EXISTENCE OF BLOW-UP SOLUTIONS FOR A DEGENERATE PARABOLIC-ELLIPTIC KELLER–SEGEL SYSTEM WITH LOGISTIC SOURCE

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Abstract. This paper deals with existence of finite-time blow-up solutions to a degenerate parabolic–elliptic Keller–Segel system with logistic source. Recently, finite-time blow-up was established for a degenerate Jäger–Luckhaus system with logistic source. However, blow-up solutions of the aforementioned system have not been obtained. The purpose of this paper is to construct blow-up solutions of a degenerate Keller–Segel system with logistic source.

1. Introduction and main result

In this paper we consider the quasilinear degenerate Keller–Segel system with logistic source,

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta u^m - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa, \quad x \in \Omega, \ t > 0, \\
0 &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \ t > 0, \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{aligned}
\]

where \( \Omega := B_R(0) \subset \mathbb{R}^n \ (n \geq 3) \) be a ball with some \( R > 0; \ m \geq 1, \ \chi > 0, \ \lambda > 0, \ \mu > 0 \) and \( \kappa > 1; \nu \) is the outward normal vector to \( \partial \Omega; \ u_0 \in L^\infty(\Omega) \) is nonnegative and radially symmetric. This system describes a situation such that a cellular slime moves towards higher concentrations of the chemical substance.

In the case \( m = 1 \), Winkler \[10\] obtained initial data leading to finite-time blow-up under a smallness condition for \( \kappa > 1 \) in three- or higher-dimensional cases. In the case \( m \in \left[ 1, 2 - \frac{2}{n} \right) \), for the system such that the diffusion term is replaced with \( \nabla \cdot ((u + 1)^{m-1} \nabla u) \), Black, Fuest and Lankeit showed that solutions blow up in finite time under the condition that \( \kappa < 1 + \min \left\{ \frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)} \right\} \) in \[1\] Theorem 1.2 (ii)]. On the other hand, a difficulty is caused in (1.1) by the degenerate diffusion term \( \Delta u^m \) because in the case of nondegenerate diffusion

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classical solutions can be considered, whereas in the case of degenerate diffusion class.,
classical solutions are not always obtained. In such circumstances, it had not been
clear whether blow-up of solutions to (1.1) occurs.

Regarding this difficulty, existence of blow-up solutions was recently established
in \[8\] for the following Jäger–Luckhaus system with \(\varepsilon = 0\),

\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \Delta(u + \varepsilon)^m - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa, \quad x \in \Omega, \ t > 0, \\
0 &= \Delta v - M(t) + u, \quad x \in \Omega, \ t > 0,
\end{aligned}
\]

where \(M(t) := \frac{1}{|\Omega|} \int_\Omega u(x,t) \, dx\). This system was studied in \[1, 3, 7, 9\]; in the case
\(m = 1\) and \(\varepsilon = 0\), finite-time blow-up was shown under smallness conditions for
\(\kappa\) in the three- and higher-dimensional cases in \[1, 9\] (in the case
\(M(t) = v\), see \[10\]); these conditions were improved in \[3\]; in the case \(m \neq 1\), the condition
\(\kappa < \min \{2, \frac{n}{2}\}\) was generalized to the condition that \(\kappa < \min \{2, (2 - m) \frac{n}{2}\}\)
if \(m \geq 0\) or \(\kappa < \min \{2, n\}\) if \(m < 0\) in \[7\]. After that, in the case of degenerate
diffusion (\(\varepsilon = 0\)), finite-time blow-up solutions was constructed in a framework of
weak solutions in \[8\].

In contrast, for the degenerate Keller–Segel system with logistic source there is
no result on blow-up. The purpose is to prove existence of blow-up solutions to
(1.1) in a framework of weak solutions under the same condition as in \[1, \text{Theorem}
1.2 \text{ (ii)}\]. Referring to the method in \[8\], we introduce moment solutions as follows.

