} \le \frac{c}{t^\delta}\|u_0\|_X\,. \tag{2.10}$$

Let us consider linear nonhomogeneous problem

$$\begin{cases} \frac{du}{dt} = Au + g(t)\,, \\ u|_{t=0} = u_0 \in L^2(\Omega)\,, \end{cases} \tag{2.11}$$

where $g \in L^2(0, T; L^2(\Omega))$ is a given function.

We consider mild solution of (2.11), i.e. $u \in C([0,T]; L^2(\Omega))$,

$$u(t) = T(t)u_0 + \int_0^t T(t-s)g(s)\,ds\,. \tag{2.12}$$

It is known [17] that mild solution of (2.11) is a weak solution of (2.11), i.e. $u \in L^2(0,T;H_0^1)$ such that $\forall\, v \in H_0^1$, $\forall\, \eta \in C_0^\infty(0,T)$

$$\int_0^T (u(t),v)\eta\,ds + \int_0^T (A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}v)\eta\,ds = \int_0^T (g(t),v)\eta\,ds\,, \tag{2.13}$$

where $(\cdot,\cdot)$ is a scalar product in L^2, $\|u\| = \sqrt{(u,u)}$. Moreover, every weak solution u of (2.11) is a mild continuous of (2.11) in $[0,T]$. Additionally, if $u_0 \in H_0^1$ then $u \in C([0,T];H_0^1)\bigcap L^2(0,T;H^2)$, $u_t \in L^2(0,T;L^2)$. All this facts help us to prove the global existence result.

Now we are in position to construct the semi processes family, generated by the equation (2.1).

Let $h \in L^\infty(0,+\infty;X)$ and let $\Sigma(h) \subset L^2_{loc}(0,+\infty;X)$, $(\Sigma(0) = \{0\})$ be an arbitrary shift invariant (i.e. $\forall\, d \in \Sigma(h)$, $\forall\, s \geq 0$ $d(s+\cdot) \in \Sigma(h)$) topological space generated by h.

Let us consider the problem (2.1) where h is replaced by $d \in \Sigma(h)$

$$\begin{cases} \frac{\partial u}{\partial t} = Au + f(u) + d(t,x)\,, \quad (t,x) \in (0,\infty)\times\Omega \\ u|_{\partial\Omega} = 0\,, \\ u(0,x) = u_0(x)\,, \end{cases} \tag{2.14}$$

From the previous arguments we deduce that every solution of (2.14) is global. We denote by

$$S_d(t,\tau,u_\tau)$$

the solution of (2.14) at the moment $t \geq \tau$ with initial data $(\tau,u_\tau) \in [0,\infty)\times X$. Then the family $\{S_d(t,\tau,u_\tau)\}_{d\in\Sigma(h)}$ generates a semiprocess family [19], i.e. $\forall\, t \geq \tau \geq 0$ $\forall u_\tau \in X$ $\forall\, d \in \Sigma(h)$

$$\begin{aligned} S_d(\tau,\tau,u_\tau) &= u_\tau\,, \\ S_d(t,s,S_d(s,\tau,u_\tau)) &= S_d(t,\tau,u_\tau) \;\; \forall\, t \geq s \geq \tau\,, \\ S_d(t+p,\tau+p,u_\tau) &= S_{d(\cdot+p)}(t,\tau,u_\tau) \;\; \forall\, p \geq 0\,. \end{aligned}$$

Every semiprocess family satisfies the cocycle property

$$S_d(t+p,0,u) = S_{d(\cdot+p)}\big(t,0,S_d(p,0,u)\big)\,.$$

In particular, for $d \equiv 0$

$$S_0(t+p,0,u) = S_0\big(t,0,S_0(p,0,u)\big)\,,$$

i.e. S_0 is a semigroup.

