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PADOVAN AND PERRIN NUMBERS AS PRODUCTS OF TWO GENERALIZED LUCAS NUMBERS

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Abstract. Let P_m and E_m be the m-th Padovan and Perrin numbers respectively. Let r, s be non-zero integers with $r \geq 1$ and $s \in \{-1, 1\}$, let $\{U_n\}_{n\geq 0}$ be the generalized Lucas sequence given by $U_{n+2} = rU_{n+1} + sU_n$, with $U_0 = 0$ and $U_1 = 1$. In this paper, we give effective bounds for the solutions of the following Diophantine equations

$$P_m = U_n U_k \quad \text{and} \quad E_m = U_n U_k\,,$$

where m, n and k are non-negative integers. Then, we explicitly solve the above Diophantine equations for the Fibonacci, Pell and balancing sequences.

1. Introduction

Let (u_n) and (v_n) be two linear recurrent sequences. The problem of finding the common terms of (u_n) and (v_n) was treated in [6], [8], [10]. The authors proved, under some assumption, that the Diophantine equation

$$u_n = v_m$$

has only finitely many integer solutions (n, m). The aim of this paper is to study the common terms of Padovan, Perrin, and the product of two generalized Lucas sequences.

The Padovan sequence $\{P_n\}_{n\geq 0}$ is defined by $P_0 = P_1 = P_2 = 1$, and

$$P_{n+3} = P_{n+1} + P_n\,, \quad \text{for} \quad n \geq 0\,.$$

See the sequence A000931 in the OEIS. The first few terms of Padovan sequence are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, \ldots.$$

The Perrin sequence $\{E_n\}_{n\geq 0}$ is given by $E_0 = 3$, $E_1 = 0$, $E_2 = 2$, and

$$E_{n+3} = E_{n+1} + E_n\,, \quad \text{for} \quad n \geq 0\,.$$

Its first few terms are

$$3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, \ldots$$

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(sequence A001608 in the OEIS). We recall some facts and properties of the Padovan and the Perrin sequences which will be used later. The characteristic equation of the two sequences is

$$x^3 - x - 1 = 0$$

with roots α, β, $\lambda = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12}$$

by

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \quad \text{and} \quad r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Let

$$\begin{aligned} a &= \frac{(1-\beta)(1-\lambda)}{(\alpha-\beta)(\alpha-\lambda)} = \frac{1+\alpha}{-\alpha^2 + 3\alpha + 1} \\ b &= \frac{(1-\alpha)(1-\lambda)}{(\beta-\alpha)(\beta-\lambda)} = \frac{1+\beta}{-\beta^2 + 3\beta + 1} \\ c &= \frac{(1-\alpha)(1-\beta)}{(\lambda-\alpha)(\lambda-\beta)} = \frac{1+\lambda}{-\lambda^2 + 3\lambda + 1} = \overline{b}. \end{aligned}$$

The Binet's formulas for P_n and E_n respectively are

$$P_n = a\alpha^n + b\beta^n + c\lambda^n, \quad \text{for } n \geq 0, \tag{1.1}$$

and

$$E_n = \alpha^n + \beta^n + \lambda^n, \quad \text{for } n \geq 0. \tag{1.2}$$

The minimal polynomial of a over the integers is

$$23x^3 - 23x^2 + 6x - 1, \tag{1.3}$$

whose roots are a, b, c with $|a|$, $|b|$, $|c| < 1$ (see [3]). Numerically, the following estimates hold:

$$\begin{aligned} 1.32 < \alpha &< 1.33, \\ 0.86 < |\lambda| = |\beta| = \alpha^{-\frac{1}{2}} &< 0.87, \\ 0.72 < a &< 0.73, \\ 0.24 < |b| = |c| &< 0.25. \end{aligned}$$

It is easy to see that the contribution of the complex conjugate roots β and λ, in the right-hand side of (1.1), is very small. In particular, setting

$$e_1(m) := P_m - a\alpha^m = \lambda\beta^m + c\gamma^m, \quad \text{then} \quad |e_1(m)| < \frac{1}{\alpha^{m/2}}, \tag{1.4}$$

for $m \geq 1$. Furthermore, by induction, one can prove that

$$\alpha^{m-3} \leq P_m \leq \alpha^{m-1}, \quad \text{for } m \geq 1. \tag{1.5}$$

Also, it follows that the difference between the right hand side of equation (1.2) and α^m becomes quite small as m increases. More specifically, letting

$$e_2(m) := E_m - \alpha^m = \beta^m + \lambda^m, \quad \text{then} \quad |e_2(m)| < \frac{2}{\alpha^{m/2}}\,, \quad \text{for} \quad m \geq 1\,. \tag{1.6}$$

Similarly, one has

$$\alpha^{m-2} \leq E_m \leq \alpha^{m+1}\,, \quad \text{for} \quad m \geq 2\,. \tag{1.7}$$

Recall that the generalized Lucas sequence $\{U_n\}_{n\geq 0}$ and its companion sequence $\{V_n\}_{n\geq 0}$ are defined with initial values $U_0 = 0$, $U_1 = 1$, $V_0 = 2$, $V_1 = r$, by

$$U_{n+1} = rU_n + sU_{n-1} \quad \text{and} \quad V_{n+1} = rV_n + sV_{n-1}\,, \quad \text{for} \quad n \geq 0\,,$$

where r and s are integers such that $\Delta = r^2 + 4s > 0$. The Binet's formulas for these sequences are given by

$$U_n = \frac{\delta^n - \gamma^n}{\delta - \gamma} \quad \text{and} \quad V_n = \delta^n + \gamma^n\,, \tag{1.8}$$

where $\delta = \dfrac{r+\sqrt{\Delta}}{2}$ and $\gamma = \dfrac{r-\sqrt{\Delta}}{2}$. For more details on the generalized Lucas sequence see [9]. The aim in this paper is the study of the two Diophantine equations

$$P_m = U_nU_k\,, \tag{1.9}$$

and

$$E_m = U_nU_k\,, \tag{1.10}$$

where m, n and k are non-negative integers such that $n \leq k$. Note that if $n = 0$, then equation (1.10) has infinitely many solutions of the form $(m, n, k) = (1, 0, k)$. So, for the Diophantine equation (1.10) it remains to see what happen for $n \geq 1$. Here are our main results.

Theorem 1.1. *If (k, m, n) is a positive integer solution of equation* (1.9)*, then*

$$m < \frac{\log\delta}{\log\alpha}\,(n+k) + 3 \tag{1.11}$$

and

$$k\log\delta - \log(1+\sqrt{\Delta}) < 2.5\cdot 10^{13}(1+\log B)(\log\delta)\times\rho_1\,, \tag{1.12}$$

where ρ_1 is given by

$$\rho_1 = \log\left(23^2(\Delta^2+3\Delta)^3\right) + 1.5\cdot 10^{14}(1+\log B)\log\delta\log(23\cdot\Delta^3)$$

with

$$B = \frac{\log\delta}{\log\alpha}\,(n+k) + 3\,.$$

Theorem 1.2. *If (k, m, n) is a positive integer solution of equation* (1.10)*, then*

$$m < \frac{\log\delta}{\log\alpha}\,(n+k) + 2 \tag{1.13}$$

and

$$k\log\delta - \log(1+\sqrt{\Delta}) < 2.5\cdot 10^{13}(1+\log B)(\log\delta)\times\rho_2\,, \tag{1.14}$$

where ρ_2 *is given by*

$$\rho_2 = \log\left((\Delta^2+3\Delta)^3\right) + 4.5\cdot 10^{14}(1+\log B)\cdot\log\delta\cdot\log\Delta$$

with

$$B = \frac{\log\delta}{\log\alpha}\,(n+k)+2\,.$$

The proofs of our main theorems are mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originally introduced by Baker and Davenport in [1]. We organize this paper as follows. In Section 2, we recall some important results for the proofs of the main theorems. Sections 3 and 4 are devoted to the proofs of our main results and we finish with Section 5, where we give some applications.

2. Useful tools

In this section, we gather the tools we need to prove Theorems 1.1 and 1.2 and other results.

