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STABILIZATION IN DEGENERATE PARABOLIC EQUATIONS IN DIVERGENCE FORM AND APPLICATION TO CHEMOTAXIS SYSTEMS

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Abstract. This paper presents a stabilization result for weak solutions of degenerate parabolic equations in divergence form. More precisely, the result asserts that the global-in-time weak solution converges to the average of the initial data in some topology as time goes to infinity. It is also shown that the result can be applied to a degenerate parabolic-elliptic Keller-Segel system.

1. Introduction: stabilization result

Let $\Omega \subset \mathbb{R}^N$ ($N \in \mathbb{N}$) be a bounded domain with smooth boundary $\partial\Omega$. Then we consider the initial-boundary value problem for the degenerate parabolic equation,

$$\begin{cases} u_t = \nabla \cdot (f(u)\nabla u + g(u,x,t))\,, & x \in \Omega,\ t > 0\,,\\ (f(u)\nabla u + g(u,x,t)) \cdot \nu = 0\,, & x \in \partial\Omega,\ t > 0\,,\\ u(x,0) = u_0(x)\,, & x \in \Omega\,, \end{cases} \tag{1.1}$$

where f is supposed to be a non-negative function satisfying

$$f \in C([0,\infty)) \cap C^2((0,\infty))\,, \tag{1.2}$$

$$f(\sigma) \ge k_0\sigma^{m-1} \text{ with some } k_0 > 0,\ m \ge 1\ (\forall \sigma \ge 0)\,,\ \limsup_{\sigma \searrow 0} \sigma f'(\sigma) < \infty\,, \tag{1.3}$$

and moreover, g is assumed to be a vector-valued function approximated by $g_\varepsilon \in C([0,\infty) \times \overline{\Omega} \times [0,\infty); \mathbb{R}^N) \cap C^{1,1,0}([0,\infty) \times \Omega \times (0,\infty); \mathbb{R}^N)$ with some

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$\{\varepsilon\} \subset (0,1)$ fulfilling $\varepsilon \to 0$ such that for all $T > 0$,

$$\begin{aligned} (1.4) \qquad & 0 \le w_\varepsilon,\ w \in L^\infty(0,T;L^\infty(\Omega)), \\ & w_\varepsilon \to w \text{ a.e. on } \Omega \times (0,T) \text{ and weakly* in } L^\infty(0,T;L^\infty(\Omega)) \\ & \quad \Rightarrow\ g_\varepsilon(w_\varepsilon,\cdot,\cdot) \to g(w,\cdot,\cdot) \text{ weakly in } L^2(0,T;(L^2(\Omega))^N), \\ (1.5) \qquad & 0 \le w \in L^\infty(0,\infty;L^\infty(\Omega)) \text{ with } \|w\|_{L^\infty(0,\infty;L^\infty(\Omega))} \le c \\ & \quad \Rightarrow\ \|g_\varepsilon(w,\cdot,\cdot)\|_{L^2(0,\infty;(L^2(\Omega))^N)} \le M(c), \end{aligned}$$

where $M(c) \ge 0$ is a constant depending on c.

We first state the definition of weak solutions to (1.1) as follows:

Definition 1.1. A non-negative function $u(x,t)$ defined in $\Omega \times (0,\infty)$ is called *a global weak solution* of (1.1) if the following conditions are satisfied for all $T > 0$:

- $u \in L^\infty(0,T;L^\infty(\Omega))$,
- $\displaystyle\int_0^u f(\sigma)\,d\sigma \in L^2(0,T;H^1(\Omega))$, $g(u,x,t) \in L^2(0,T;(L^2(\Omega))^N)$,
- u fulfills (1.1) in the distributional sense: for every $\varphi \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))$ with $\operatorname{supp}\varphi(x,\cdot) \subset [0,T)$ (a.a. $x \in \Omega$),

$$\begin{aligned} &\int_0^T \int_\Omega \left(\nabla \left(\int_0^u f(\sigma)\,d\sigma \right) \cdot \nabla\varphi - g(u,x,t)\cdot\nabla\varphi - u\varphi_t \right) dx\,dt \\ &\quad = \int_\Omega u_0(x)\varphi(x,0)\,dx. \end{aligned}$$