It is known [8] that under conditions (2.2), (2.3) the semigroup S_0 processes a global attractor $\Theta \subset X$, that is

1) Θ is a compact set;
2) $\Theta = S_0(t,0,\Theta)\ \ \forall\, t\geq 0$;
3) for every bounded set $B\subset X$

$$\sup_{u\in B} \mathrm{dist}_X(S_0(t,0,u),\Theta)\to 0 \quad \text{as} \quad t\to\infty\,.$$

In the sequel we denote for $\Sigma=\Sigma(h)$, $B\subset X$

$$S_\Sigma(t,\tau,B)=\bigcup_{d\in\Sigma}\bigcup_{u\in B} S_d(t,\tau,u)\,.$$

Definition 2.2. A compact set $\Theta_\Sigma\subset X$ is called a uniform attractor of the semiprocess family $\{S_d\}_{d\in\Sigma}$ if for every bounded set $B\subset X$ we have

$$dist_X(S_\Sigma(t,0,B),\Theta_\Sigma)\to 0 \quad \text{as} \quad t\to\infty\,, \tag{2.15}$$

and Θ_Σ is the minimum among all closed sets satisfying (2.15).

The following well known result provides conditions for existence of uniform attractor.

Lemma 2.3 ([3]). *Let $\{S_d\}_{d\in\Sigma}$ be a semiprocess family with a first countable space Σ, and*

1) *there exists a bounded set $B_0\subset X$ such that for every bounded set $B\subset X$,*

$$\exists T=T(B)\quad \forall\, t\geq T \qquad S_\Sigma(t,0,B)\subset B_0\,;$$

2) $\forall\, d_n\subset\Sigma\ \forall t_n\to\infty\ \forall$ *bounded* $\{u_n\}\subset X$ *the sequence* $\{S_{d_n}(t_n,0,u_n)\}$ *is precompact in X.*

Then $\{S_d\}_{d\in\Sigma}$ has a uniform attractor Θ_Σ.

If, additionally, for all $t\geq 0$ the map

$$X\times\Sigma\ni (u,d)\to S_d(t,0,u)\in X \tag{2.16}$$

is continuous, then Θ_Σ is negatively invariant, i.e.

$$\forall\, t\geq 0\quad \Theta_\Sigma\subset S_\Sigma(t,0,\Theta_\Sigma)\,. \tag{2.17}$$

Remark 2.4. From (2.17) we get inclusion: $\Theta_\Sigma\subset B_0$.

Assume that

$$h(t,x)=\sum_{j=1}^{K} h_j(t)\varphi_j(x)\,, \tag{2.18}$$

where $K\geq 1$, $h_j\in L^\infty(0,+\infty)$, $\varphi_j\in X$.

Let us put

$$W := cl_{\left(L^{2,w}_{loc}(0,+\infty)\right)^K}\left\{(h_1(\cdot+s),\ldots,h_K(\cdot+s))\; s\geq 0\right\}\,,$$

$$\Sigma=\Sigma(h)=\Big\{\sum_{j=1}^{K} d_j(t)\varphi_j(x)\mid \{d_1,\ldots,d_K\}\in W\Big\}\,. \tag{2.19}$$

It is known [22] that the set

$$W_g := cl_{L^{2,w}_{loc}(0,+\infty)} \{g(\cdot + s) \mid s \geq 0\}$$

is compact in $L^{2,w}_{loc}(0,+\infty) \Leftrightarrow \|g\|_+ := \sup_{t\geq 0} \int\limits_t^{t+1} \|g(s)\|^2_X ds < \infty$. Moreover, such a set is shift-invariant, and $\forall \xi \in W_g$

$$\|\xi\|_+ \leq \|g\|_+ .$$

Therefore, the set Σ defined by (2.19) is shift-invariant, and

$$\forall d \in \Sigma(h) \quad \|d\|_+ \leq \|h\|_\infty . \tag{2.20}$$

Theorem 2.5. *Assume that conditions* (2.2), (2.3), (2.18) *take place. Then the semiprocess family* $\{S_d\}_{d\in\Sigma}$ *generated by mild solutions of the problem* (2.1), *has a uniform attractor* A_Σ, *which satisfies* (2.17).