2.1. **Linear forms in logarithms.** Here we recall a result from the theory of lower bounds for nonzero linear forms in logarithms of algebraic numbers. Let η be an algebraic number of degree d, let $a>0$ be the leading coefficient of its minimal polynomial over $\mathbb{Z}$ and let $\eta=\eta^{(1)},\ldots,\eta^{(d)}$ denote its conjugates. The logarithmic height of η is defined by

$$h(\eta) = \frac{1}{d}\Big(\log a + \sum_{j=1}^{d}\log\max\left(1,\left|\eta^{(j)}\right|\right)\Big)\,.$$

This height has the following basic properties. For η_1,η_2 algebraic numbers and $m\in\mathbb{Z}$ we have

$$h(\eta_1\pm\eta_2)\le h(\eta_1)+h(\eta_2)+\log 2\,, \tag{2.1}$$

$$h(\eta_1\eta_2^{\pm 1})\le h(\eta_1)+h(\eta_2)\,, \tag{2.2}$$

$$h(\eta_1^m) = |m|h(\eta_1)\,. \tag{2.3}$$

Now let $\mathbb{L}$ be a real number field of degree $d_{\mathbb{L}}$, $\eta_1,\ldots,\eta_t\in\mathbb{L}$ and $b_1,\ldots,b_t\in\mathbb{Z}\setminus\{0\}$. Let $B\ge\max\{|b_1|,\ldots,|b_t|\}$ and

$$\Lambda = \eta_1^{b_1}\cdots\eta_t^{b_t}-1\,.$$

Let $A_1,\ldots,A_t$ be real numbers with

$$A_i\ge\max\{d_{\mathbb{L}}h(\eta_i),|\log\eta_i|,0.16\}\,,\quad i=1,2,\ldots,t\,.$$

The first tool we need is the following result due to Matveev [7]. Here we use the version of Bugeaud, Mignotte and Siksek [2, Theorem 9.4].

Theorem 2.1. *Assume that* $\Lambda\ne 0$. *Then*

$$\log|\Lambda| > -1.4\cdot 30^{t+3}\cdot t^{4.5}\cdot d_{\mathbb{L}}^2\cdot(1+\log d_{\mathbb{L}})\cdot(1+\log B)\cdot A_1\ldots A_t\,.$$

Also, we need the following lemma due to Guzmán and Luca.

Lemma 2.2 (Lemma 7 of [5]). *If $l \geq 1$, $H > (4l^2)^l$ and $H > L/(\log L)^l$, then*

$$L < 2^l H (\log H)^l \,.$$

The following lemma plays an important role for solving the Diophantine equations (1.9) and (1.10).

Lemma 2.3. *The n-th term of the generalized Lucas sequence $\{U_n\}$, with $s \in \{-1, 1\}$, satisfies the inequalities*

$$\delta^{n-2} \leq U_n < \delta^n \,, \tag{2.4}$$

for $n \geq 2$.

Proof. We will prove Lemma 2.3 in two cases according to $s \in \{-1, 1\}$.
Case 1: $s = 1$. We split into two parts which we prove independently as follows.

First, we will prove inequality $U_n < \delta^n$, for all $n \geq 1$. If $n = 1$, then $U_1 = 1 < \dfrac{r + \sqrt{r^2+4}}{2} = \delta^1$. Thus $U_1 < \delta^1$. If $n = 2$, then $U_2 = rU_1 + U_0 = r < \delta r + 1 = \delta^2$, since δ is a root of equation $x^2 = rx + 1$. Thus, $U_2 < \delta^2$. Now by induction, we will prove that the inequality $U_n < \delta^n$ still holds, for all $n \geq 3$. Assuming that it holds until $n = k$, then we will prove that it holds for $n = k + 1$.

$$U_{k+1} = rU_k + U_{k-1} < r\delta^k + \delta^{k-1} = \delta^{k-1}(r\delta + 1) = \delta^{k-1}\delta^2 = \delta^{k+1} \,.$$

Thus, we obtain $U_{k+1} < \delta^{k+1}$.

Next, we will prove the inequality $U_n \geq \delta^{n-2}$, for all $n \geq 1$. If $n = 1$, $U_1 = 1$ and $\delta^{-1} = \dfrac{2}{r + \sqrt{r^2+4}} < 1$. Thus, we get $U_1 > \delta^{-1}$. If $n = 2$, $U_2 = r \geq 1$ and $\delta^0 = 1$. Then, $U_2 \geq \delta^0$. Now by induction, we will prove that inequality $U_n \geq \delta^{n-2}$ still holds, for all $n \geq 3$. Assuming that it holds until $n = k$, then we will prove that it holds for $n = k + 1$.

$$U_{k+1} = rU_k + U_{k-1} > r\delta^{k-2} + \delta^{k-3} = \delta^{k-3}(r\delta + 1) = \delta^{k-3}\delta^2 = \delta^{k-1} \,.$$

It follows that $U_{k+1} > \delta^{k-1}$.

Case 2: $s = -1$. Note that in this case $\gamma = \dfrac{r - \sqrt{r^2-4}}{2} > 0$. We have

$$U_n = \frac{\delta^n - \gamma^n}{\sqrt{\Delta}} < \frac{\delta^n}{\sqrt{\Delta}} < \delta^n \,,$$

which is the required upper bound of (2.4). As

$$\frac{\gamma}{\delta} = \frac{r - \sqrt{r^2-4}}{r + \sqrt{r^2-4}} < 1 \,,$$

we have

$$\frac{U_n}{U_{n-1}} = \delta \left(\frac{1 - (\gamma/\delta)^n}{1 - (\gamma/\delta)^{n-1}} \right) > \delta$$

for all $n \geq 2$, and thus $U_n > \delta U_{n-1}$. By iterating recursively this inequality, we obtain $U_n > \delta^{n-1} U_1 = \delta^{n-1} > \delta^{n-2}$. This completes the proof. □

2.2. **Dujella and Pethő Lemma.** Our next tool is a version of the reduction method of Baker and Davenport [1]. We use a slight variant of the version given by Dujella and Pethő [4]. For a real number x, we write $\|x\|$ for the distance from x to the nearest integer.

Lemma 2.4. *Let M be a positive integer, p/q be a convergent of the continued fraction expansion of the irrational number τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let*

$$\varepsilon := \|\mu q\| - M \cdot \|\tau q\| \, .$$

If $\varepsilon > 0$, then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w} \tag{2.5}$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\varepsilon)}{\log B} \, .$$

3. Proof of Theorem 1.1

First, combining the right side of (2.4) with (1.5), we obtain

$$\alpha^{m-3} \leq P_m = U_n U_k < \delta^{n+k} \, .$$

Taking the logarithm of both sides, we get that

$$(m-3)\log\alpha < (n+k)\log\delta,$$

which leads to inequality (1.11) of Theorem 1.1. Next, we examine (1.9) in two different steps.

Step 1. Using (1.4) and (1.8), we rewrite equation (1.9) as

$$\frac{\delta^n - \gamma^n}{\sqrt{\Delta}} \cdot \frac{\delta^k - \gamma^k}{\sqrt{\Delta}} = a\alpha^m + e_1(m)$$

to obtain

$$\frac{\delta^{n+k}}{\Delta} - a\alpha^m = e_1(m) + \frac{\delta^n\gamma^k + \gamma^n\delta^k - \gamma^{n+k}}{\Delta} \, . \tag{3.1}$$

Taking absolute values on both sides of (3.1), we get

$$\Big|\frac{\delta^{n+k}}{\Delta} - a\alpha^m\Big| \leq |e_1(m)| + \frac{\delta^n|\gamma|^k}{\Delta} + \frac{|\gamma|^n\delta^k}{\Delta} + \frac{|\gamma|^{n+k}}{\Delta} \, . \tag{3.2}$$

Dividing both sides of (3.2) by $\dfrac{\delta^{n+k}}{\Delta}$ and using the fact that $|\gamma| = 1/\delta$ and $n \leq k$, we obtain

$$\begin{aligned}\Big|1 - \frac{\Delta a\alpha^m}{\delta^{n+k}}\Big| &\leq \frac{\Delta|e_1(m)|}{\delta^{n+k}} + \frac{1}{\delta^{2k}} + \frac{1}{\delta^{2n}} + \frac{1}{\delta^{2(n+k)}} \\ &< \frac{\Delta}{\delta^{2n}} + \frac{1}{\delta^{2n}} + \frac{1}{\delta^{2n}} + \frac{1}{\delta^{4n}} < \frac{3+\Delta}{\delta^{2n}} \, .\end{aligned}$$

