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FINITE-TIME BLOW-UP IN A TWO-SPECIES CHEMOTAXIS-COMPETITION MODEL WITH SINGLE PRODUCTION

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Abstract. This paper is concerned with blow-up of solutions to a two-species chemotaxis-competition model with production from only one species. In previous papers there are a lot of studies on boundedness for a two-species chemotaxis-competition model with productions from both two species. On the other hand, finite-time blow-up was recently obtained under smallness conditions for competitive effects. Now, in the biological view, the production term seems to promote blow-up phenomena; this implies that the lack of the production term makes the solution likely to be bounded. Thus, it is expected that there exists a solution of the system with single production such that the species which does not produce the chemical substance remains bounded, whereas the other species blows up. The purpose of this paper is to prove that this conjecture is true.

1. Introduction and main result

In this paper we deal with the two-species chemotaxis-competition model with single production,

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + \mu_1 u(1 - u^{\kappa_1 - 1} - a_1 v^{\lambda_1 - 1})\,, \\
\dfrac{\partial v}{\partial t} = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + \mu_2 v(1 - a_2 u^{\lambda_2 - 1} - v^{\kappa_2 - 1})\,, \\
0 = d_3\Delta w + \alpha u - \gamma w\,, \\
(\nabla u\cdot\nu)|_{\partial\Omega} = (\nabla v\cdot\nu)|_{\partial\Omega} = (\nabla w\cdot\nu)|_{\partial\Omega} = 0\,, \\
u(x,0) = u_0(x),\ v(x,0) = v_0(x)
\end{cases}
\tag{1.1}
$$

in a ball $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \ge 3, R > 0)$. Here, ν is the outward normal vector to $\partial\Omega$; d_1, d_2, d_3, χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, $\gamma > 0$ and κ_1, κ_2, λ_1, $\lambda_2 > 1$; u_0, $v_0 \in C^0(\overline{\Omega})$ are nonnegative and radially symmetric. This system describes a situation in which multi species move toward higher concentrations of the signal substance (which is produced by the spesies), and compete with each other.

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In a two-species chemotaxis-competition model obtained on replacing the third equation in (1.1) by

$$0 = d_3\Delta w + \alpha u + \beta v - \gamma w \quad (\beta > 0)\,,$$

boundedness and stabilization in the case $\kappa_1 = \kappa_2 = \lambda_1 = \lambda_2 = 2$ were established under smallness conditions for χ_1 and χ_2 in [2, 5, 7, 8]; more related works can be found in [1, 9]. On the other hand, a result on finite-time blow-up in the two-species chemotaxis system was recently obtained in [6, Theorem 4.1] under the condition

$$\max\{\kappa_1, \lambda_1, \kappa_2, \lambda_2\} < \begin{cases} \frac{7}{6} & \text{if } \ n \in \{3, 4\}\,, \\ 1 + \frac{1}{2(n-1)} & \text{if } \ n \geq 5\,. \end{cases}$$

Now, in the biological view, the production term seems to promote blow-up phenomena; this implies that the lack of the production term makes the solution likely to be bounded. Thus, since the third equation in (1.1) lacks the production term βv, it is expected that there exists a solution of (1.1) such that v remains bounded, whereas u blows up. The purpose of this paper is to prove that this conjecture is true.

The main results read as follows. The first theorem gives blow-up in (1.1).

Theorem 1.1. *Let d_1, d_2, d_3, χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, $\gamma > 0$ and κ_1, κ_2, λ_1, $\lambda_2 > 1$. Assume that κ_1 and λ_1 satisfy that*

$$\max\{\kappa_1, \lambda_1\} < \begin{cases} \frac{7}{6} & \textit{if } \ n \in \{3, 4\}\,, \\ 1 + \frac{1}{2(n-1)} & \textit{if } \ n \geq 5\,. \end{cases} \tag{1.2}$$

Then, for all $L > 0$, $M_0 > 0$ and $\widetilde{M_0} \in (0, M_0)$ there exists $r_\star \in (0, R)$ with the following property: If

$$u_0,\ v_0 \in C^0(\overline{\Omega}) \ \textit{are nonnegative and radially symmetric} \tag{1.3}$$

and

$$\int_\Omega (u_0(x) + v_0(x))\, dx = M_0 \quad \textit{and} \quad \int_{B_{r_\star}(0)} u_0(x)\, dx \geq \widetilde{M_0} \tag{1.4}$$

as well as

$$u_0(x) + v_0(x) \leq L|x|^{-n(n-1)} \quad \textit{for all } \ x \in \Omega, \tag{1.5}$$

then there exist $T^* < \infty$ and exactly one triplet (u, v, w) of (1.1) which blows up in finite time in the sense that

$$\lim_{t \nearrow T^*} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty. \tag{1.6}$$

Remark 1.2. This result means that whether blow-up in (1.1) occurs or not can be determined by the parameters which come only from the first equation.