Proof. For every d with $\|d\|_+ < \infty$ inequality (2.7) implies

$$\|u(t)\|_X \leq Me^{-\lambda t}\|u_0\|_X + \frac{MC_1}{\lambda} + \|d\|_+^{\frac{1}{2}}(1-e^{-\lambda})^{-\frac{1}{2}} .$$

So, from (2.20) for every $d \in \Sigma(h)$ we get that for all bounded $B \subset X$ $\exists T = T(B)$ $\forall t \geq T$

$$S_\Sigma(t,0,B) \subset B_0 = \{u \in X \mid \|u\|_X \leq 1 + C\} , \tag{2.21}$$

for some positive constant C, which does not depend on B. Therefore, assumption 1) from Lemma 2.3 takes place. Moreover, for every bounded $B \subset X$ and every $u(\cdot)$ with $u(0) = u_0$ there exists $K = K(B)$ such that for all $d \in \Sigma$ and all $u_0 \in B$, $t \geq 0$

$$\|f(u(t))\|_\infty \leq K .$$

Then due to (2.10) for $t > 0$ and $\delta \in \left(\frac{1}{2}, 1\right)$

$$\|u(t)\|_{C^{1+\alpha}} \leq \frac{C}{t^\delta}\|u_0\|_X + \int\limits_0^t \frac{C}{s^\delta} K ds + \int\limits_0^t \frac{C}{s^\delta}\|h\|_\infty ds \leq r(t) . \tag{2.22}$$

Due to compact embedding $C^{1+\alpha} \Subset X$ and inclusions: for $\{d_n\} \subset \Sigma$, $t_n \to \infty$, $\|u_0^n\|_X \leq r$

$$\xi_n = S_{d_n}(t_n, 0, u_0^n) = S_{d_n}(t_n, t_n - 1, S_{d_n}(t_n - 1, 0, u_0^n)) =$$

$$= S_{d_n(\cdot + t_n - 1)}(1, 0, S_{d_n}(t_n - 1, 0, u_0^n)) \subset S_\Sigma(1, 0, B_0)$$

for sufficiently large $n \geq 1$, where B_0 is taken from (2.21). So, we conclude that $\{\xi_n\}$ is precompact in X, and, therefore, semiprocess family $\{S_d\}_{d\in\Sigma}$ possesses a uniform attractor Θ_Σ. $\square$

Let us prove (2.17). For this aim we prove the following result.

Lemma 2.6. *Assume that for* $d^n = (d_1^n, \dots, d_k^n)$, $\quad d = (d_1, \dots, d_k)$

$$d^n \to d \quad in \ \left(L^{2,w}_{\mathrm{loc}}(0,+\infty)\right)^K, \quad u_0^n \to u_0 \ in \ X\,. \tag{2.23}$$

Then for all $t \in [0,T]$ *we have*

$$u_n(t) = S_{d_n}(t,0,u_0) \to u(t) = S_d(t,0,u_0) \quad in \ X\,. \tag{2.24}$$

Proof. Due to (2.21) both $\{u_n\}$ and $\{f(u_n)\}$ are bounded in $\mathbb{C}([0,T];X)$. Let us consider u_n as a weak solution of (2.11) with right side

$$g_n(t) = f(u_n) + \sum_{j=1}^{K} d_j^n(t)\varphi_j\,.$$

Then $\{g_n\}$ is bounded in $L^2(0,T;X)$, $\{u_n\}$ is bounded in $L^2(0,T;H_0^1)$, $\{u_{n_t}\}$ is bounded in $L^2(0,T;H^{-1})$. So, due to Aubin-Lions Lemma there exists a function $u \in \mathbb{C}([0,T];L^2)$ such that up to subsequence:

$$u_n \to u \quad \text{weakly in} \quad L^2(0,T;H^{-1})\,,$$

$$u_n \to u \text{ in } L^2(0,T;L^2) \quad \text{and almost everywhere (a.e.) in} \quad (0,T)\times\Omega\,,$$

$$\forall\, t \in [0,T] \ u_n(t) \to u(t) \quad \text{weakly in} \quad L^2\,. \tag{2.25}$$