We next give the following approximate problem:

$$(1.6) \qquad \begin{cases} (u_\varepsilon)_t = \nabla\cdot\left(f(u_\varepsilon+\varepsilon)\nabla u_\varepsilon + g_\varepsilon(u_\varepsilon,x,t)\right), & x \in \Omega,\ t > 0, \\ \big(f(u_\varepsilon+\varepsilon)\nabla u_\varepsilon + g_\varepsilon(u_\varepsilon,x,t)\big)\cdot\nu = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases}$$

where $g_\varepsilon \in C([0,\infty)\times\overline{\Omega}\times[0,\infty);\mathbb{R}^N) \cap C^{1,1,0}([0,\infty)\times\Omega\times(0,\infty);\mathbb{R}^N)$ with some $\{\varepsilon\} \subset (0,1)$ fulfilling $\varepsilon \to 0$ is an approximation of g, which also appears in (1.4), (1.5). The initial data $u_{0\varepsilon}$ is the regularization of u_0 such that $u_{0\varepsilon} \in C_0^\infty(\Omega)$ and $u_{0\varepsilon} \to u_0$ in $L^p(\Omega)$ as $\varepsilon \to 0$ for any $p \in [1,\infty)$. For example, we define it as $u_{0\varepsilon} := [\zeta_\varepsilon(\rho_\varepsilon * \widetilde{u_0})]|_\Omega$, where $\widetilde{u_0}$ denotes the zero extension of u_0 on $\mathbb{R}^N$. The function ρ_ε is the mollifier such that $0 \le \rho_\varepsilon \in C_0^\infty(\mathbb{R}^N)$, $\operatorname{supp}\rho_\varepsilon \subset \overline{B(0,\varepsilon)}$, $\int_{\mathbb{R}^N}\rho_\varepsilon(x)\,dx = 1$, and ζ_ε is the cut-off function defined as $\zeta_\varepsilon(x) := \zeta(\varepsilon x)$, where ζ is a fixed function in $C_0^\infty(\mathbb{R}^N)$ such that $0 \le \zeta \le 1$, $\zeta(x) = 1$ ($|x| \le 1$), $\zeta(x) = 0$ ($|x| \ge 2$). We assume that (1.6) possesses global classical solutions $u_\varepsilon \in C^0(\overline{\Omega}\times[0,\infty)) \cap C^{2,1}(\overline{\Omega}\times(0,\infty))$.

We now present a stabilization result established in [7].

Theorem 1.2. *Let f, g satisfy* (1.2), (1.3), (1.4), (1.5) *and $u_0 \in L^\infty(\Omega)$, $u_0 \ge 0$. Let u_ε be a global classical solution of* (1.6)*. Suppose that there exists a constant $u_{\max} > 0$, which is independent of ε and t, such that*

$$\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le u_{\max} \quad \textit{for all } t > 0.$$

Then there exists a global weak solution to (1.1)*, which is given by*

$$u_\varepsilon \to u \ \textit{weakly* in } L^\infty(0,\infty;L^\infty(\Omega)) \ \textit{as } \varepsilon \to 0$$

for some subsequence of $\{\varepsilon\}$*, satisfying*

$$\begin{aligned}
&u \in C_{w-*}([0,\infty);L^\infty(\Omega)),\\
&\|u(t)\|_{L^\infty(\Omega)} \le u_{\max} \quad \textit{for all } t \ge 0,\\
&u(t) \to \overline{u_0} \quad \textit{weakly* in } L^\infty(\Omega) \quad \textit{as } t \to \infty,
\end{aligned}$$

where $\overline{u_0} := \int_\Omega u_0(x)\,dx$*.*

The above theorem is applicable to some degenerate parabolic equations with drift terms in divergence form, whereas a similar result on stabilization in the case of non-divergence form with reaction terms has already been developed by [9]. In [7] we applied Theorem 1.2 to a parabolic–*parabolic* Keller-Segel system with degenerate diffusion. In this paper we give another application.