From this, it follows that

$$\left|1 - \frac{\Delta a \alpha^m}{\delta^{n+k}}\right| < \frac{3+\Delta}{\delta^{2n}}. \tag{3.3}$$

Now, let us apply Theorem 2.1 to

$$\Lambda_1 := 1 - \alpha^m \delta^{-(n+k)} \Delta a$$

with $t = 3$,

$$(\eta_1, b_1) := (\alpha, m)\,, \quad (\eta_2, b_2) := (\delta, -n-k)\,, \quad \text{and} \quad (\eta_3, b_3) := (\Delta a, 1)\,.$$

Note that the numbers η_1, η_2 and η_3 are positive real numbers and elements of the field $\mathbb{L} = \mathbb{Q}(\alpha, \delta)$. It is obvious that the degree of the field $\mathbb{L}$ over $\mathbb{Q}$ is 6. So $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 6$. Now, we show that $\Lambda_1 \neq 0$. Indeed, if $\Lambda_1 = 0$, then

$$\Delta a \cdot \alpha^m \cdot 2^{n+k} = (r + \sqrt{\Delta})^{n+k} = x + y\sqrt{\Delta}$$

for some positive integers x and y, which is impossible. Thus $\Lambda_1 \neq 0$. Moreover,

$$h(\eta_1) = h(\alpha) = \frac{1}{3}\log\alpha, \quad h(\eta_2) = h(\delta) = \frac{1}{2}\log\delta$$

and

$$h(\eta_3) = h(\Delta a) \le h(\Delta) + h(a) = \log\Delta + \frac{1}{3}\log 23\,.$$

Now we choose

$$\max\left\{6h(\eta_1), |\log\eta_1|, 0.16\right\} = 2\log\alpha = A_1\,.$$

Note that for the generalized Lucas sequences, if $s \in \{-1, 1\}$ and $r \ge 1$, then $\delta \ge (1+\sqrt{5})/2$. So we can choose

$$\max\{6h(\eta_2), |\log\eta_2|, 0.16\} = 3\log\delta = A_2\,.$$

Furthermore, we have

$$\max\left\{6h(\eta_3), |\log\eta_3|, 0.16\right\} = 2\log\left(23\delta^3\right) = A_3\,.$$

Also, we have $\max\{|-(n+k)|, |m|, |1|\} \le \dfrac{\log\delta}{\log\alpha}(n+k) + 3$. So we can take $B := \dfrac{\log\delta}{\log\alpha}(n+k) + 3$. Using Theorem 2.1, we get

$$\begin{aligned}\log|\Lambda_1| &> -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2 \cdot (1+\log 6) \cdot (1+\log B)\\ &\quad \times 2\log\alpha \cdot 3\log\delta \cdot \log\left(23^2\Delta^6\right)\\ &> -5 \times 10^{13} \cdot (1+\log B) \cdot \log\delta \cdot \log\left(23\Delta^3\right)\,. \end{aligned} \tag{3.4}$$

Combining this with (3.3), we get

$$2n\log\delta - \log(3+\Delta) < 5 \times 10^{13} \cdot (1+\log B) \cdot \log\delta \cdot \log\left(23\Delta^3\right)\,. \tag{3.5}$$

Step 2. Rearranging equation (1.9) as

$$\frac{\delta^k}{\sqrt{\Delta}} - \frac{a\alpha^m}{U_n} = \frac{\gamma^k}{\sqrt{\Delta}} + \frac{e_1(m)}{U_n} \tag{3.6}$$

and taking absolute values on both sides of (3.6), we get

$$\Big|\frac{\delta^k}{\sqrt{\Delta}} - \frac{a\alpha^m}{U_n}\Big| \leq \frac{|\gamma|^k}{\sqrt{\Delta}} + \frac{|e_1(m)|}{U_n}$$

which leads to

$$\Big|\frac{\delta^k}{\sqrt{\Delta}} - \frac{a\alpha^m}{U_n}\Big| \leq \frac{1}{\sqrt{\Delta}\delta^k} + \frac{1}{U_n}\,. \tag{3.7}$$

Dividing both sides of (3.7) by $\dfrac{\delta^k}{\sqrt{\Delta}}$, for $n \geq 1$, we obtain

$$\begin{aligned}\Big|1 - \frac{\sqrt{\Delta}}{\delta^k}\cdot\frac{a\alpha^m}{U_n}\Big| &\leq \frac{1}{\delta^{2k}} + \frac{\sqrt{\Delta}}{\delta^k U_n}\\ &\leq \frac{1}{\delta^k} + \frac{\sqrt{\Delta}}{\delta^k} = \frac{1+\sqrt{\Delta}}{\delta^k}\,. \end{aligned}\tag{3.8}$$

From this, it follows that

$$\Big|1 - \frac{a\sqrt{\Delta}}{U_n}\cdot\delta^{-k}\alpha^m\Big| \leq \frac{1+\sqrt{\Delta}}{\delta^k}\,. \tag{3.9}$$

Taking $t = 3$,

$$(\eta_1, b_1) := (\alpha, m)\,, \quad (\eta_2, b_2) := (\delta, -k) \quad \text{and} \quad (\eta_3, b_3) := \Big(\frac{a\sqrt{\Delta}}{U_n}, 1\Big)\,,$$

we can apply Theorem 2.1 to

$$\Lambda_2 := 1 - \frac{a\sqrt{\Delta}}{U_n}\cdot\delta^{-k}\alpha^m\,.$$

The numbers η_1, η_2 and η_3 are positive real numbers and elements of the field $\mathbb{L} = \mathbb{Q}(\alpha, \delta)$ and so $d_{\mathbb{L}} = 6$. Now, we show that $\Lambda_2 \neq 0$. Indeed, if $\Lambda_2 = 0$, then

$$2^k \cdot a\Delta \cdot \alpha^m = (r + \sqrt{\Delta})^k \cdot \sqrt{\Delta}\cdot U_n = x + y\sqrt{\Delta}\,,$$

for some positive integers x and y, which is impossible. So $\Lambda_2 \neq 0$. Note that $h(\eta_1) = \dfrac{1}{3}\log\alpha$, $h(\eta_2) = \dfrac{1}{2}\log\delta$ and from (2.4) and (2.2), we get

$$\begin{aligned} h(\eta_3) &= h\Big(\frac{a\sqrt{\Delta}}{U_n}\Big)\\ &\leq h(a) + h(\sqrt{\Delta}) + h(U_n) = \frac{1}{3}\log 23 + \frac{1}{2}\log\Delta + \log U_n\\ &\leq \frac{1}{3}\log 23 + \frac{1}{2}\log\Delta + n\log\delta\,. \end{aligned}$$

Here we take $A_1 := 2\log\alpha$, $A_2 := 3\log\delta$ and $A_3 := 2\log 23 + 3\log\Delta + 6n\log\delta$. Since $B \geq \max\{|m|, |-k|, |1|\}$, we can take $B := \dfrac{\log\delta}{\log\alpha}(n+k) + 3$. Thus, taking into account the inequality (3.9) and using Theorem 2.1, we obtain

$$\log\left(\frac{1+\sqrt{\Delta}}{\delta^k}\right) \geq \log|\Lambda_2| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 6^2\cdot(1+\log 6) \times(1+\log B)\cdot 2\log\alpha\cdot 3\log\delta\cdot\left(\log\left(23^2\cdot\Delta^3\right)+6n\log\delta\right).$$

We deduce that

$$k\log\delta < 2.5\cdot 10^{13}\cdot(1+\log B)\cdot\log\delta\cdot\left(\log\left(23^2\cdot\Delta^3\right)+6n\log\delta\right) + \log(1+\sqrt{\Delta})\,. \tag{3.10}$$

Inserting inequality (3.5) into inequality (3.10), we get inequality (1.12) of Theorem 1.1. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 follows essentially the same lines as that of Theorem 1.1, and we will therefore avoid some details. First, combining the right side of (2.4) with (1.7), we have

$$\alpha^{m-2} \leq E_m = U_n U_k < \delta^{n+k}\,.$$

By taking the logarithm of both sides, we get that

$$(m-2)\log\alpha < (n+k)\log\delta\,.$$

This proves inequality (1.13) of Theorem 1.2. Next, we examine (1.10) in two different steps.