Then $f(u_n(t,x)) \to f(u(t,x))$ a.e. and, therefore,

$$g_n \to g = f(u) + \sum_{j=1}^{K} d_j(t)\varphi \text{ weakly in } L^2(0,T;L^2)\,.$$

So, u is a weak solution of (2.11) with the right hand side g. Thus, due to the previous arguments we have that u is a mild solution of (2.11) in L^2 and, therefore, a mild solution of (2.1) in L^2. Then u is a mild solution of (2.1) in X. Indeed, due to the (2.22) and (2.25) $\forall\, t \in [0,T]$ $\ u_n(t) \to u(t)$ in X. Then for all $t \in [0,T]$ $u(t,\cdot) \in X \ \Rightarrow \ f(u(t,\cdot)) \in X \ \Rightarrow g \in L^2(0,T;X) \ \Rightarrow u(t) \in S_d(t,0,u_0)$. Lemma is proved.

Property (2.24) implies (2.16), and, therefore, (2.17). Theorem is proved. □

3. Application to the robust stability theory

In this section we want to obtain asymptotic gain property (2.6).

Theorem 3.1. *Under conditions* (2.2), (2.3), (2.18) *problem* (2.1) *for* $\|h\|_\infty \le R_0$ *possesses asymptotic gain property w.r.t. global attractor* Θ *of the undisturbed* ($h \equiv 0$) *system.*

Proof. Let us assume that we have the limit property

$$dist\left(\Theta_{\Sigma(h)}, \Theta\right) \to 0 \text{ as } \|h\|_\infty \to 0\,. \tag{3.1}$$

Let us prove that (3.1) implies (2.6). Indeed, according to construction $\Sigma(0) = \{0\}$, and $h \in \Sigma(h)$. So, for $u_0 \in X$, $z \in \Theta_{\Sigma(h)}$, $t > 0$, $u(t) = S_h(t, 0, u_0)$ we have: for $\theta \in \Theta$:

$$\|u(t) - \theta\|_X \le \|u(t) - z\|_X + \|z - \theta\|_X \Rightarrow$$

$$\inf_{\theta\in\Theta} \|u(t) - \theta\|_X \le \|u(t) - z\|_X + \inf_{\theta\in\Theta} \|z - \theta\|_X \Rightarrow$$

$$\inf_{\theta\in\Theta} \|u(t) - \theta\|_X \le \inf_{z\in\Theta_{\Sigma(h)}} \|u(t) - z\|_X + \sup_{z\in\Theta_{\Sigma(h)}} \inf_{\theta\in\Theta} \|z - \theta\|_X \Rightarrow$$

$$\|u(t)\|_\Theta \le \operatorname{dist}_X \left(u(t), \Theta_{\Sigma(h)}\right) + \operatorname{dist}_X \left(\Theta_{\Sigma(h)}, \Theta\right) \Rightarrow$$

$$\|S_h(t, 0, u_0)\|_\Theta \le \operatorname{dist}_X \left(S_{\Sigma(h)}(t, 0, u_0), \Theta_{\Sigma(h)}\right) + \operatorname{dist}_X \left(\Theta_{\Sigma(h)}, \Theta\right) .$$

The first summand in the right part of this inequality tends to zero for every h. Let us put

$$\gamma(s) := \sup_{\|h\|_\infty\le s} dist_X(A_{\Sigma(h)}, A) + s\,.$$

Due to (3.1) $\gamma \in K$ and $\operatorname{dist}_X(\Theta_{\Sigma(h)}, \Theta) \le \gamma(\|h\|_\infty)$, so we have the required result. Let us prove (3.1). Assume that (3.1) does not take place. It means that there exists $h_n \to 0$ in $L^\infty(0, +\infty; X)$, there exist $\varepsilon > 0$ and $z_n \in \Theta_{\Sigma(h_n)}$ such that

$$\operatorname{dist}(z_n, \Theta) \ge \varepsilon\,. \tag{3.2}$$

From Theorem 2.5 we have that $\Theta_{\Sigma(h)} \subset K$, where compact K depends on R_0 (see estimation (2.22)). Then

$$z_n \in \Theta_{\Sigma(h_n)} \subset S_{\Sigma(h_n)}(t, 0, \Theta_{\Sigma(h_n)}) \subset S_{\Sigma(h_n)}(t, 0, K)\,.$$