**Definition 1.1.** Let \(T \in (0, \infty]\). A pair \((u, v)\) of nonnegative and radially symmetric functions defined on \(\Omega \times (0, T)\) is called a moment solution of (1.1) on \([0, T)\)

(i) \(u \in C^0_{W, -\lambda}([0, T); L^\infty(\Omega)) \cap L^\infty([0, T); L^\infty(\Omega)), \quad u^m \in L^2(0, T; H^1(\Omega))\) if \(T < \infty; \ u^m \in L^2_{\text{loc}}(0, T; H^1(\Omega))\) if \(T = \infty, \varepsilon\)

\(v \in L^\infty_{\text{loc}}([0, T); H^1(\Omega))\),

(ii) for all \(\varphi \in L^2(0, T; H^1(\Omega)) \cap W^{1,1}(0, T; L^2(\Omega))\) with \(\text{supp} \varphi(x, \cdot) \subset [0, T)\)

(a.a. \(x \in \Omega)) ,

\[
\int_0^T \int_\Omega (\nabla u^m \cdot \nabla \varphi - \chi u \nabla v \cdot \nabla \varphi - (\lambda u - \mu u^\kappa) \varphi - u \varphi_t) \, dx \, dt \\
= \int_\Omega u_0(x) \varphi(x, 0) \, dx,
\]

\[
\int_0^T \int_\Omega (\nabla v \cdot \nabla \varphi + v \varphi - u \varphi) \, dx \, dt = 0,
\]

(iii) \((u, v)\) satisfies the following moment inequality:

\[
\phi(t) - \phi(0) \geq K \int_0^t \phi^2(\tau) \, d\tau \quad \text{for all } t \in (0, T),
\]
where
\[
\phi(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w(s,t)\,ds \quad \text{for } t \in (0,T),
\]
\[
w(s,t) := \int_0^{s_1} \rho^{n-1}u(\rho,t)\,d\rho \quad \text{for } s \in [0,R^n] \text{ and } t \in (0,T)
\]
with some \(s_0 \in (0,R^n), \gamma \in (0,1)\) and \(K = K(R,m,\chi,\mu,\kappa,\gamma,s_0) > 0\).

We next define maximal moment solutions, which are ensured by Zorn’s lemma as in the proof of [6, Lemma 2.4].

**Definition 1.2.** Define the set \(S\) as
\[
S := \{(T,u,v) \mid T \in (0,\infty), (u,v) \text{ is a moment solution of } (1.1) \text{ on } [0,T]\}
\]
which is not empty as shown in the proof of Theorem 1.3, with the order relation \(\preceq\) given by
\[
(T_1,u_1,v_1) \preceq (T_2,u_2,v_2) \iff T_1 \leq T_2, u_2\mid_{(0,T_1)} = u_1, v_2\mid_{(0,T_1)} = v_1.
\]
Then Zorn’s lemma assures some maximal element \((T_{\text{max}},u,v) \in S\), and \((u,v)\) is called a maximal moment solution of (1.1) on \([0,T_{\text{max}}]\).

Now we state the main theorem, in which (1.2) is the same condition in [1, Theorem 1.2 (ii)].

**Theorem 1.3.** Let \(m \in [1,2 - \frac{2}{n},\chi > 0, \lambda > 0, \mu > 0 \text{ and } \kappa > 1\). Assume that
\[
\kappa < 1 + \min \left\{ \frac{(m-1)n+1}{2(n-1)}, \frac{n-2-(m-1)n}{n(n-1)} \right\}.
\]
Then for all \(M_0 > 0\) and \(L > 0\) there exist \(\sigma_0 > 0, \eta_0 \in (0,M_0)\) and \(r_\ast \in (0,R)\) with the following property: If
\[
u_0 \in L^\infty(\Omega) \text{ is nonnegative and radially symmetric and}
\]
\[
\int_\Omega \nu_0(x)\,dx = M_0 \quad \text{and} \quad \int_{B_{r_\ast}(0)} \nu_0(x)\,dx \geq M_0 - \eta_0
\]
as well as
\[
\nu_0(x) \leq L|x|^{-p} \quad \text{for a.a. } x \in \Omega,
\]
where \(p := \frac{n(n-1)}{(m-1)n+1} + \sigma_0\), then there exists a moment solution of (1.1) on \([0,T_{\text{max}}]\) which blows up at \(T_{\text{max}} < \infty\) in the sense that
\[
\limsup_{t \searrow T_{\text{max}}} \|\nu(\cdot,t)\|_{L^\infty(\Omega)} = \infty.
\]