Step a. We need to rewrite equation (1.10) into the form

$$\frac{\delta^n-\gamma^n}{\sqrt{\Delta}}\cdot\frac{\delta^k-\gamma^k}{\sqrt{\Delta}} = \alpha^m + e_2(m)$$

to obtain

$$\frac{\delta^{n+k}}{\Delta} - \alpha^m = e_2(m) + \frac{\delta^n\gamma^k+\gamma^n\delta^k-\gamma^{n+k}}{\Delta}\,. \tag{4.1}$$

Taking absolute values on both sides of (4.1), we get

$$\left|\frac{\delta^{n+k}}{\Delta} - \alpha^m\right| \leq |e_2(m)| + \frac{\delta^n|\gamma|^k}{\Delta} + \frac{|\gamma|^n\delta^k}{\Delta} + \frac{|\gamma|^{n+k}}{\Delta}\,. \tag{4.2}$$

Dividing both sides of (4.2) by $\dfrac{\delta^{n+k}}{\Delta}$ and using the fact that $|\gamma| = 1/\delta$ and $n \leq k$, we easily get

$$\left|1 - \frac{\Delta\alpha^m}{\delta^{n+k}}\right| < \frac{3+\Delta}{\delta^{2n}}\,. \tag{4.3}$$

Put

$$\Lambda_3 := 1 - \alpha^m\delta^{-(n+k)}\Delta\,.$$

Now, let us apply Theorem 2.1 to Λ_3, with $t = 3$,

$$(\eta_1, b_1) := (\alpha, m)\,,\quad (\eta_2, b_2) := (\delta, -n-k)\quad\text{and}\quad(\eta_3, b_3) := (\Delta, 1)\,.$$

Also, the numbers η_1, η_2 and η_3 are positive real numbers and elements of the field $\mathbb{L} = \mathbb{Q}(\alpha, \delta)$. It is obvious that the degree of the field $\mathbb{L}$ over $\mathbb{Q}$ is 6, i.e. $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 6$. Now, let us show that $\Lambda_3 \neq 0$. On the contrary, assume that $\Lambda_3 = 0$. Then

$$\Delta \cdot \alpha^m \cdot 2^{n+k} = (r + \sqrt{\Delta})^{n+k} = x + y\sqrt{\Delta}$$

for some positive integers x and y, which is not possible. Thus $\Lambda_3 \neq 0$. Moreover,

$$h(\eta_1) = h(\alpha) = \frac{1}{3}\log\alpha, \quad h(\eta_2) = h(\delta) = \frac{1}{2}\log\delta$$

and

$$h(\eta_3) = h(\Delta) = \log\Delta\,.$$

Now we can choose

$$A_1 := 2\log\alpha, \quad A_2 := 3\log\delta \quad \text{and} \quad A_3 := 6\log\Delta\,.$$

Also, we have $\max\{|-(n+k)|, |m|, |1|\} \leq \dfrac{\log\delta}{\log\alpha}(n+k) + 2$. So we can take $B := \dfrac{\log\delta}{\log\alpha}(n+k) + 2$. Using Theorem 2.1, we get

$$\begin{aligned}\log|\Lambda_3| &> -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2 \cdot (1 + \log 6) \cdot (1 + \log B) \\ &\quad \times 2\log\alpha \cdot 3\log\delta \cdot 6\log\Delta \\ &> -1.5 \times 10^{14} \cdot (1 + \log B) \cdot \log\delta \cdot \log\Delta\,. \end{aligned} \tag{4.4}$$

Combining this with (4.3), we get

$$2n\log\delta - \log(3 + \Delta) < 1.5 \times 10^{14} \cdot (1 + \log B) \cdot \log\delta \cdot \log\Delta\,. \tag{4.5}$$

Step b. Rearranging equation (1.10) as

$$\frac{\delta^k}{\sqrt{\Delta}} - \frac{\alpha^m}{U_n} = \frac{\gamma^k}{\sqrt{\Delta}} + \frac{e_2(m)}{U_n} \tag{4.6}$$

and taking the absolute value on both sides of (4.6), we get

$$\Big|\frac{\delta^k}{\sqrt{\Delta}} - \frac{\alpha^m}{U_n}\Big| \leq \frac{|\gamma|^k}{\sqrt{\Delta}} + \frac{|e_2(m)|}{U_n}$$

which leads to

$$\Big|\frac{\delta^k}{\sqrt{\Delta}} - \frac{\alpha^m}{U_n}\Big| \leq \frac{1}{\sqrt{\Delta}\delta^k} + \frac{1}{U_n}\,. \tag{4.7}$$

Dividing both sides of (4.7) by $\dfrac{\delta^k}{\sqrt{\Delta}}$, we obtain for $n \geq 1$

$$\begin{aligned}\Big|1 - \frac{\sqrt{\Delta}}{\delta^k} \cdot \frac{\alpha^m}{U_n}\Big| &\leq \frac{1}{\delta^{2k}} + \frac{\sqrt{\Delta}}{\delta^k U_n} \\ &\leq \frac{1}{\delta^k} + \frac{\sqrt{\Delta}}{\delta^k} = \frac{1 + \sqrt{\Delta}}{\delta^k}\,. \end{aligned} \tag{4.8}$$

From this, it follows that

$$\left|1-\frac{\sqrt{\Delta}}{U_n}\cdot\delta^{-k}\alpha^m\right|\leq\frac{1+\sqrt{\Delta}}{\delta^k}\,. \tag{4.9}$$

Put

$$\Lambda_4:=1-\frac{\sqrt{\Delta}}{U_n}\cdot\delta^{-k}\alpha^m\,.$$

Now, we have everything ready to apply Theorem 2.1 to Λ_4 with the following data: $t=3$,

$$(\eta_1,b_1):=(\alpha,m)\,,\quad (\eta_2,b_2):=(\delta,-k)\quad\text{and}\quad(\eta_3,b_3):=\Big(\frac{\sqrt{\Delta}}{U_n},1\Big)\,.$$

The numbers η_1,η_2 and η_3 are positive real numbers and elements of the field $\mathbb{L}=\mathbb{Q}(\alpha,\delta)$ and so $d_{\mathbb{L}}=6$. Now, we show that $\Lambda_4\neq 0$. On the contrary, assume that $\Lambda_4=0$. We have

$$2^k\cdot\Delta\cdot\alpha^m=(r+\sqrt{\Delta})^k\cdot\sqrt{\Delta}\cdot U_n=x+y\sqrt{\Delta}\,,$$

for some positive integers x and y, which is impossible. Hence, $\Lambda_4\neq 0$. Again, $h(\eta_1)=\dfrac{1}{3}\log\alpha$, $h(\eta_2)=\dfrac{1}{2}\log\delta$ and from (2.4) and (2.2), we get

$$\begin{aligned}h(\eta_3)=h\Big(\frac{\sqrt{\Delta}}{U_n}\Big)&\leq h(\sqrt{\Delta})+h(U_n)=\frac{1}{2}\log\Delta+\log U_n\\&\leq\frac{1}{2}\log\Delta+n\log\delta\,.\end{aligned}$$

So, we take $A_1:=2\log\alpha$, $A_2:=3\log\delta$ and $A_3:=3\log\Delta+6n\log\delta$. Since $B\geq\max\{|m|,|-k|,|1|\}$, we can take $B:=\dfrac{\log\delta}{\log\alpha}(n+k)+2$. Thus, taking into account inequality (4.9) and using Theorem 2.1, we obtain

$$\begin{aligned}\log\Big(\frac{1+\sqrt{\Delta}}{\delta^k}\Big)\geq\log|\Lambda_4|&>-1.4\cdot 30^6\cdot 3^{4.5}\cdot 6^2\cdot(1+\log 6)\\&\times(1+\log B)\cdot 2\log\alpha\cdot 3\log\delta\cdot(3\log\Delta+6n\log\delta)\end{aligned}$$

or

$$\begin{aligned}k\log\delta<2.5\cdot 10^{13}\cdot(1+\log B)\cdot\log\delta\cdot(3\log\Delta+6n\log\delta)\\+\log(1+\sqrt{\Delta})\,.\end{aligned}\tag{4.10}$$

Inserting inequality (4.5) into inequality (4.10), we get inequality (1.14) of Theorem 1.2. This completes the proof of Theorem 1.2.