Theorem 1.1 gives existence of a constant $T^* > 0$ and a classical solution (u, v, w) of (1.1) on $[0, T^*)$ such that $\lim_{t\nearrow T^*}(\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty$. Then we consider the next question

whether $\lim_{t\nearrow T^*} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty$ *and* $\lim_{t\nearrow T^*} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty$ *hold.*

The second theorem is concerned with simultaneous blow-up in (1.1).

Theorem 1.3. *Let d_1, d_2, d_3, χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, $\gamma > 0$ and κ_1, κ_2, λ_1, $\lambda_2 > 1$. Then the following holds*:
(i) *Assume that u_0, $v_0 \in C^0(\overline{\Omega})$ are nonnegative. Let $T \in (0, \infty]$ and let (u, v, w) be a classical solution of* (1.1) *on $[0, T)$. Then (u, v, w) satisfies that*

$$\text{if } \lim_{t\nearrow T} \|v(\cdot,t)\|_{L^\infty(\Omega)} = \infty, \text{ then } \lim_{t\nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

(ii) *Assume that κ_1 and λ_1 satisfy* (1.2). *Moreover, suppose that $\lambda_2 \geq 2$ and*

$$0 < \chi_2 < \begin{cases} \frac{a_2 d_3 \mu_2}{\alpha} & \text{if } \lambda_2 = 2\,, \\ \infty & \text{if } \lambda_2 > 2\,. \end{cases} \tag{1.7}$$

Then there are initial data $u_0, v_0 \in C^0(\overline{\Omega})$ and $T^* < \infty$ such that the corresponding solution (u, v, w) of (1.1) *on $[0, T^*)$ satisfies*

$$\lim_{t\nearrow T^*} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty \quad \text{and} \quad \sup_{t\in(0,T^*)} \|v(\cdot,t)\|_{L^\infty(\Omega)} < \infty\,.$$

Remark 1.4. This theorem means that if v blows up at time T then u also blows up at T, and moreover there is a solution such that u blows up at T but v is bounded in $\Omega \times (0, T)$; thus this result gives a positive answer to the conjecture.

This paper is organized as follows. In order to show Theorem 1.1, we will derive a differential inequality for some moment-type function in Section 2. Section 3 is devoted to the proof of Theorem 1.3.

2. Proof of Theorem 1.1

We first state a result on local existence of solutions to (1.1).

Lemma 2.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ $(n \geq 3)$ be a ball with some $R > 0$, and let d_1, d_2, d_3, χ_1, χ_2, μ_1, μ_2, a_1, a_2, α, $\gamma > 0$ and κ_1, κ_2, λ_1, $\lambda_2 > 1$. Assume that $u_0, v_0 \in C^0(\overline{\Omega})$ are nonnegative. Then there exist $T_{\max} \in (0, \infty]$ and a unique triplet (u, v, w) of functions*

$$u,\ v,\ w \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))\,,$$

which solves (1.1) *classically. Moreover, u, $v \geq 0$ in $\Omega \times (0, T_{\max})$ and*

$$\text{if } T_{\max} < \infty\,, \quad \text{then} \quad \lim_{t\nearrow T_{\max}} (\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)}) = \infty. \tag{2.1}$$

Also, if u_0, v_0 are radially symmetric, then so are u, v, w for any $t \in (0, T_{\max})$.

Proof. This lemma is shown by a standard fixed point argument as in [3, 7]. □

In this section we assume that u_0, $v_0 \in C^0(\overline{\Omega})$ are nonnegative and radially symmetric and that (u, v, w) is a classical solution of (1.1) on $[0, T_{\max})$ given by Lemma 2.1. Moreover, we regard $u(x, t)$, $v(x, t)$ and $w(x, t)$ as functions of $r := |x|$ and t. Also, we introduce the functions U, V and W as

$$U(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho,t)\, d\rho \quad \text{and} \quad V(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} v(\rho,t)\, d\rho$$

as well as

$$W(s,t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} w(\rho,t)\, d\rho$$

for $s \in [0, R^n]$ and $t \in [0, T_{\max})$, and define ϕ_U and ψ_U as

$$\phi_U(t) := \int_0^{s_0} s^{-b}(s_0 - s) U(s,t)\, ds$$

and

$$\psi_U(t) := \int_0^{s_0} s^{-b}(s_0 - s) U(s,t) U_s(s,t)\, ds$$

for $t \in [0, T_{\max})$ with some $s_0 \in (0, R^n)$ and $b \in (0,1)$. We note that ϕ_U belongs to $C^0([0,T_{\max})) \cap C^1((0,T_{\max}))$. To obtain the differential inequality for ϕ_U, we first give the following lemma.