Therefore, $z_n = u_n(t) = S_{d_n}(t, 0, \xi_n)$, where $\xi_n \to \xi$ in X, $\|d_n\|_+ \le \|h_n\|_\infty \to 0$. Then from Lemma 2.6

$$u_n(t) \to u(t) = S_0(t, 0, \xi) \subset S_0(t, 0, B_0)\,. \tag{3.3}$$

Due to the uniform attraction we can choose $t > 0$ such that

$$dist_X(S_0(t, 0, B_0), \Theta) < \frac{\varepsilon}{2}\,.$$

Then from (3.3)

$$z_n \to u(t) \in O_{\frac{\varepsilon}{2}}(\Theta),$$

that is a contradiction with (3.2). Theorem is proved. □

References

[1] Asrorov, F., Sobchuk, V., Kurylko, O., *Finding of bounded solutions to linear impulsive systems*, East-Europ. J. Enterprise Technol. **6** (4(102)) (2019), 14–20.

[2] Barabash, O., Dakhno, N., Shevchenko, H., Sobchuk, V., *Unmanned aerial vehicles flight trajectory optimisation on the basis of variational enequality algorithm and projection method*, Proceeding 2019 IEEE 5th International Conference "Actual Problems of Unmanned Aerial Vehicles Developments" (APUAVD), National Aviation University, Kyiv, Ukraine, 2019, pp. 136–139.

[3] Chepyzkov, V.V., Vishik, M.I., *Attractors for equations of mathematical physics*, vol. 49, AMS Colloquium Publications, 2002.

[4] Dashkovskiy, S., Feketa, P., Kapustyan, O., Romaniuk, I., *Invariance and stability of global attractors for multi-valued impulsive dynamical systems*, J. Math. Anal. Appl. **458** (1) (2018), 193–218.

[5] Dashkovskiy, S., Kapustyan, O., Romaniuk, I., *Global attractors of impulsive parabolic inclusions*, Discrete Contin. Dyn. Syst. Ser. B **22** (5) (2017), 1875–1886.

[6] Dashkovskiy, S., Kapustyan, O., Schmid, J., *A local input-to-state stability result w.r.t. attractors of nonlinear reaction-diffusion equations*, Math. Control Signals Systems **32** (3) (2020), 309–326.

[7] Dashkovskiy, S., Mironchenko, A., *Input-to-state stability of infinite-dimensional control systems*, Math. Control Signal Systems **25** (2013), 1–35.

[8] Haraux, A., Kirane, M., *Estimation C^1 pour des problemes paraboliques semi-lineaires*, Ann. Fac. Sci. Toulouse Math. **5** (1983), 265–280.

[9] Kapustyan, O.V., Kapustian, O.A., Gorban, N.V., Khomenko, O.V., *Strong global attractor for the three-dimensional Navier-Stokes system of equations in unbounded domain of channel type*, J. Automat. Inform. Sci. **47** (11) (2015), 48–59.

[10] Kapustyan, O.V., Kasyanov, P.O., Valero, J., *Structure of the global attractor for weak solutions of a reaction-diffusion equation*, Appl. Math. Inform. Sci. **9** (5) (2015), 2257–2264.

[11] Kichmarenko, O., Stanzhytskyi, O., *Sufficient conditions for the existence of optimal controls for some classes of functional-differential equations*, Nonlinear Dyn. Syst. Theory **18** (2) (2018), 196–211.

[12] Manthey, R., Zausinger, T., *Stochastic equations in L^2_ρ*, Stochastic Rep. **66** (1977), 370–373.