5. The study of some applications

In this section, we explicitly study equations (1.9) and (1.10) with specific cases of Lucas sequences namely the Fibonacci, Pell and balancing sequences. For the proofs, we use the assumption $k\geq 2$ according to Lemma 2.3.

5.1. **The Fibonacci sequence.** The Fibonacci sequence (F_n) is a particularity of the Lucas sequences which corresponds to $r = s = 1$. In this case we have $\Delta = 5$ and $\delta = (1+\sqrt{5})/2$. The following is our main result in this case.

Theorem 5.1.

1) *The set of solutions* (m, n, k) *of the Diophantine equation*

$$P_m = F_n F_k \tag{5.1}$$

in non-negative integers m, n *and* k *with* $1 \le n \le k$ *is*

$$\left\{\begin{array}{c} (0,1,1),\ (0,1,2),\ (0,2,2),\ (1,1,1),\ (1,1,2),\ (1,2,2),\ (2,1,1),\\ (2,1,2),\ (2,2,2),\ (3,1,3),\ (3,2,3),\ (4,1,3),\ (4,2,3),\ (5,1,4),\\ (5,2,4),\ (6,3,3),\ (7,1,5),\ (7,2,5),\ (9,4,4),\ (11,3,6),\ (12,1,8),\\ (12,2,8) \end{array}\right\}.$$

2) *The set of solutions* (m, n, k) *of the Diophantine equation*

$$E_m = F_n F_k \tag{5.2}$$

in non-negative integers m, n, k *with* $1 \le n \le k$ *is*

$$\left\{\begin{array}{c} (0,1,4),\ (0,2,4),\ (2,1,3),\ (2,2,3),\ (3,1,4),\ (3,2,4),\ (4,1,3),\\ (4,2,3),\ (5,1,5),\ (5,2,5),\ (6,1,5),\ (6,2,5),\ (8,3,5),\ (13,4,7),\\ (15,3,9) \end{array}\right\}.$$

Proof. 1) From Theorem 1.1 and since $n \le k$, we get

$$B := \frac{\log \delta}{\log \alpha}(n+k) + 3 < 5k\,, \quad \text{for} \quad k \ge 2\,. \tag{5.3}$$

Combining (5.3) with (1.12), we get

$$k < 2.5 \cdot 10^{13} \cdot (1 + \log 5k)(17.4 + 5.8 \cdot 10^{14}(1 + \log 5k)) + 2.5\,. \tag{5.4}$$

Factoring the right side of inequality (5.4) by $(\log 5k)^2$ while using $k \ge 2$, we get

$$k < 3 \times 10^{28} (\log 5k)^2\,. \tag{5.5}$$

Now, we apply Lemma 2.2 with $l = 2$, $L = 5k$ and $H = 1.5 \cdot 10^{29}$. So, we get $k < 5.5 \times 10^{32}$. Now, let us try to reduce the upper bound on k by applying Lemma 2.4. Let

$$z_1 := m \log \alpha - (n+k)\log \delta + \log 5a\,.$$

If $z_1 > 0$, then by (3.3), we have the inequalities

$$|z_1| = z_1 < e^{z_1} - 1 = |1 - e^{z_1}| < 8 \cdot \delta^{-2n}$$

since $x < e^x - 1$ for $x > 0$. If $z_1 < 0$, then

$$1 - e^{z_1} = |1 - e^{z_1}| < 8 \cdot \delta^{-2n} < \frac{1}{2}\,, \quad \text{for} \quad n \ge 3\,.$$

From this, we get $e^{z_1} > \dfrac{1}{2}$ and therefore, we obtain

$$e^{|z_1|} = e^{-z_1} < 2\,.$$

Consequently, we get

$$|z_1| < e^{|z_1|} - 1 = e^{|z_1|}|1 - e^{z_1}| < 16 \cdot \delta^{-2n}\,.$$

In both cases, the inequalities

$$0 < |z_1| < 16 \cdot \delta^{-2n}$$

hold. That is,

$$0 < |m \log\alpha - (n+k)\log\delta + \log 5a| < 16 \cdot \delta^{-2n}.$$

Dividing these inequalities by $\log\delta$, we get

$$0 < \left| m\frac{\log\alpha}{\log\delta} - (n+k) + \frac{\log 5a}{\log\delta} \right| < 33.3 \cdot \delta^{-2n}\,. \tag{5.6}$$

Now, we show that $\dfrac{\log\alpha}{\log\delta}$ is irrational. On the contrary, assume that

$$\frac{\log\alpha}{\log\delta} = \frac{p}{q}$$

for some positive integers p and q. This shows that $\alpha^q = \delta^p$, which implies $2^p\alpha^q = x + y\sqrt{5}$, for some positive integers x and y. This last equality is not possible. Moreover $m < 5k < 2.8 \times 10^{33}$. So, we have everything ready to apply Lemma 2.4 to inequality (5.6) with the following data

$$\tau := \frac{\log\alpha}{\log\delta}\,, \quad \mu := \frac{\log 5a}{\log\delta}\,, \quad A := 33.3\,, \quad B := \delta\,, \quad M := 2.8 \times 10^{33}\,,$$

and $w := 2n$. For the last part of the proof, we use Mathematica to apply Lemma 2.4. For the computations, if the first convergent such that $q > 6M$ does not satisfy the condition $\varepsilon > 0$, then we use the next convergent until we find the one that satisfies the conditions. Then we found that the denominator of the 65-th convergent

$$\frac{p_{65}}{q_{65}} = \frac{3532061372497206065034856425053368 7}{6044353741907918847246854201902732 7}$$

of τ exceeds $6M$. Thus, we can say that inequality (5.6) has no solution for

$$2n = w \geq \frac{\log(Aq_{65}/\varepsilon)}{\log\delta} \geq \frac{\log(Aq_{65}/0.277632)}{\log\delta} \geq 176.376\,.$$

Therefore, we obtain

$$n \leq 88\,.$$

Substituting this upper bound for n into (3.10), we obtain

$$k < 6.64 \times 10^{15} \cdot (1 + \log 5k) + 2.5\,,$$

which implies $k < 2.85 \times 10^{17}$. Now, let

$$z_2 := m\log\alpha - k\log\delta + \log\left(\frac{a\sqrt{5}}{F_n}\right).$$

If $z_2 > 0$, then by (3.9), we get

$$|z_2| = z_2 < e^{z_2} - 1 = |1 - e^{z_2}| < (1+\sqrt{5}) \cdot \delta^{-k}\,.$$

If $z_2 < 0$, then we have

$$1 - e^{z_2} = |1 - e^{z_2}| < (1 + \sqrt{5}) \cdot \delta^{-k} < \frac{1}{4}, \quad \text{for} \quad k \geq 6,$$

in which case we get $e^{z_2} > \dfrac{3}{4}$ and

$$e^{|z_2|} = e^{-z_2} < \frac{4}{3}.$$

Thus it follows that

$$|z_2| < e^{|z_2|} - 1 = e^{|z_2|}|1 - e^{z_2}| < \frac{4}{3} \cdot (1 + \sqrt{5}) \cdot \delta^{-k}.$$

This means that the inequalities

$$0 < |z_2| < 4.4 \cdot \delta^{-k}$$

are always true. That is,

$$0 < \left| m \log \alpha - k \log \delta + \log\left(\frac{a\sqrt{5}}{F_n}\right) \right| < 4.4 \cdot \delta^{-k}.$$

Dividing both sides of the above inequalities by $\log \delta$, we obtain

$$0 < \left| m \frac{\log \alpha}{\log \delta} - k + \frac{\log\left(a\sqrt{5}/F_n\right)}{\log \delta} \right| < 9.2 \cdot \delta^{-k}. \tag{5.7}$$

Note that $m < 5k < 1.43 \cdot 10^{18}$. So considering the fact that $n \leq 88$, we apply Lemma 2.4 to inequalities (5.7) with the following data

$$\tau := \frac{\log \alpha}{\log \delta}, \quad \mu := \frac{\log\left(a\sqrt{5}/F_n\right)}{\log \delta}, \quad A := 9.2, \quad B := \delta, \quad M := 1.43 \cdot 10^{18},$$

and $w := k$. Thus, with the help of Mathematica we found that the denominator of the 38-th convergent

$$\frac{p_{38}}{q_{38}} = \frac{289247483585778209742}{494984068896025620125}$$

of τ exceeds $6M$. Therefore, we can say that inequality (5.7) has no solution for

$$k = w \geq \frac{\log(Aq_{38}/\varepsilon)}{\log \delta} \geq \frac{\log(Aq_{38}/0.00293878)}{\log \delta} \geq 115.749.$$

We conclude that

$$k \leq 115.$$

Thus, it remains to check that (5.1) holds for $1 \leq n \leq 88$, $1 \leq k \leq 115$ and $0 \leq m \leq 575$. A quick inspection using Mathematica reveals that Diophantine equation (5.1) has only the solutions listed in part 1) of Theorem 5.1.