2. Application to chemotaxis systems

Consider the following degenerate parabolic–*elliptic* Keller-Segel system:

$$\tag{2.1}\begin{cases}
u_t = \nabla\cdot(D(u)\nabla u - u\nabla v), & x\in\Omega,\ t>0,\\
0 = \Delta v - v + u, & x\in\Omega,\ t>0,\\
\big(D(u)\nabla u + S(u)\nabla v\big)\cdot\nu = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\
u(x,0) = u_0(x), & x\in\Omega,
\end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N\in\mathbb{N})$ is a bounded domain with smooth boundary $\partial\Omega$. Assume that the diffusivity function D fulfills the following conditions:

(2.2) $\quad D \in C([0,\infty)) \cap C^2((0,\infty))$,

(2.3) $\quad D(\sigma) \ge k_0\sigma^{m-1}$ $(\sigma \ge 0)$ with some $k_0 > 0,\ m \ge 1,\ \limsup_{\sigma\searrow 0} \sigma D'(\sigma) < \infty$

and that the initial data (u_0, v_0) satisfies

$$\tag{2.4} u_0 \ge 0, \quad u_0 \in L^\infty(\Omega).$$

We define weak solutions of (2.1).

Definition 2.1. A couple (u,v) of non-negative functions satisfying the following is called *a global weak solution* of (2.1):

- $u \in L^\infty(0,T;L^\infty(\Omega))$, $\int_0^u D(\sigma)\,d\sigma \in L^2(0,T;H^1(\Omega))$ for all $T>0$,
- $v \in L^\infty(0,T;W^{1,\infty}(\Omega))$ for all $T>0$,

- (u,v) fulfills (2.1) in the distributional sense: for all $\varphi \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^1(\Omega))$ with $\operatorname{supp}\varphi(x,\cdot) \subset [0,T)$ (a.a. $x \in \Omega$),

$$\begin{aligned}\int_0^T \int_\Omega \Big(\nabla\Big(\int_0^u D(\sigma)\,d\sigma \Big) \cdot \nabla\varphi - u\nabla v \cdot \nabla\varphi - u\varphi_t \Big)\,dx\,dt \\ = \int_\Omega u_0(x)\varphi(x,0)\,dx\,,\end{aligned}$$

$$\int_0^T \int_\Omega (\nabla v \cdot \nabla\varphi + v\varphi - u\varphi)\,dx\,dt = 0\,.$$

In this section we deal with the sub-critical case that $2 - \frac{2}{N} < m$, where $m = 2 - \frac{2}{N}$ is the critical exponent whether (2.1) possesses a global bounded solution or not. In view of the results in [10] which dealt with a general quasilinear chemotaxis term with $N \geq 3$, solutions are global and bounded if $2 - \frac{2}{N} < m$, whereas there are many initial data producing unbounded solutions if $m < 2 - \frac{2}{N}$. A similar situation is found in the parabolic-parabolic system: for boundedness in the case $2 - \frac{2}{N} < m$, see [5,12,14] on bounded domains, [6,13] on the whole space; for blow-up in the case $m \leq 2 - \frac{2}{N}$, see [2,4,11] and [16].