[13] Mironchenko, A., Prieur, Ch., *Input-to-state stability of infinite-dimensional systems: recent results and open questions*, SIAM Rev. **62** (2020), 529–614.

[14] Mironchenko, A., Wirtz, F., *Characterization of input-to-state stability for infinite-dimensional systems*, IEEE Trans. Automat. Control **63** (6) (2018), 1602–1617.

[15] Nakonechnyi, A.G., Mashchenko, S.O., Chikrii, V.K, *Motion control under conflict condition*, J. Automat. Inform. Sci. **50** (1) (2018), 54–75.

[16] Nakonechnyi, O.G., Kapustian, O.A., Chikrii, A.O., *Approximate guaranteed mean square estimates of functionals on solutions of parabolic problems with fast oscillating coefficients under nonlinear observations*, Cybernet. Systems Anal. **55** (5) (2019), 785–795.

[17] Pazy, A., *Semigroups of linear operators and applications to PDE*, Springer-Verlag New York, 1983.

[18] Pichkur, V.V., Sobchuk, V.V., *Mathematical models and control design of a functionally stable technological process*, J. Optim. Differ. Equ. Appl. (JODEA) **21** (1) (2021), 1–11.

[19] Robinson, J., *Infinite-dimensional dynamical systems. An introduction to dissipative parabolic PDEs and the theory of global attractors*, Cambridge University Press, 2001.

[20] Samoilenko, A.M., Stanzhitskii, A.N., *On the averaging of differential equations on an infinite interval*, Differ. Equ. **42** (4) (2006), 505–511.

[21] Schmid, J., Kapustyan, O., Dashkovskiy, S., *Asymptotic gain results for attractors of semilinear systems*, Math. Control Relat. Fields **12** (3) (2022), 763–788.

[22] Sell, G., You, Y., *Dynamics of evolutionary equations*, Springer New York, NY, 2000.

[23] Sontag, E.D., *Smooth stabilization implies coprime factorization*, IEEE Trans. Automat. Control **34** (4) (1989), 435–443.

[24] Sontag, E.D., *Mathematical control theory. Deterministic finite-dimensional systems*, Springer, N.Y., 1998.

[25] Stanzhitskii, A.M., *Investigation of invariant sets of Itô stochastic systems with the use of Lyapunov functions*, Ukrainian Math. J. **53** (2) (2001), 323–327.

[26] Stanzhyts'kyi, O., *Investigation of exponential dichotomy of Ito stochastic systems by using quadratic forms*, Ukrainian Math. J. **53** (11) (2001), 1882–1894.

Department of Integral and Differential Equations,
Mechanics and Mathematics Faculty,
Taras Shevchenko National University of Kyiv,
Volodymyrska Street, 60, 01601 Kyiv, Ukraine
E-mail: kapustyanav@gmail.com

Department of System Analysis and Decision Making Theory,
Faculty of Computer Science and Cybernetics,
Taras Shevchenko National University of Kyiv,
Volodymyrska Street, 60, 01601 Kyiv, Ukraine
and
Dipartimento di Ingegneria e Scienze dell'Informazione e Matematica,
Universita degli Studi dell'Aquila,
Via Vetorio, Coppito 1, 67100 L'Aquila, Italy
E-mail: olenakapustian@knu.ua

Department of General Mathematics, Mechanics and Mathematics Faculty,
Taras Shevchenko National University of Kyiv,
Volodymyrska Street, 60, 01601 Kyiv, Ukraine
E-mail: ostanzh@gmail.com

Department of Algebra and Differential Equations,
Faculty of Mathematics and Digital Technologies, Uzhhorod National University,
Narodna Square, 3, 88000 Uzhhorod, Ukraine
and
Department of Mathematical Analysis,
Faculty of Natural Sciences and Health,
The John Paul II Catholic University of Lublin,
Al. Raclawickie 14, 20-950 Lublin, Poland
E-mail: korol.ihor@gmail.com