Lemma 2.2. *Let $s_0 \in (0, R^n)$ and $b \in (0,1)$. Then*

$$\begin{aligned}
\phi_U'(t) \geq{}& d_1 n^2 \int_0^{s_0} s^{2-\frac{2}{n}-b}(s_0 - s) U_{ss}\, ds \\
&+ \frac{\alpha\chi_1 n}{d_3}\psi_U(t) - \frac{\gamma\chi_1 n}{d_3}\int_0^{s_0} s^{-b}(s_0 - s) U_s W\, ds \\
&- \mu_1 n^{\kappa_1 - 1}\int_0^{s_0} s^{-b}(s_0 - s)\left(\int_0^s U_s^{\kappa_1}(\sigma,t)\, d\sigma\right) ds \\
&- a_1\mu_1 n^{\lambda_1 - 1}\int_0^{s_0} s^{-b}(s_0 - s)\left(\int_0^s U_s(\sigma,t) V_s^{\lambda_1 - 1}(\sigma,t)\, d\sigma\right) ds \\
=:{}& I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \tag{2.2}$$

for all $t \in (0, T_{\max})$.

Proof. By straightforward computations we can derive (2.2) (see [6, (4.17)]). □

We next estimate the third term on the right-hand side of (2.2).

Lemma 2.3. *Let $b \in (0, \min\{1, 2 - \frac{4}{n}\})$. For all $L > 0$ and all $M_0 > 0$ there exists $C > 0$ such that if u_0, v_0 satisfy* (1.3) *and $\int_\Omega (u_0(x) + v_0(x))\, dx = M_0$ as well as* (1.5)*, then*

$$I_3 \geq -C s_0^{\frac{2}{n}} \psi_U(t) - C s_0^{1-b+\frac{2}{n}} \tag{2.3}$$

for all $s_0 \in (0, R^n)$ and $t \in (0, \min\{1, T_{\max}\})$.

Proof. As in [10, estimate (4.5)], by integration by parts we have

$$I_3 \geq -(b+1)\frac{\gamma\chi_1 n}{d_3} s_0 \int_0^{s_0} s^{-b-1} U W\, ds$$

for all $t \in (0, T_{\max})$. Furthermore, by the structure of the third equation in (1.1), a result similar to [10, Lemma 4.8] is established, so that we attain (2.3). □

With regard to Lemma 2.3, by virtue of the structure of the third equation in (1.1), a term including $\psi_V(t)$ does not appear unlike [6, Lemma 4.4], where $\psi_V(t) := \int_0^{s_0} s^{-b}(s_0 - s)V(s,t)V_s(s,t)\,ds$. Thus we derive a differential inequality for only ϕ_U to show blow-up.

Lemma 2.4. *Assume that $\kappa_1 > 1$ and $\lambda_1 > 1$ satisfy* (1.2). *Then there exists $b \in \left(1 - \frac{2}{n}, \min\{1, 2 - \frac{4}{n}\}\right)$ with the property that for all $L > 0$ and $M_0 > 0$ one can find $C_1 > 0$, $C_2 > 0$ and $s_1 \in (0, R^n)$ such that if u_0, v_0 satisfy* (1.3) *and $\int_\Omega (u_0(x) + v_0(x))\,dx = M_0$ as well as* (1.5), *then*

$$\phi_U'(t) \geq C_1 s_0^{-(3-b)} \phi_U^2(t) - C_2 s_0^{1-b+\frac{2}{n}} \tag{2.4}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{1, T_{\max}\})$.