We would like to turn to the asymptotic behavior of global solutions. To the best of our knowledge, there are few papers on this topic, e.g., the sub-critical parabolic–*parabolic* case is studied in [1,3,8] and [15]. For instance, the solution (u,v) of non-degenerate systems converges to $(\overline{u_0},\overline{u_0})$ in $(L^\infty(\Omega))^2$, where $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx$, under some smallness condition for initial data ([1,3,15]), whereas, when $m \geq 2$, an energy solution (u,v) tends to a non-negative stationary solution (U,V) which is potentially non-constant or constant equilibria ([8]). From these results, solvability has already been achieved for $2-\frac{2}{N} < m$ and stabilization has not been achieved in the case that $2-\frac{2}{N} < m < 2$. In [7] we could establish stabilization in the fully parabolic version of (2.1) by applying Theorem 1.2. However, there seems to be still room for consideration in the parabolic–*elliptic* Keller–Segel system (2.1). In this section, we will extend the range of the application of Theorem 1.2.

In stating the main theorem, we use the constant in the Poincaré inequality through the embedding $W^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ for any $\alpha \geq \frac{2N}{N+2}$:

$$\|\psi - \overline{\psi}\|^2_{L^2(\Omega)} \leq k_{\mathcal{P}} \|\nabla\psi\|^2_{L^\alpha(\Omega)} \ (\forall \psi \in W^{1,\alpha}(\Omega)), \tag{2.5}$$

where $\overline{\psi} := \frac{1}{|\Omega|}\int_\Omega \psi$ and $k_{\mathcal{P}} = k_{\mathcal{P}}(\alpha,N,\Omega)$ is a positive constant.

Theorem 2.2. *Let D satisfy the conditions* (2.2), (2.3) *with*

$$2 - \frac{2}{N} < m \leq 2\,.$$

Let (u_0,v_0) satisfy (2.4) *and assume that*

$$\|u_0\|^{2-m}_{L^1(\Omega)} < \frac{k_0}{k_{\mathcal{P}}}\,, \tag{2.6}$$

where $k_{\mathcal{P}}$ is the same one as in (2.5) *with $\alpha = \frac{2}{3-m}$. Then, there exists a global weak solution (u, v) of* (2.1) *which satisfies*

$$
\begin{aligned}
&u \in C_{w-*}([0, \infty); L^\infty(\Omega)),\\
&\|u(t)\|_{L^\infty(\Omega)} \le u_{\max} \quad \textit{for all } t \ge 0,\\
&\|v(t)\|_{W^{1,\infty}(\Omega)} \le v_{\max} \quad \textit{for all } t \ge 0,\\
&u(t) \to \overline{u_0} \quad \textit{weakly* in } L^\infty(\Omega) \textit{ as } t \to \infty,\\
&v(t) \to \overline{u_0} \quad \textit{strongly in } L^\infty(\Omega) \textit{ as } t \to \infty,
\end{aligned}
\tag{2.7}
$$

where $u_{\max}, v_{\max} \ge 0$ are constants that appear in Lemma 2.3 and $\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0$.

As in Theorem 1.2, we consider the approximate problem

$$
\begin{cases}
(u_\varepsilon)_t = \nabla \cdot (D(u_\varepsilon + \varepsilon)\nabla u_\varepsilon) - \nabla \cdot (u_\varepsilon \nabla v_\varepsilon)\,, & x \in \Omega,\ t > 0\,,\\
0 = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon\,, & x \in \Omega,\ t > 0\,,\\
\frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0 & x \in \partial\Omega,\ t > 0\,,\\
u_\varepsilon(x, 0) = [\zeta_\varepsilon(\rho_\varepsilon * \widetilde{u_0})]|_\Omega\,, & x \in \Omega\,,
\end{cases}
\tag{2.8}
$$

where $\widetilde{u_0}$ denotes the zero extension of u_0 on $\mathbb{R}^N$, ρ_ε is the mollifier and ζ_ε is the cut-off function.

We first give existence of global bounded solutions to the approximate problem (2.8), which can be proved by the same way as in [6] for the fully parabolic case; note that in the parabolic-elliptic case it suffices to replace Δv with $v - u$ instead of the use of the maximal Sobolev regularity in [6, (28)].