In order to prove Theorem 1.3, we will construct a moment solution. To this end, we derive a moment inequality for a solution of a problem approximate to (1.1). The key to obtaining the inequality is to establish a pointwise estimate for an approximate solution (Lemma 2.1).
2. Proof of Theorem 1.3

To show finite-time blow-up of solutions to (1.1), for the present we focus on the following approximate problem:

\[
\begin{aligned}
\frac{\partial u_\varepsilon}{\partial t} &= \Delta (u_\varepsilon + \varepsilon)^m - \chi \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + \lambda u_\varepsilon - \mu u_\varepsilon^\kappa, & x \in \Omega, \ t > 0, \\
0 &= \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega, \ t > 0, \\
\frac{\partial u_\varepsilon}{\partial \nu} &= 0, & x \in \partial \Omega, \ t > 0, \\
\varepsilon(x, 0) &= u_{0\varepsilon}(x), & x \in \Omega,
\end{aligned}
\]

(2.1)

where \( \varepsilon \in (0, 1) \), and \( u_{0\varepsilon} := (\rho_\varepsilon \ast \overline{u_0})|_{\Omega} \) with

\[
\overline{u_0}(x) := \begin{cases} u_0(x) & \text{if } x \in \Omega, \\ 0 & \text{otherwise,} \end{cases}
\]

\[
\rho_\varepsilon(x) := \frac{1}{\varepsilon^n} \left( \int_{\mathbb{R}^n} \rho(y) \, dy \right)^{-1} \rho \left( \frac{x}{\varepsilon} \right), \quad \rho(x) := \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}
\]

We note that the solution \((u_\varepsilon, v_\varepsilon)\) of (2.1) on \([0, T_\varepsilon)\) is obtained by a standard fixed point argument (see e.g. [11]), where \( T_\varepsilon \) is the maximal existence time for the solution \((u_\varepsilon, v_\varepsilon)\). We know that \( \rho_\varepsilon \) is nonnegative and radially symmetric. Thus, for the initial data \( u_0 \) satisfying (1.3), \( u_{0\varepsilon} \) is nonnegative and radially symmetric. Moreover, we see that \( u_{0\varepsilon} \rightarrow u_0 \) in \( L^1(\Omega) \) as \( \varepsilon \searrow 0 \) and that on passing to a subsequence if necessary, \( u_{0\varepsilon} \rightarrow u_0 \) a.a. \( x \in \Omega \) as \( \varepsilon \searrow 0 \). Furthermore, as in [5, Section 2.2] and [8, Lemmas 2.2 and 2.3], we can find \( T_0 > 0 \) and \( K_0 > 0 \) such that for all \( \varepsilon \in (0, 1) \),

\[
T_0 \leq T_\varepsilon \quad \text{and} \quad \sup_{t \in (0, T_0)} \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq K_0.
\]

(2.2)

In order to establish a moment inequality, an estimate for \( u_\varepsilon \) is a cornerstone. In a degenerate Jäger–Luckhaus system with logistic source the key is radial monotonicity of an approximate solution (see [8, Lemma 2.7]). However, in our case it is difficult to obtain this property due to the structure of the second equation in (2.1). For this reason, instead of monotonicity, based on [10, Lemma 3.3] and [1, Lemma 5.2], we show a pointwise estimate for \( u_\varepsilon \).