Proof. Let us fix $\varepsilon > 0$ such that

$$2\varepsilon \leq 1 - \frac{2}{n}\,. \tag{2.5}$$

Moreover, we can take $b \in \left(1 - \frac{2}{n}, \min\{1, 2 - \frac{4}{n}\}\right)$ such that

$$(n-1)(\max\{\kappa_1, \lambda_1\} - 1) < \frac{b}{2}\,, \tag{2.6}$$

because (1.2) ensures that $(n-1)(\min\{\kappa_1, \lambda_1\} - 1) < \frac{1}{3} = \frac{1}{2}\left(2 - \frac{4}{n}\right)$ if $n = 3$, and that $(n-1)(\min\{\kappa_1, \lambda_1\} - 1) < \frac{1}{2}$ if $n \geq 4$. Noting that (1.3), (1.5) and the condition $\int_\Omega (u_0(x) + v_0(x))\,dx = M_0$, from [6, Lemma 4.2] we can find c_1, $c_2 > 0$ such that

$$I_1 \geq -c_1 s_0^{\frac{3-b}{2} - \frac{2}{n}} \sqrt{\psi_U(t)}$$

and

$$I_4 \geq -c_2 s_0^{-(n-1)(\kappa_1 - 1) + \frac{3-b}{2} - \varepsilon} \sqrt{\psi_U(t)}$$

for all $t \in (0, \min\{1, T_{\max}\})$. Moreover, thanks to [6, Lemma 4.5], there exists $c_3 > 0$ satisfying

$$I_5 \geq -c_3 s_0^{-(n-1)(\lambda_1 - 1) + \frac{3-b}{2} - \varepsilon} \sqrt{\psi_U(t)}$$

for all $t \in (0, \min\{1, T_{\max}\})$. Hence, plugging these inequalities and Lemma 2.3 into (2.2) entails that

$$\begin{aligned} \phi_U'(t) \geq \frac{\alpha \chi_1 n}{d_3} \psi_U(t) &- c_4 s_0^{\frac{2}{n}} \psi_U(t) - c_4 s_0^{1-b+\frac{2}{n}} \\ &- c_1 s_0^{\frac{3-b}{2} - \frac{2}{n}} \sqrt{\psi_U(t)} \\ &- c_2 s_0^{-(n-1)(\kappa_1 - 1) + \frac{3-b}{2} - \varepsilon} \sqrt{\psi_U(t)} \\ &- c_3 s_0^{-(n-1)(\lambda_1 - 1) + \frac{3-b}{2} - \varepsilon} \sqrt{\psi_U(t)} \end{aligned}$$

for all $t \in (0, \min\{1, T_{\max}\})$ with some $c_4 > 0$. By Young's inequality we infer that

$$\begin{aligned}\phi'_U(t) &\geq c_5\psi_U(t) - c_4 s_0^{\frac{2}{n}}\psi_U(t)\\ &\quad - c_6 s_0^{1-b+\frac{2}{n}}\left(s_0^{2-\frac{6}{n}} + 1 + s_0^{2-\frac{2}{n}-2(n-1)(\kappa_1-1)-2\varepsilon} + s_0^{2-\frac{2}{n}-2(n-1)(\lambda_1-1)-2\varepsilon}\right)\end{aligned}$$

for all $t \in (0, \min\{1, T_{\max}\})$ with some $c_5 > 0$ and $c_6 > 0$. Now let us choose $s_1 \in (0, R^n)$ such that $c_4 s_1^{\frac{2}{n}} \leq \frac{c_5}{2}$. Noting from (2.5) and (2.6) that

$$2 - \frac{2}{n} - 2(n-1)(\min\{\kappa_1, \lambda_1\} - 1) - 2\varepsilon > 1 - b > 0\,,$$

we have from the relation $2 - \frac{6}{n} \geq 0$ that

$$\phi'_U(t) \geq \frac{c_5}{2}\psi_U(t) - c_7 s_0^{1-b+\frac{2}{n}} \tag{2.7}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{1, T_{\max}\})$ with some $c_7 > 0$, where we have used the relations $c_4 s_0^{\frac{2}{n}} < c_4 s_1^{\frac{2}{n}} \leq \frac{c_5}{2}$ and $s_0 < R^n$. Now from [10, Lemma 4.4] there exists $c_8 > 0$ satisfying that $\psi_U(t) \geq c_8 s_0^{-(3-b)}\phi_U^2(t)$ for all $t \in (0, T_{\max})$, which together with (2.7) yields (2.4). $\square$