Lemma 2.3. *Assume that D satisfy the conditions* (2.2), (2.3) *with $2 - \frac{2}{N} < m$. Then for any initial data satisfying* (2.4), *there exists a pair $(u_\varepsilon, v_\varepsilon)$ of non-negative functions*

$$
u_\varepsilon, v_\varepsilon \in C^0(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))\,,
\tag{2.9}
$$

which solves (2.8) *classically, and $(u_\varepsilon, v_\varepsilon)$ fulfills*

$$
\|u_\varepsilon(t)\|_{L^\infty(\Omega)} \le u_{\max},\ \|v_\varepsilon(t)\|_{W^{1,\infty}(\Omega)} \le v_{\max} \quad \textit{for all } t \in (0, T)\,,
$$

where $u_{\max}, v_{\max}$ are positive constants which are independent of t, ε. Moreover, there exist a subsequence $\{\varepsilon_n\}_n \subset \{\varepsilon\}$ and non-negative functions

$$
u \in L^\infty(0, \infty; L^\infty(\Omega)),\ v \in L^\infty(0, \infty; W^{1,\infty}(\Omega))
$$

such that

$$
\begin{aligned}
&u_{\varepsilon_n} \to u \quad \textit{weakly* in } L^\infty(0, \infty; L^\infty(\Omega)),\\
&u_{\varepsilon_n} \to u \quad \textit{a.e. on } \Omega \times (0, \infty),\\
&v_{\varepsilon_n} \to v \quad \textit{weakly* in } L^\infty(0, \infty; W^{1,\infty}(\Omega))
\end{aligned}
\tag{2.10}
$$

as $n \to \infty$.

In order to apply Theorem 1.2, we will verify the conditions (1.2)–(1.5) with

$$f(\sigma) = D(\sigma)\,, \quad g(w,x,t) = w\nabla v\,, \quad g_\varepsilon(w,x,t) = w\nabla v_\varepsilon\,,$$

where $\varepsilon := \varepsilon_n$ for large n. In the following proof, c_i $(i = 1, 2, \cdots)$ denote positive constants independent of t and ε.

Proof of Theorem 2.2. We first observe that (1.2) and (1.3) are satisfied by (2.2) and (2.3). In view of (2.9) we can define $g_\varepsilon \in C([0,\infty) \times \overline{\Omega} \times [0,\infty); \mathbb{R}^N) \cap C^{1,1,0}([0,\infty) \times \Omega \times (0,\infty); \mathbb{R}^N)$ as

$$g_\varepsilon(w,x,t) := w\nabla v_\varepsilon\,,$$

where $\{\varepsilon\} \subset (0,1)$ fulfilling $\varepsilon \to 0$ is defined as $\varepsilon := \varepsilon_n$ appearing in Lemma 2.3 for large n. From now on we omit the proviso that $\varepsilon \to 0$.

Next, we will confirm (1.4). Let w_ε, w be non-negative functions which belong to $L^\infty(0,T; L^\infty(\Omega))$ for all $T > 0$ and satisfy

$$w_\varepsilon \to w \text{ a.e. on } \Omega \times (0,T) \text{ and weakly* in } L^\infty(0,T; L^\infty(\Omega))\,.$$

Since there exists c_1 such that $\|w_\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega))} \le c_1$, we see from the Lebesgue dominated convergence theorem that $w_\varepsilon \to w$ strongly in $L^2(0,T; L^2(\Omega))$. Combining this convergence with (2.10) ensures that

$$g_\varepsilon(w_\varepsilon, \cdot, \cdot) \to w\nabla v = g(w,x,t) \quad \text{weakly in } L^2(0,T; (L^2(\Omega))^N)\,.$$

We next consider (1.5). Let $w \in L^\infty(0,\infty; L^\infty(\Omega))$ with $\|w\|_{L^\infty(0,\infty;L^\infty(\Omega))} \le c_2$. Then we have

$$\int_0^\infty \int_\Omega |g_\varepsilon(w)|^2\,dx\,dt \le c_2^2 \int_0^\infty \int_\Omega |\nabla v_\varepsilon|^2\,dx\,dt\,. \tag{2.11}$$