2) Referring to Theorem 1.2 and using the fact that $n \leq k$, we obtain

$$B := \frac{\log \delta}{\log \alpha}(n + k) + 2 < 5k. \tag{5.8}$$

By combining (5.8) with (1.14), we get

$$k < 2.5 \cdot 10^{13} \cdot (1 + \log 5k)(11.1 + 3.5 \cdot 10^{14}(1 + \log 5k)) + 2.5. \tag{5.9}$$

Factoring the right side of inequality (5.9) by $(\log 5k)^2$ while using $k \geq 2$, we obtain

$$k < 1.26 \times 10^{28} (\log 5k)^2 \,. \tag{5.10}$$

A judicious application of Lemma 2.2 with $l = 2$, $L = 5k$ and $H = 6.3 \cdot 10^{28}$ allows us to find that $k \leq 2.22 \cdot 10^{32}$. Now, let us try to reduce the upper bound on k by applying Lemma 2.4. From (4.3) we can put

$$z_3 := m \log \alpha - (n+k) \log \delta + \log \Delta \,.$$

If $z_3 > 0$, it is easy to see that

$$|z_3| = z_3 < e^{z_3} - 1 = |1 - e^{z_3}| < 8 \cdot \delta^{-2n}$$

whereas if $z_3 < 0$ we get

$$0 < |z_3| < 16 \cdot \delta^{-2n} \,.$$

So, we obtain that inequalities

$$0 < \left| m \frac{\log \alpha}{\log \delta} - (n+k) + \frac{\log 5}{\log \delta} \right| < 33.3 \cdot \delta^{-2n} \tag{5.11}$$

hold for all $n \geq 3$. We apply Lemma 2.4 to inequalities (5.11) with the following data

$$\tau := \frac{\log \alpha}{\log \delta} \,, \quad \mu := \frac{\log 5}{\log \delta} \,, \quad A := 33.3 \,, \quad B := \delta \,, \quad M := 1.11 \times 10^{33} \,,$$

and $w := 2n$. Therefore, using Mathematica we found that the denominator of the 64-th convergent

$$\frac{p_{64}}{q_{64}} = \frac{67076299875112138515427836277206240}{114786477749345061824556991553 79417}$$

of τ exceeds $6M$. It follows that inequalities (5.11) have no solution for

$$2n = w \geq \frac{\log(Aq_{64}/\varepsilon)}{\log \delta} \geq \frac{\log(Aq_{64}/0.175498)}{\log \delta} \geq 173.877 \,.$$

Thus, we obtain

$$n \leq 86 \,.$$

By inserting this upper bound of n in (4.10), we get

$$k < 6.33 \times 10^{15}(1 + \log 5k) + 2.5 \,,$$

which leads to $k < 2.71 \cdot 10^{17}$. Put

$$z_4 := m \log \alpha - k \log \delta + \log \left(\frac{\sqrt{5}}{F_n} \right) .$$

We easily obtain that

$$0 < \left| m \frac{\log \alpha}{\log \delta} - k + \frac{\log \left(\sqrt{5}/F_n \right)}{\log \delta} \right| < 9.2 \cdot \delta^{-k} \,. \tag{5.12}$$

Note that $m < 5k < 1.36 \cdot 10^{18}$. So considering the fact that $n \leq 86$, we may apply Lemma 2.4 to inequalities (5.12) with the following data

$$\tau := \frac{\log \alpha}{\log \delta} \,, \quad \mu := \frac{\log \left(\sqrt{5}/F_n \right)}{\log \delta} \,, \quad A := 9.2 \,, \quad B := \delta \,, \quad M := 1.36 \cdot 10^{18} \,,$$

and $w := k$. Thus, with the help of a Mathematica we found that the denominator of the 38-th convergent

$$\frac{p_{38}}{q_{38}} = \frac{289247483585778209742}{494984068896025620125}$$

of τ exceeds $6M$. Therefore, we can say that inequalities (5.12) have no solution for

$$k = w \geq \frac{\log(Aq_{38}/\varepsilon)}{\log \delta} \geq \frac{\log(Aq_{38}/0.00685216)}{\log \delta} \geq 113.99\,.$$

We conclude that

$$k \leq 113\,.$$

Thus, it remains to check that (5.1) holds for $1 \leq n \leq 86$, $1 \leq k \leq 113$ and $0 \leq m \leq 565$. A quick inspection using Mathematica reveals that Diophantine equation (5.2) has only the solutions listed in part 2) of Theorem 5.1. This completes the proof of Theorem 5.1. $\square$

In light of Theorem 5.1, we can deduce the following result.

Corollary 5.2.

1) *The only solutions of* $P_m = F_n^2$ *in non-negative integers* n, m *are* $P_0 = F_1^2 = 1$, $P_0 = F_2^2 = 1$, $P_1 = F_1^2 = 1$, $P_1 = F_2^2 = 1$, $P_2 = F_1^2 = 1$, $P_2 = F_2^2 = 1$, $P_6 = F_3^2 = 4$ *and* $P_9 = F_4^2 = 9$.
2) *The Diophantine equation* $E_m = F_n^2$ *has no solution in non-negative integers* n, m *such that* $n \geq 1$.

5.2. **The Pell sequence.** Considering $(r, s) = (2, 1)$ we obtain the Pell sequence $(\mathcal{P}_n)$. In this case we have $\Delta = 8$ and $\delta = 1 + \sqrt{2}$. The following is our main result in this case.

Theorem 5.3.

1) *The only solutions* (m, n, k) *of the Diophantine equation*

$$P_m = \mathcal{P}_n \mathcal{P}_k \tag{5.13}$$

in non-negative integers m, n *and* k *with* $1 \leq n \leq k$ *are* $(0, 1, 1)$, $(1, 1, 1)$, $(2, 1, 1)$, $(3, 1, 2)$, $(4, 1, 2)$, $(6, 2, 2)$, $(7, 1, 3)$, *and* $(10, 1, 4)$.

2) *The only solutions of the Diophantine equation*

$$E_m = \mathcal{P}_n \mathcal{P}_k \tag{5.14}$$

in non-negative integers m, n, k *with* $1 \leq n \leq k$ *are* $(2, 1, 2)$, $(4, 1, 2)$, $(5, 1, 3)$, $(6, 1, 3)$, $(8, 2, 3)$, $(9, 1, 4)$, *and* $(12, 1, 5)$.