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Thanks to Lemma 2.4, there exist $c_1 > 0$, $c_2 > 0$ and $s_1 \in (0, R^n)$ such that

$$\phi'_U(t) \geq c_1 s_0^{-(3-b)}\phi_U^2(t) - c_2 s_0^{1-b+\frac{2}{n}}$$

for all $s_0 \in (0, s_1)$ and $t \in (0, \min\{1, T_{\max}\})$. Let us pick $s_0 \in (0, s_1)$ fulfilling

$$\sqrt{\frac{c_2}{c_1}}s_0^{\frac{1}{n}} + \frac{2}{c_1}s_0 \leq \frac{\widetilde{M}_0}{2^{3-b}\omega_n}\,.$$

Then it follows that

$$\frac{\widetilde{M}_0}{2^{3-b}\omega_n}s_0^{2-b} \geq \sqrt{\frac{c_2}{c_1}}s_0^{2-b+\frac{1}{n}} + \frac{2}{c_1}s_0^{3-b}\,.$$

Moreover, put

$$r_\star := \left(\frac{s_0}{4}\right)^{\frac{1}{n}} \in (0, R)\,,$$

and select initial data u_0, v_0 satisfy (1.3), (1.4) and (1.5). By [10, estimate (5.5)], we can verify that

$$\phi_U(0) \geq \frac{\widetilde{M}_0}{2^{3-b}\omega_n}s_0^{2-b}.$$

As in the proof of [4, Lemma 4.6] (with $d_1(s_0) = c_1 s_0^{-(3-b)}$, $d_2(s_0) = c_2 s_0^{1-b-\frac{2}{n}}$ and $\phi(s_0) = \frac{\widetilde{M}_0}{2^{3-b}\omega_n}s_0^{2-b}$), we can derive that $T_{\max} \leq \frac{1}{2}$. Therefore, from (2.1) we arrive at (1.6), which completes the proof. $\square$

3. Proof of Theorem 1.3

In the following, we let $T \in (0, \infty]$ and let (u, v, w) be a classical solution of (1.1) on $[0, T)$ with $u_0, v_0 \in C^0(\overline{\Omega})$ being nonnegative. Now we put

$$\mathcal{L}\widetilde{v} := d_2 \Delta \widetilde{v} - \chi_2 \nabla \widetilde{v} \cdot \nabla w$$

for $\widetilde{v} \in C^2(\overline{\Omega})$. Then we note from the second and third equations in (1.1) that

$$\begin{aligned} \frac{\partial v}{\partial t} &= \mathcal{L}v - \chi_2 v \Delta w + \mu_2 v(1 - a_2 u^{\lambda_2 - 1} - v^{\kappa_2 - 1}) \\ &= \mathcal{L}v + \frac{\alpha \chi_2}{d_3} uv - \frac{\gamma \chi_2}{d_3} vw + \mu_2 v(1 - a_2 u^{\lambda_2 - 1} - v^{\kappa_2 - 1}) \\ &\le \mathcal{L}v + \frac{\alpha \chi_2}{d_3} uv + \mu_2 v - a_2 \mu_2 u^{\lambda_2 - 1} v - \mu_2 v^{\kappa_2} \end{aligned} \tag{3.1}$$

for all $x \in \Omega$ and $t \in (0, T)$. By using this inequality we will show the following two lemmas which play an important role in the proof of Theorem 1.3.

Lemma 3.1. *The solution (u, v, w) satisfies that if $\lim_{t \nearrow T} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty$, then $\lim_{t \nearrow T} \|v(\cdot, t)\|_{L^\infty(\Omega)} < \infty$.*

Proof. Assume that $\|u(\cdot, t)\|_{L^\infty(\Omega)} \le c_1$ for all $t \in (0, T)$ with some $c_1 > 0$. Then, from (3.1) we see that

$$\frac{\partial v}{\partial t} \le \mathcal{L}v + \Big(\frac{\alpha \chi_2}{d_3} c_1 + \mu_2 \Big) v - \mu_2 v^{\kappa_2}$$

for all $x \in \Omega$ and $t \in (0, T)$. Let us next choose $\overline{v} \in (0, \infty)$ such that $\|v_0\|_{L^\infty(\Omega)} \le \overline{v}$, and denote by $y \colon [0, \infty) \to \mathbb{R}$ the function solving

$$\begin{cases} y'(t) = \left(\dfrac{\alpha \chi_2}{d_3} c_1 + \mu_2 \right) y(t) - \mu_2 y^{\kappa_2}(t)\,, \quad t > 0\,, \\ y(0) = \overline{v}\,. \end{cases}$$