Lemma 2.1. Let \( m \in \left[1, 2 - \frac{2}{n}\right) \), \( \chi > 0 \), \( \lambda > 0 \), \( \mu > 0 \), \( \kappa > 1 \), \( M_0 > 0 \) and \( L > 0 \). Moreover, for any \( \sigma_0 > 0 \), set \( p := \frac{n(n-1)}{(m-1)m+1} + \sigma_0 \) and assume that \( u_0 \) satisfies (1.3), (1.5) and \( \int_{\Omega} u_0(x) \, dx = M_0 \) and that there exist \( T_0 > 0 \) and \( K_0 > 0 \) fulfilling (2.2). Then there exist \( \varepsilon_0 \in (0, 1) \) and \( L_1 > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_0) \),

\[
(2.3) \quad u_\varepsilon(x, t) \leq L_1 |x|^{-p}
\]

for all \( x \in \Omega \) and \( t \in (0, T_0) \).
**Proof.** Putting \( \tilde{u}_\varepsilon(x,t) := e^{-\lambda t}u_\varepsilon(x,t) \), we can derive from \((2.1)\) that
\[
\begin{cases}
\frac{\partial \tilde{u}_\varepsilon}{\partial t} \leq \nabla \cdot (m(e^{\lambda t} \tilde{u}_\varepsilon + \varepsilon)^{m-1}\nabla \tilde{u}_\varepsilon - \chi \tilde{u}_\varepsilon \nabla v_\varepsilon), & x \in \Omega, \ t > 0, \\
(m(e^{\lambda t} \tilde{u}_\varepsilon + \varepsilon)^{m-1}\nabla \tilde{u}_\varepsilon - \chi \tilde{u}_\varepsilon \nabla v_\varepsilon) \cdot \nu = 0, & x \in \partial \Omega, \ t > 0, \\
\tilde{u}_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega.
\end{cases}
\] (2.4)

Next, let \( \sigma_0 > 0 \). We can take \( \xi > 0 \) small enough and \( \varepsilon_0 \in (0,1) \) such that \( u_{0\varepsilon} \leq u_0 + \xi \) for a.a. \( x \in \Omega \) and all \( \varepsilon \in (0,\varepsilon_0) \). By virtue of this inequality, \((1.5)\) and the fact that \( \lambda \leq R \), it follows that
\[
u_{0\varepsilon} \leq L|x|^{-p} + \xi R^p |x|^{-p} = (L + \xi R^p)|x|^{-p}
\] for all \( x \in \Omega \) and \( \varepsilon \in (0,\varepsilon_0) \). Also, from the condition \( \int_\Omega u_0 \, dx = M_0 \), we obtain that
\[
\int_\Omega u_{0\varepsilon} \, dx \leq M_0 + \xi |\Omega| =: \tilde{M}_0
\] for all \( \varepsilon \in (0,\varepsilon_0) \). On the other hand, integrating the first equation in \((2.1)\) over \( \Omega \), we infer that
\[
\int_\Omega \frac{d}{dt} u_\varepsilon \, dx = \lambda \int_\Omega u_\varepsilon \, dx - \mu \int_\Omega u_\varepsilon^\mu \, dx \leq \lambda \int_\Omega u_\varepsilon \, dx,
\]
which ensures that
\[
\int_\Omega u_\varepsilon \, dx \leq e^{\lambda t} \int_\Omega u_{0\varepsilon} \, dx \leq e^{\lambda T_0} \tilde{M}_0
\] for all \( t \in (0,T_0) \). Moreover, we see from the second equation in \((2.1)\) that
\[
r^{n-1}(v_\varepsilon)_r = \int_0^r \rho^{n-1}v_\varepsilon \, d\rho - \int_0^r \rho^{n-1}u_\varepsilon \, d\rho \leq \frac{1}{\omega_n} \left( \int_\Omega v_\varepsilon \, dx + \int_\Omega u_\varepsilon \, dx \right)
\]
for all \( r \in (0,R) \) and \( t \in (0,T_0) \), where \( \omega_n := n|B_1(0)| \). Here, since we integrate the second equation in \((2.1)\) over \( \Omega \) to guarantee that
\[
\int_\Omega u_\varepsilon \, dx = \int_\Omega v_\varepsilon \, dx,
\]
the above inequality and \((2.7)\) yields
\[
r^{n-1}(v_\varepsilon)_r \leq \frac{2}{\omega_n} e^{\lambda T_0} \tilde{M}_0 =: c_1
\]
for all \( r \in (0,R) \) and \( t \in (0,T_0) \). Picking \( \theta_0 > n \) so large satisfying \( m - 1 > \frac{1}{\theta_0} - \frac{1}{n} \) and \( p = \frac{n(n-1)}{(m-1)n+1} + \sigma_0 > \frac{(n-1)}{(m-1)\theta_0 - \frac{1}{n}} \), we have
\[
\int_\Omega |x|^{\theta_0(n-1)}|\nabla v_\varepsilon(x,t)|^{\theta_0} \, dx = \omega_n \int_0^R r^{(\theta_0+1)(n-1)}(v_\varepsilon)_r(t) \, dx \, d\rho
\]
\[
\leq \frac{1}{n} \omega_n c_1 R^n
\]
for all \( t \in (0,T_0) \). From this inequality and \((2.4)-(2.6)\) we therefore can apply [2, Theorem 1.1] to obtain \((2.3)\). □
We next derive a moment inequality for an approximate solution of (2.1).