Set

$$z_\varepsilon(t) := v_\varepsilon(t) - \overline{v_\varepsilon(t)} = v_\varepsilon(t) - \frac{1}{|\Omega|}\int_\Omega v_\varepsilon(t).$$

Then, due to $\overline{v_\varepsilon(t)} = \overline{u_{0\varepsilon}}$, which is obtained by integrating the second equation in (2.8) over Ω, z_ε satisfies

$$\begin{cases} 0 = \Delta z_\varepsilon - z_\varepsilon + (u_\varepsilon - \overline{u_{0\varepsilon}})\,, & x \in \Omega,\ t > 0\,, \\ \nabla z_\varepsilon \cdot \nu = 0\,, & x \in \partial\Omega\,. \end{cases} \tag{2.12}$$

Testing the equation in (2.12) by $u_\varepsilon - \overline{u_{0\varepsilon}}$ and z_ε, we obtain

$$0 = -\int_\Omega \nabla u_\varepsilon \cdot \nabla v_\varepsilon\,dx - \int_\Omega (u_\varepsilon - \overline{u_{0\varepsilon}})z_\varepsilon\,dx + \int_\Omega (u_\varepsilon - \overline{u_{0\varepsilon}})^2\,dx,$$
$$0 = -\int_\Omega (|\nabla z_\varepsilon|^2 + |z_\varepsilon|^2)\,dx + \int_\Omega (u_\varepsilon - \overline{u_{0\varepsilon}})z_\varepsilon\,dx\,.$$

From the first equation in (2.8) we see that

$$\frac{d}{dt}\int_\Omega (u_\varepsilon \log u_\varepsilon - u_\varepsilon)\,dx = -\int_\Omega \frac{D(u_\varepsilon + \varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx + \int_\Omega \nabla v_\varepsilon \cdot \nabla u_\varepsilon\,dx\,.$$

Adding the above three identities, we have

$$\begin{aligned}(2.13)\quad &\frac{d}{dt}\int_\Omega (u_\varepsilon \log u_\varepsilon - u_\varepsilon)\,dx \\ &= -\int_\Omega (|\nabla z_\varepsilon|^2 + |z_\varepsilon|^2)\,dx - \int_\Omega \frac{D(u_\varepsilon+\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx + \int_\Omega (u_\varepsilon - \overline{u_{0\varepsilon}})^2\,dx\,.\end{aligned}$$

The condition (2.6) and $m \le 2$ help us to control the last term on the right-hand side of the above identity by the second term on the same side. The fact that $W^{1,\frac{2}{3-m}}(\Omega) \hookrightarrow L^2(\Omega)$ as $2 - \frac{2}{N} < m$ and (2.5) provide the constant $k_\mathcal{P}$ such that

$$\|u_\varepsilon(t) - \overline{u_{0\varepsilon}}\|^2_{L^2(\Omega)} \le k_\mathcal{P}\|\nabla u_\varepsilon(t)\|^2_{L^{\frac{2}{3-m}}(\Omega)}\,.$$

From Hölder's inequality along with $\|u_\varepsilon(t)\|_{L^1(\Omega)} \le \|u_0\|_{L^1(\Omega)}$ ($\forall t \ge 0$) we infer

$$\|\nabla u_\varepsilon(t)\|^2_{L^{\frac{2}{3-m}}(\Omega)} \le \left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{2-m}}\,dx\right)(\|u_0\|_{L^1(\Omega)} + \varepsilon|\Omega|)^{2-m}.$$

Thanks to (2.3), it clearly holds that

$$\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+\varepsilon)^{2-m}}\,dx \le \frac{1}{k_0}\int_\Omega \frac{D(u_\varepsilon+\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx\,.$$