Proof. 1) Using Theorem 1.1 and since $n \leq k$, we get

$$B := \frac{\log \delta}{\log \alpha}(n + k) + 3 < 8k\,, \quad \text{for} \quad k \geq 2\,. \tag{5.15}$$

Combining (5.15) with (1.12), we get

$$k < 2.5 \cdot 10^{13} \cdot (1 + \log 8k)(19.71 + 1.24 \cdot 10^{15}(1 + \log 8k)) + 1.6\,. \tag{5.16}$$

By factoring the right side of inequality (5.16) by $(\log 8k)^2$ while using $k \geq 2$, we obtain

$$k < 5.8 \times 10^{28} (\log 8k)^2 \,. \tag{5.17}$$

We apply Lemma 2.2 with $l = 2$, $L = 8k$ and $H = 4.64 \cdot 10^{29}$, in which case we obtain $k < 1.1 \times 10^{33}$. We will now apply Lemma 2.4 to reduce this large upper bound of k. So let

$$z_5 := m \log \alpha - (n + k) \log \delta + \log 8a \,.$$

If $z_5 > 0$, then by (3.3), we have the inequality

$$|z_5| = z_5 < e^{z_5} - 1 = |1 - e^{z_5}| < 11 \cdot \delta^{-2n} \,.$$

If $z_5 < 0$, then

$$1 - e^{z_5} = |1 - e^{z_5}| < 11 \cdot \delta^{-2n} < \frac{2}{5} \,, \quad \text{for} \quad n \geq 2 \,.$$

From this, we deduce that $e^{z_5} > \dfrac{3}{5}$ and therefore, we see that

$$e^{|z_5|} = e^{-z_5} < \frac{5}{3} \,.$$

Consequently, we get

$$|z_5| < e^{|z_5|} - 1 = e^{|z_5|} |1 - e^{z_5}| < 18.4 \cdot \delta^{-2n}.$$

In both cases, the inequalities

$$0 < |z_5| < 18.4 \cdot \delta^{-2n}$$

hold. That is,

$$0 < |m \log \alpha - (n + k) \log \delta + \log 8a| < 18.4 \cdot \delta^{-2n}.$$

Dividing these inequalities by $\log \delta$, we get

$$0 < \left| m \frac{\log \alpha}{\log \delta} - (n + k) + \frac{\log 8a}{\log \delta} \right| < 20.9 \cdot \delta^{-2n} \,. \tag{5.18}$$

Moreover, $m < 8k < 8.8 \times 10^{33}$. Hence, since the conditions of Lemma 2.4 are satisfied, we may now apply it to inequalities (5.18) with the following data

$$\tau := \frac{\log \alpha}{\log \delta} \,, \quad \mu := \frac{\log 8a}{\log \delta} \,, \quad A := 20.9 \,, \quad B := \delta \,, \quad M := 8.8 \times 10^{33} \,,$$

and $w := 2n$. We use Mathematica to apply Lemma 2.4 and we found that the denominator of the 62-th convergent

$$\frac{p_{62}}{q_{62}} = \frac{2030412055728949200740186956475250 1}{6363990987519141726436737909357711 6}$$

of τ exceeds $6M$. Thus, we can say that inequalities (5.18) have no solution for

$$2n = w \geq \frac{\log(Aq_{62}/\varepsilon)}{\log \delta} \geq \frac{\log(Aq_{62}/0.239005)}{\log \delta} \geq 95.9974 \,.$$

So

$$n \leq 47 \,.$$

Substituting this upper bound for n into (3.10), we obtain

$$k < 6.53 \times 10^{15} \cdot (1 + \log 8k) + 1.6\,,$$

which implies $k < 2.83 \times 10^{17}$. We now set

$$z_6 := m \log \alpha - k \log \delta + \log\left(\frac{a\sqrt{8}}{\mathcal{P}_n}\right).$$

If $z_6 > 0$, then by (3.9), we get the inequality

$$|z_6| = z_2 < e^{z_6} - 1 = |1 - e^{z_6}| < (1 + \sqrt{8}) \cdot \delta^{-k}\,.$$

If $z_6 < 0$, then

$$1 - e^{z_6} = |1 - e^{z_6}| < (1 + \sqrt{8}) \cdot \delta^{-k} < \frac{2}{7}\,, \quad \text{for} \quad k \geq 3\,.$$

So, we obtain $e^{z_6} > \dfrac{5}{7}$ and

$$e^{|z_6|} = e^{-z_6} < \frac{7}{5}\,.$$

Thus it follows that

$$|z_6| < e^{|z_6|} - 1 = e^{|z_6|}|1 - e^{z_6}| < \frac{7}{5} \cdot (1 + \sqrt{8}) \cdot \delta^{-k}\,.$$

This means that the inequalities

$$0 < |z_6| < 5.36 \cdot \delta^{-k}$$

always hold. That is,

$$0 < \left|m \log \alpha - k \log \delta + \log\left(\frac{a\sqrt{8}}{\mathcal{P}_n}\right)\right| < 5.36 \cdot \delta^{-k}\,.$$

Dividing both sides of the above inequalities by $\log \delta$, we obtain

$$0 < \left|m \frac{\log \alpha}{\log \delta} - k + \frac{\log\left(a\sqrt{8}/\mathcal{P}_n\right)}{\log \delta}\right| < 6.1 \cdot \delta^{-k}\,. \tag{5.19}$$

Note that $m < 8k < 2.27 \cdot 10^{18}$. We can apply Lemma 2.4 to (5.19) with the following data

$$\tau := \frac{\log \alpha}{\log \delta}\,, \quad \mu := \frac{\log\left(a\sqrt{8}/\mathcal{P}_n\right)}{\log \delta}\,, \quad A := 6.1, \quad B := \delta\,, \quad M := 2.27 \cdot 10^{18}\,,$$

and $w := k$ by considering $n \leq 47$. Thus, with the help of Mathematica we found that the denominator of the 32-nd convergent

$$\frac{p_{32}}{q_{32}} = \frac{124622671453435719 57}{390609165135935769 26}$$

of τ exceeds $6M$. Therefore, we can say that inequalities (5.19) have no solution for

$$k = w \geq \frac{\log(Aq_{32}/\varepsilon)}{\log \delta} \geq \frac{\log(Aq_{32}/0.000971797)}{\log \delta} \geq 61.105\,.$$

We conclude that

$$k \leq 61\,.$$

Thus, it remains to check that (5.13) holds for $1 \le n \le 47$, $1 \le k \le 61$, and $0 \le m \le 488$. A quick inspection using Mathematica reveals that Diophantine equation (5.13) has only the solutions listed in part 1) of Theorem 5.3.

2) Theorem 1.2 with $n \le k$ leads to

$$B := \frac{\log \delta}{\log \alpha}(n+k) + 2 < 8k\,. \tag{5.20}$$

By combining (5.20) with (1.14), we see that

$$k < 3.82 \times 10^{28} \left(\log 8k\right)^2 \,, \quad \text{for} \quad k \ge 2\,. \tag{5.21}$$

Applying Lemma 2.2 with $l = 2$, $L = 8k$ and $H = 3.06{\cdot}10^{29}$, we obtain $k \le 7.06{\cdot}10^{32}$. Now, let us try to reduce the upper bound of k by applying Lemma 2.4. From (4.3), we can put

$$z_7 := m \log \alpha - (n+k)\log \delta + \log \Delta\,.$$

If $z_7 > 0$, it is easy to see that

$$|z_7| = z_7 < e^{z_7} - 1 = |1 - e^{z_7}| < 11 \cdot \delta^{-2n}$$

and if $z_7 < 0$, we get

$$0 < |z_7| < 18.4 \cdot \delta^{-2n}\,.$$

So, we obtain the following inequalities which are true in all cases for $n \ge 2$

$$0 < \left| m\frac{\log \alpha}{\log \delta} - (n+k) + \frac{\log 8}{\log \delta}\right| < 20.9 \cdot \delta^{-2n}\,. \tag{5.22}$$

We apply Lemma 2.4 to inequalities (5.22) with the following data

$$\tau := \frac{\log \alpha}{\log \delta}\,, \quad \mu := \frac{\log 8}{\log \delta}\,, \quad A := 20.9\,, \quad B := \delta\,, \quad M := 5.7 \times 10^{33}\,,$$

and $w := 2n$. Therefore, using Mathematica we found that the denominator of the 61-th convergent

$$\frac{p_{61}}{q_{61}} = \frac{147479251798514830052840594406506 65}{4622493383269723869514096531532964 1}$$

of τ exceeds $6M$. It follows that inequalities (5.22) have no solution for

$$2n = w \ge \frac{\log(Aq_{61}/\varepsilon)}{\log \delta} \ge \frac{\log(Aq_{61}/0.293383)}{\log \delta} \ge 95.4021\,.$$