Lemma 2.2. Let $m \in \left[1, 2 - \frac{2}{n}\right]$, $\chi > 0$, $\lambda > 0$, $\mu > 0$ and $\kappa > 1$. Assume that (1.2) is satisfied and that there exist $T_0 > 0$ and $K_0 > 0$ fulfilling (2.2). Then for all $M_0 > 0$ and $L > 0$ there exist $\eta_0 \in (0, M_0)$ and $r_0 \in (0, R)$ which satisfy the following property: If $u_0$ satisfies (1.3)–(1.5) with some $\sigma_0 > 0$, then there exist $\epsilon_0 \in (0, 1)$ and $K > 0$ such that for any $\epsilon \in (0, \epsilon_0)$,

$$
(2.8) \quad \phi_\epsilon(t) - \phi_\epsilon(0) \geq K \int_0^t \phi_\epsilon^2(\tau) \, d\tau
$$

for all $t \in (0, T_0)$, where

$$
\phi_\epsilon(t) := \int_0^{s_0} s^{-\gamma}(s_0 - s)w_\epsilon(s, t) \, ds \quad \text{for } t \in (0, T_\epsilon),
$$

$$
w_\epsilon(s, t) := \int_0^{s_0^\frac{1}{m}} \rho^{n-1} u_\epsilon(\rho, t) \, d\rho \quad \text{for } s \in [0, R^n] \text{ and } t \in (0, T_\epsilon)
$$

with some $s_0 \in (0, R^n)$ and $\gamma \in (0, 1)$.

Proof. Let us first put $p := \frac{n(n-1)}{(m-1)n+1} + \sigma_0$, where we choose $\sigma_0 > 0$ sufficiently small fulfilling that $\kappa < 1 + \min \left\{ \frac{n}{2p}, \frac{n-2}{n} - (m-1) \right\}$. Furthermore, we select $\gamma \in \left( \max \left\{ \frac{2p}{n}, 1 - \frac{2}{n} - \frac{p}{n}(m-1) \right\}, \min \left\{ 2 - \frac{4}{n}, -\frac{2p}{n}(m-1), 1 \right\} \right)$. Also, noting that $u_{0, \epsilon} \to u_0$ in $L^1(\Omega)$ as $\epsilon \to 0$, we fix $\xi_0 > 0$ small enough and pick $\epsilon_0 \in (0, 1)$ given by Lemma 2.1 satisfying