Connecting the above three estimates, we obtain

$$(2.14)\quad \|u_\varepsilon(t) - \overline{u_{0\varepsilon}}\|^2_{L^2(\Omega)} \le \frac{k_\mathcal{P}}{k_0}(\|u_0\|_{L^1(\Omega)} + \varepsilon|\Omega|)^{2-m}\int_\Omega \frac{D(u_\varepsilon+\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx\,.$$

By virtue of (2.6), if we take ε_0 small enough to fit

$$\frac{k_\mathcal{P}}{k_0}(\|u_0\|_{L^1(\Omega)} + \varepsilon|\Omega|)^{2-m} - 1 < 0 \;(\varepsilon \in (0,\varepsilon_0))\,,$$

then (2.13) together with (2.14) warrants that for $\varepsilon \in (0,\varepsilon_0)$,

$$\frac{d}{dt}\int_\Omega (u_\varepsilon \log u_\varepsilon - u_\varepsilon)\,dx \le -\int_\Omega (|\nabla z_\varepsilon|^2 + |z_\varepsilon|^2)\,dx - c_3\int_\Omega \frac{D(u_\varepsilon+\varepsilon)}{u_\varepsilon}|\nabla u_\varepsilon|^2\,dx\,,$$

where $c_3 = 1 - \frac{k_\mathcal{P}}{k_0}(\|u_0\|_{L^1(\Omega)} + \varepsilon_0|\Omega|)^{2-m} > 0$. Integrating this inequality with respect to the time variable provides c_4 such that for $\varepsilon \in (0,\varepsilon_0)$,

$$\begin{aligned}\int_0^\infty\int_\Omega |\nabla v_\varepsilon|^2\,dx\,dt &= \int_0^\infty\int_\Omega |\nabla z_\varepsilon|^2\,dx\,dt \\ &\le -\int_\Omega (u_\varepsilon \log u_\varepsilon - u_\varepsilon)\,dx + \int_\Omega (u_{0\varepsilon}\log u_{0\varepsilon} - u_{0\varepsilon})\,dx \\ (2.15)\qquad &\le c_4\end{aligned}$$

in light of boundedness of $f(\xi) = |\xi\log\xi - \xi|$ for $\xi \in [0, u_{\max}]$. Plugging (2.15) into (2.11), we deduce that (1.5) holds. Thus, we can apply Theorem 1.2 to the parabolic-elliptic Keller-Segel system (2.1), so that there exists a global weak solution (u, v) fulfilling

$$\begin{aligned}
&u \in C_{w-*}([0,\infty); L^\infty(\Omega))\,,\\
&\|u(t)\|_{L^\infty(\Omega)} \le u_{\max} \quad \text{for all } t \ge 0\,,\\
&u(t) \to \overline{u_0} \quad \text{weakly* in } L^\infty(\Omega) \text{ as } t \to \infty\,.
\end{aligned}$$

Moreover, from the Sobolev embedding $W^{2,N+1}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and elliptic regularity as well as $\|u(t)\|_{L^\infty(\Omega)} \le u_{\max}$ $(t \ge 0)$ we have

$$\|v(t)\|_{W^{1,\infty}(\Omega)} \le c_5\|v(t)\|_{W^{2,N+1}(\Omega)} \le c_6\|u(t)\|_{L^{N+1}(\Omega)} \le c_6|\Omega|^{\frac{1}{N+1}}u_{\max} = v_{\max}$$

with some $c_5, c_6 > 0$. We finally verify (2.7). Since $u(t) \to \overline{u_0}$ weakly in $L^{N+1}(\Omega)$ as $t \to \infty$, the compactness of $(I-\Delta)^{-1}$ from $L^{N+1}(\Omega)$ in $W^{1,N+1}(\Omega)$ implies that

$$v(t) - \overline{u_0} = (I-\Delta)^{-1}(u(t) - \overline{u_0}) \to 0 \quad \text{strongly in } W^{1,N+1}(\Omega) \quad \text{as } t \to \infty\,,$$

and also strongly in $L^\infty(\Omega)$ by the Sobolev embedding theorem, which implies (2.7). This completes the proof. □

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