Thus, we have

$$n \le 47\,.$$

By inserting this upper bound of n in (4.10), we get

$$k < 6.37 \times 10^{15}(1 + \log 8k) + 1.6\,,$$

which leads to $k < 2.76 \cdot 10^{17}$. Put

$$z_8 := m \log \alpha - k \log \delta + \log\left(\frac{\sqrt{8}}{\mathcal{P}_n}\right).$$

One can see that

$$0 < \left| m\frac{\log\alpha}{\log\delta} - k + \frac{\log\left(\sqrt{8}/\mathcal{P}_n\right)}{\log\delta} \right| < 6.1\cdot\delta^{-k}\,. \tag{5.23}$$

Note that $m < 8k < 2.21\cdot 10^{18}$. So considering the fact that $n \le 47$, we apply Lemma 2.4 to inequality (5.23) with the following data

$$\tau := \frac{\log\alpha}{\log\delta}\,,\quad \mu := \frac{\log\left(\sqrt{8}/\mathcal{P}_n\right)}{\log\delta}\,,\quad A := 6.1,\quad B := \delta\,,\quad M := 2.21\cdot 10^{18}\,,$$

and $w := k$. Thus, with the help of Mathematica we found that the denominator of the 32-nd convergent

$$\frac{p_{32}}{q_{32}} = \frac{12462267145343571957}{39060916513593576926}$$

of τ exceeds $6M$. Therefore, we can say that the inequalities (5.23) have no solution for

$$k = w \ge \frac{\log(Aq_{32}/\varepsilon)}{\log\delta} \ge \frac{\log(Aq_{32}/0.0406095)}{\log\delta} \ge 56.87\,.$$

Therefore, $k \le 56$ holds in all cases. Thus, it remains to check that equation (5.14) holds for $1 \le n \le 47$, $1 \le k \le 56$, and $0 \le m \le 448$. By a fast computation with Mathematica in these ranges, we see that the Diophantine equation (5.14) has only the solutions listed in part 2) of Theorem 5.3. This completes the proof of Theorem 5.3. □

From Theorem 5.3, we deduce the following result.

Corollary 5.4.

1) *The only solutions of $P_m = \mathcal{P}_n^2$ in non-negative integers n, m are $P_0 = \mathcal{P}_1^2 = 1$, $P_1 = \mathcal{P}_1^2 = 1$, $P_2 = \mathcal{P}_1^2 = 1$, and $P_6 = \mathcal{P}_2^2 = 4$.*
2) *The Diophantine equation $E_m = \mathcal{P}_n^2$ has no solution in non-negative integers n, m such that $n \ge 1$.*

5.3. **The balancing sequence.** The positive integer solutions n of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+\mathcal{R})$$

are called *balancing numbers* with corresponding balancers $\mathcal{R}$. The sequence of *balancing numbers* is denoted by $(B_n)_{n\ge 0}$ and can be viewed as a particular Lucas sequence with $(r,s) = (6,-1)$. In this case we have $\Delta = 32$ and $\delta = 3 + 2\sqrt{2}$. Here is our result.

Theorem 5.5.

1) *The only solutions (m, n, k) of the Diophantine equation*

$$P_m = B_nB_k \tag{5.24}$$

in non-negative integers m, n and k with $1 \le n \le k$ are $(0,1,1)$, $(1,1,1)$, and $(2,1,1)$.

2) *The Diophantine equation*

$$E_m = B_n B_k \tag{5.25}$$

has no solution in non-negative integers m, n, k with $1 \leq n \leq k$.

Proof. 1) Using Theorem 1.1 and since $n \leq k$, for $k \geq 2$, we get

$$B := \frac{\log \delta}{\log \alpha}(n+k) + 3 < 15k\,. \tag{5.26}$$

Combining (5.15) with (1.12), we get

$$k < 1.5 \times 10^{29} \left(\log 15k\right)^2 \,, \quad \text{for} \quad k \geq 2\,. \tag{5.27}$$

We now apply Lemma 2.2 with $l = 2$, $L = 15k$ and $H = 2.3 \cdot 10^{30}$ and get $k < 3 \times 10^{33}$, a large effective upper bound for k. So, in order to apply Lemma 2.4 for reducing this upper bound of k we put

$$z_9 := m \log \alpha - (n+k) \log \delta + \log 32a\,.$$

Following the same procedure as $z_9 < 0$ or $z_9 > 0$ while considering the relation (3.3), we get the inequalities

$$0 < |m \log \alpha - (n+k) \log \delta + \log 32a| < 38.89 \cdot \delta^{-2n} \quad \text{for} \quad n \geq 2.$$

Dividing these inequalities by $\log \delta$, we get

$$0 < \left| m \frac{\log \alpha}{\log \delta} - (n+k) + \frac{\log 32a}{\log \delta} \right| < 22.1 \cdot \delta^{-2n}\,. \tag{5.28}$$

Note that $m < 15k < 4.5 \times 10^{34}$. So, we choose the following data

$$\tau := \frac{\log \alpha}{\log \delta}\,, \quad \mu := \frac{\log 32a}{\log \delta}\,, \quad A := 22.1\,, \quad B := \delta\,, \quad M := 4.5 \times 10^{34}\,,$$

and $w := 2n$ to apply Lemma 2.4 to inequalities (5.28). We use Mathematica to apply Lemma 2.4 and we found that the denominator of the 68-th convergent

$$\frac{p_{68}}{q_{68}} = \frac{96670935546009581734992195271063798 3}{605998137997637025634218535626811911 0}$$

of τ exceeds $6M$. Therefore, the inequalities (5.28) have no solution for

$$2n = w \geq \frac{\log(Aq_{68}/\varepsilon)}{\log \delta} \geq \frac{\log(Aq_{68}/0.427688)}{\log \delta} \geq 50.285\,.$$

So

$$n \leq 25\,.$$

Substituting this upper bound for n into (3.10), we obtain

$$k < 7.1 \times 10^{15} \cdot (1 + \log 15k) + 1.08\,,$$

which implies $k < 3.13 \times 10^{17}$. We put now

$$z_{10} := m \log \alpha - k \log \delta + \log\left(\frac{a\sqrt{32}}{B_n}\right).$$

By (3.9), it is easy to show that

$$0 < |z_{10}| < 8.33 \cdot \delta^{-k}\,,$$

which leads to

$$0 < \left| m\frac{\log\alpha}{\log\delta} - k + \frac{\log\left(a\sqrt{32}/B_n\right)}{\log\delta}\right| < 4.73\cdot\delta^{-k}\,. \tag{5.29}$$

We have $m < 15k < 4.7\cdot 10^{18}$. We can apply Lemma 2.4 to inequalities (5.29) with

$$\tau := \frac{\log\alpha}{\log\delta}\,,\quad \mu := \frac{\log\left(a\sqrt{32}/B_n\right)}{\log\delta}\,,\quad A := 4.73\,,\quad B := \delta\,,\quad M := 4.7\cdot 10^{18}\,,$$

and $w := k$ by considering $n \le 25$. Thus, with the help of Mathematica we found that the denominator of the 39-th convergent

$$\frac{p_{39}}{q_{39}} = \frac{8102922801676034924}{5079454442454669015 7}$$

of τ exceeds $6M$. Therefore, we can say that the inequalities (5.29) have no solution for

$$k = w \ge \frac{\log(Aq_{39}/\varepsilon)}{\log\delta} \ge \frac{\log(Aq_{39}/0.0191931)}{\log\delta} \ge 28.8649\,.$$

It follows that

$$k \le 28\,.$$

Thus, it remains to check that (5.24) holds for $1 \le n \le 25$, $1 \le k \le 28$ and $0 \le m \le 420$. A quick inspection using Mathematica in the ranges $1 \le n \le 25$, $1 \le k \le 28$, and $0 \le m \le 420$ reveals that the Diophantine equation (5.24) has only the solutions listed in part 1) of Theorem 5.5.

2) The proof in this case is similar to those of cases 2) of Theorems 5.1 and 5.3. Thus, we omit the details of the calculations. But at the end we get $1 \le k \le 28$, $1 \le n \le 25$, and $0 \le m \le 420$. Inspecting solutions of (5.25) in these ranges yields no solution. This proves Theorem 5.5. □

Corollary 5.6. *The number* 1 *is the only Padovan number that is both a balancing number and a product of two balancing numbers.*

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