$$
\int_{\Omega} u_{0, \epsilon} \geq M_0 - \xi_0
$$

for all $\epsilon \in (0, \epsilon_0)$. In order to obtain (2.8), we shall show that there exist $c_1 > 0$, $c_2 > 0$, $\theta \in (0, 2)$ and $s_1 \in (0, R^n)$ such that for any $\epsilon \in (0, \epsilon_0)$ and $s_0 \in (0, s_1),$

$$
(2.9) \quad \phi_\epsilon'(t) \geq c_1 s_0^\gamma - 3 \phi_\epsilon^2(t) - c_2 s_0^{3\gamma - \gamma - \theta}
$$

for all $t \in (0, T_\epsilon)$. By straightforward computations we have from (2.1) and the definitions of $w_\epsilon$ and $\phi_\epsilon$ that

$$
\phi_\epsilon'(t) \geq mn^2 \int_0^{s_0} s^{2-\frac{2}{n}-\gamma}(s_0 - s) \left( n(w_\epsilon)_s + \epsilon \right)^{m-1} (w_\epsilon)_s \, ds
$$

$$
+ \frac{n}{\mu} \int_0^{s_0} s^{-\gamma}(s_0 - s)(w_\epsilon)_s w_\epsilon \, ds - n \int_0^{s_0} s^{-\gamma}(s_0 - s)(w_\epsilon)_s z_\epsilon \, ds
$$

$$
- n^{n-1} \mu \int_0^{s_0} s^{-\gamma}(s_0 - s) \left\{ \int_0^{s} (w_\epsilon)_s \, d\sigma \right\} \, ds
$$

for all $t \in (0, T_\epsilon)$, where $z_\epsilon(s,t) := \int_0^{s_0^\frac{1}{m}} \rho^{n-1} w_\epsilon(\rho, t) \, d\rho$ for $s \in [0, R^n]$ and $t \in (0, T_\epsilon)$. Here, we note that we can apply Lemmas 3.5, 3.8 and 3.9 to the second, third and fourth terms on the right-hand side of the above inequality. Thus, in order to derive (2.9), it is sufficient to estimate the first term. To this end, we will find $c_3 > 0$ independent of $\epsilon$ such that

$$
(2.10) \quad (n(w_\epsilon)_s + \epsilon)^m \leq c_3 s_0^{-\frac{\gamma}{m}} (w_\epsilon)_s + c_3
$$
for all \( s \in (0, R^n) \) and \( t \in (0, T_0) \), which is used after integration by parts in estimating the first term. By means of (2.3), it follows that for any \( \varepsilon \in (0, \varepsilon_0), \)
\[
w_{\varepsilon}(s, t) = \frac{1}{n} u_{\varepsilon}(s^{\frac{m}{2}}, t) \leq c_4 s^{-\frac{m}{2}}
\]
for all \( s \in (0, R^n) \) and \( t \in (0, T_0) \), where \( c_4 := \frac{B_1}{n} \). From this inequality and the fact that \( s < R^n \) as well as \( \varepsilon < 1 \), we have
\[
(n(w_{\varepsilon})_s + \varepsilon)^m \leq 2^{m-1}(n^{m}(w_{\varepsilon})_s^{m} + \varepsilon^m)
\]
\[
\leq 2^{m-1}n^m c_4^{m-1} s^{-\frac{1}{2}(m-1)}(w_{\varepsilon})_s + 2^{m-1}
\]
for all \( s \in (0, R^n) \) and \( t \in (0, T_0) \), which means that (2.10) holds. Therefore, by Lemmas 3.5, 3.6 (i), 3.8, 3.9 and 3.11] we can take \( c_5 > 0, c_6 > 0, \theta \in (0, 2) \) and \( s_1 \in (0, R^n) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) and \( s_0 \in (0, s_1), \)
\[
\phi^\epsilon(t) \geq c_5 s_0^\gamma - 3 \phi^\epsilon(t) - c_6 s_0^{3-\gamma - \theta}
\]
for all \( t \in (0, T_0) \). Furthermore, arguing as in [8 Proof of Proposition 2], we pick \( \eta_0 \in (0, M_0) \) and \( r_* \in (0, R) \) such that for any \( u_0 \) satisfying (1.3)–(1.5), the inequality \( \phi^\epsilon(t) \geq c_5^\frac{1}{2} s_0^\gamma - 3 \phi^\epsilon(t) \) holds for all \( \varepsilon \in (0, \varepsilon_0), s_0 \in (0, s_1) \) and \( t \in (0, T_0) \), which implies (2.8).