Then, by a comparison principle, we can observe that for all $x \in \Omega$ and $t \in (0, T)$,

$$v(x, t) \le y(t) \le \max \left\{ \left(\frac{\frac{\alpha \chi_2}{d_3} c_1 + \mu_2}{\mu_2} \right)^{\frac{1}{\kappa_2 - 1}}, \overline{v} \right\} =: c_2$$

holds, which implies that $\|v(\cdot, t)\|_{L^\infty(\Omega)} \le c_2$ for all $t \in (0, T)$. □

Lemma 3.2. *Assume that $\lambda_2 \ge 2$ and χ_2 satisfies* (1.7)*. Then*

$$\|v(\cdot, t)\|_{L^\infty(\Omega)} \le C \tag{3.2}$$

holds for all $t \in (0, T)$ with some $C > 0$.

Proof. When $\lambda_2 = 2$, by (3.1) and the fact $a_2 \mu_2 - \frac{\alpha \chi_2}{d_3} > 0$ (from (1.7)) we have

$$\begin{aligned} \frac{\partial v}{\partial t} &\le \mathcal{L}v + \mu_2 v - \mu_2 v^{\kappa_2} - \left(a_2 \mu_2 - \frac{\alpha \chi_2}{d_3} \right) uv \\ &\le \mathcal{L}v + \mu_2 v - \mu_2 v^{\kappa_2} \end{aligned}$$

for all $x \in \Omega$ and $t \in (0,T)$. Thus a comparison principle yields (3.2). On the other hand, in the case that $\lambda_2 > 2$, Young's inequality enables us to find some constant $c_1 > 0$ satisfying $\frac{\partial v}{\partial t} \le \mathcal{L}v + (c_1 + \mu_2)v - \mu_2 v^{\kappa_2}$ for all $x \in \Omega$ and $t \in (0,T)$. Similarly, a comparison principle yields (3.2), which concludes the proof. $\square$

Proof of Theorem 1.3. Lemma 3.1 directly entails Theorem 1.3 (i). We next show Theorem 1.3 (ii). Theorem 1.1 asserts that there are initial data $u_0, v_0 \in C^0(\overline{\Omega})$ and $T^* < \infty$ such that the corresponding solution (u,v,w) of (1.1) on $[0,T^*)$ satisfies that $\lim_{t \nearrow T^*}(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)}) = \infty$. Then, noticing from Lemma 3.2 with $T = T^*$ that $\sup_{t\in(0,T^*)} \|v(\cdot,t)\|_{L^\infty(\Omega)} < \infty$ holds, we see that $\lim_{t \nearrow T^*} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$ holds, which means that Theorem 1.3 (ii) holds. $\square$

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References

[1] Baldelli, L., Filippucci, R., *A priori estimates for elliptic problems via Liouville type theorems*, Discrete Contin. Dyn. Syst. Ser. S **13** (7) (2020), 1883–1898.

[2] Black, T., Lankeit, J., Mizukami, M., *On the weakly competitive case in a two-species chemotaxis model*, IMA J. Appl. Math. **81** (5) (2016), 860–876.

[3] Cieślak, T., Winkler, M., *Finite-time blow-up in a quasilinear system of chemotaxis*, Nonlinearity **21** (5) (2008), 1057–1076.

[4] Fuest, M., *Approaching optimality in blow-up results for Keller-Segel systems with logistic-type dampening*, NoDEA Nonlinear Differential Equations Appl. **28** (16) (2021), 17 pp.

[5] Mizukami, M., *Boundedness and stabilization in a two-species chemotaxis-competition system of parabolic-parabolic-elliptic type*, Math. Methods Appl. Sci. **41** (1) (2018), 234–249.

[6] Mizukami, M., Tanaka, Y., Yokota, T., *Can chemotactic effects lead to blow-up or not in two-species chemotaxis-competition models?*, Z. Angew. Math. Phys. **73** (239) (2022), 25 pp.

[7] Stinner, C., Tello, J.I., Winkler, M., *Competitive exclusion in a two-species chemotaxis model*, J. Math. Biol. **68** (7) (2014), 1607–1626.

[8] Tello, J.I., Winkler, M., *Stabilization in a two-species chemotaxis system with a logistic source*, Nonlinearity **25** (5) (2012), 1413–1425.

[9] Tu, X., Qiu, S., *Finite-time blow-up and global boundedness for chemotaxis system with strong logistic dampening*, J. Math. Anal. Appl. **486** (1) (2020), 25 pp.

[10] Winkler, M., *Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation*, Z. Angew. Math. Phys. **69** (69) (2018), 40 pp.

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