We are now in the position to show Theorem 1.3.

**Proof of Theorem 1.3.** We can derive results similar to [8 Lemmas 2.4 and 2.5] since the second equation in (2.1) entails that \( \Delta v_\varepsilon = v_\varepsilon - u_\varepsilon \geq -u_\varepsilon \). Thus, as in the proof of [4 Lemma 5.3] we can choose subsequence \( \{u_{\varepsilon,k}\}, \{v_{\varepsilon,k}\} (\varepsilon_k \rightarrow 0 \)
as \( k \rightarrow \infty \)) and nonnegative functions \( u, v \) such that \( u \in L^\infty(0, T_0; L^\infty(\Omega)), v^m \in L^2(0, T_0; H^1(\Omega)), \)
v \( \in L^\infty(0, T_0; W^{1,\infty}(\Omega)) \) and
\[
(2.11) \quad u_{\varepsilon,k} \rightarrow u \text{ weakly}^* \text{ in } L^\infty(0, T_0; L^\infty(\Omega)),
\]
\[
(2.12) \quad u_{\varepsilon,k} \rightarrow u \text{ in } C^0([\delta, T_0]; L^q(\Omega)) \text{ for all } \delta \in (0, T_0) \text{ and } q \in [1, \infty),
\]
\[
(2.13) \quad \nabla(u_{\varepsilon,k} + \varepsilon)^m \rightarrow \nabla u^m \text{ weakly in } L^2(0, T_0; L^2(\Omega)),
\]
\[
(2.14) \quad \nabla v_{\varepsilon,k} \rightarrow \nabla v \text{ weakly}^* \text{ in } L^\infty(0, T_0; L^\infty(\Omega))
\]
as \( k \rightarrow \infty \). Moreover, thanks to Lemma 2.2, we can take the initial data \( u_0 \) leading to (2.8). Thus, by (2.11), (2.14), we can show that \( (u, v) \) fulfills (i)–(iii) with \( T = T_0 \) in Definition 1.1 as in [8 Proof of Proposition 1]. Hence, from Definition 1.2 there exists a maximal moment solution \( (u, v) \) on \( (0, T_{\text{max}}) \). In particular, we have
\[
\phi(t) - \phi(0) \geq K \int_0^t \phi^2(\tau) \, d\tau
\]
for all \( t \in (0, T_{\text{max}}) \) with some \( K > 0 \). Putting \( \Phi(t) := \int_0^t \phi^2(\tau) \, d\tau + \phi(0) / K \) for \( t \in (0, T_{\text{max}}) \), we see that \( \Phi \in C^0([0, T_{\text{max}}) \cap C^1((0, T_{\text{max}})) \) and from the above inequality that \( \Phi'(t) \geq K^2 \Phi^2(t) \) for all \( t \in (0, T_{\text{max}}) \), which yields
\[
t \leq \frac{1}{K^2} \left( -\frac{1}{\Phi(t)} + \frac{1}{\Phi(0)} \right) \leq \frac{1}{K^2 \Phi(0)}
\]
for all \( t \in (0, T_{\text{max}}) \). This proves \( T_{\text{max}} \leq \frac{1}{K^2 \Phi(0)} < \infty \). By an extension argument as in [8 Proof of Theorem 1.1] we can obtain \( \limsup_{t \nearrow T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \), which concludes the proof.

\( \square \)
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