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WEBSTER PSEUDO-TORSION FORMULAS OF CR MANIFOLDS

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Abstract. In this article, we obtain a formula for Webster pseudo-torsion for the link of an isolated singularity of a n-dimensional complex subvariety in $\mathbb{C}^{n+1}$ and we present an alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$.

1. Introduction

The complete local invariants in the pseudoconformal geometry of a nondegenerate CR manifold M are defined on an $SU(p+1,q+1)$-bundle Y over M, which generalizes the bundle of Q-frame as a real hyperquadric [1]. To reduce the structure group, Webster singles out a real nowhere vanishing one form θ on M which annihilates the CR structure of M. A CR manifold M with such a choice θ is called a pseudohermitian manifold [6]. The contact form θ is called a pseudohermitian structure. The structure group of the Chern bundle Y is reduced to $U(p,q)$. In [6], Webster showed there is a natural connection in the bundle $T^{1,0}M$ adapted to θ. This connection can be extended to a connection to $\mathbb{C}TM$. To solve the equivalence problem of pseudohermitian manifold, Webster derived the structure equations for M, from which the Webster Ricci curvature and Webster torsion tensor are defined. In [3], the author derived a formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$. In this article, we derive a formula for Webster pseudo-torsion for the link of an isolated singularity of a n-dimensional complex subvariety in $\mathbb{C}^{n+1}$ and we present an alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$ [3]. The main idea of the alternative proof is to describe the CR structure using all Euclidean coordinates $z^1, z^2, \dots, z^{n+1}$ (see (39)). This new description of CR structure using all Euclidean coordinates is originated in [4]. In other words, we dispense with distinguishing one coordinate, say z^{n+1}, such that $\frac{\partial r}{\partial z^{n+1}} \neq 0$, as is required in Chern-Moser and subsequent works. The organization of this article is as follows. In Section 2, we review pseudohermitian geometry following Webster and Tanaka. In Section 3, we derive a key identity for Webster pseudo-torsion computation in subsequent

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sections. In Section 4, we present the alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$. In Section 5, we obtain an explicit formula for Webster pseudo-torsion for the link of an isolated singularity of a n-dimensional complex subvariety in $\mathbb{C}^{n+1}$. To the best knowledge of the author, this formula obtained in Section 5 is a new result.

2. Pseudohermitian structures

In this section, we collect the basic facts on pseudohermitian geometry. Let M be a CR manifold with structure bundle $T^{1,0}M$ satisfying $T^{1,0}M \cap \overline{T^{1,0}} = \{0\}$ and $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$. Let $T^{0,1}M := \overline{T^{1,0}}$. Set $HM = \text{Re}\,(T^{1,0}M \oplus T^{0,1}M)$. HM is a $2n$ dimensional subbundle of TM which carries a complex structure $J\colon HM \to HM$ given by $J(X+\overline{X}) = i(X-\overline{X})$ for $X \in T^{1,0}M$. Let $E \subset TM^*$ denote the real line subbundle which annihilates HM. Assuming M is orientable, E has a global nowhere vanishing section θ. A choice of such a 1-form θ is called a *pseudohermitian structure* on M. The *Levi form* of θ is the Hermitian form L_θ on $TM^{1,0}$ defined by

$$L_\theta(V,\overline{W}) = L_\theta(\overline{W},V) = -2\,\mathrm{i}\mathrm{d}\,\theta(V \wedge \overline{W})\,.$$

For a nondegenerate (resp. strongly pseudoconvex) CR manifold, L_θ is a nondegenerate (resp. positive definite) Hermitian form for any choice of θ. The choice of θ determines a unique real vector field ξ transverse to HM such that $\theta(\xi) = 1$, $\xi \rfloor d\theta = 0$. An *admissible coframe* on an open subset of M is a set of complex $(1,0)$-forms $\{\theta^1, \dots, \theta^n\}$ form basis for $TM*^{1,0}$ and satisfies $\theta^\alpha(\xi) = 0$. Then we have $d\theta = ih_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}$ for some hermitian matrix of functions $h_{\alpha\bar\beta}$. In [6], Webster showed there is a natural connection in the bundle $T^{1,0}M$ adapted to θ. This connection can be extended to a connection to $\mathbb{C}TM$. Webster showed that there are uniquely determined 1-forms ω_α^β, τ^β on M satisfying

$$d\theta = i\theta^\gamma \wedge \theta^{\overline{\gamma}}\,, \tag{1}$$

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta^{\ \alpha} + \theta \wedge \tau^\alpha\,, \tag{2}$$

$$\omega_{\alpha.}^{\ \beta} + \omega_{\overline{\beta}.}^{\ \overline{\alpha}} = 0\,, \quad \text{where } \omega_{\overline{\beta}.}^{\ \overline{\alpha}} = \overline{\omega_{\alpha.}^{\ \beta}}\,, \tag{3}$$

$$\tau^{\overline{\alpha}} = A_{\alpha\gamma}\theta^\gamma, \quad \text{where } \tau^{\overline{\alpha}} = \overline{\tau^\alpha}, \tag{4}$$

with

$$A_{\alpha\gamma} = A_{\gamma\alpha}\,, \tag{5}$$

and

$$h_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}\,. \tag{6}$$

This connection is called Webster connection. The curvature of the Webster connection, expressed in terms of the coframe is,

$$\begin{aligned}\Omega_{\beta.}^{\ \alpha} :&= d\omega_{\beta.}^{\ \alpha} - \omega_{\beta.}^{\ \gamma} \wedge \omega_{\gamma.}^{\ \alpha} - i\theta^{\bar\beta} \wedge \tau^\alpha + i\tau^{\bar\beta} \wedge \theta^\alpha, \\ &= R_{\beta\overline{\alpha}\rho\overline{\sigma}}\theta^\rho \wedge \theta^{\overline{\sigma}} + W_{\beta\overline{\alpha}\rho}\theta^\rho \wedge \theta - W_{\overline{\alpha}\beta\overline{\sigma}}\theta^{\overline{\sigma}} \wedge \theta\end{aligned} \tag{7}$$

where

$$R_{\beta\overline{\alpha}\rho\overline{\sigma}} = \overline{R}_{\alpha\bar{\beta}\sigma\bar{\rho}} = R_{\overline{\alpha}\beta\overline{\sigma}\rho}\,, \tag{8}$$

$$R_{\beta\overline{\alpha}\rho\overline{\sigma}} = R_{\rho\overline{\alpha}\beta\overline{\sigma}}\,, \tag{9}$$

$$W_{\overline{\alpha}\rho\overline{\sigma}} = W_{\overline{\sigma}\rho\overline{\alpha}}\,, \tag{10}$$

since by (6), $\Omega_{\beta.}^{\ \alpha} = \Omega_{\beta\overline{\alpha}}$. By (4), (7), we have

$$\begin{aligned} d\omega_{\beta.}^{\ \alpha} - \omega_{\beta.}^{\ \gamma} \wedge \omega_{\gamma.}^{\ \alpha} = -iA_{\beta\gamma}\theta^{\gamma} \wedge \theta^{\alpha} + R_{\beta\overline{\alpha}\rho\overline{\sigma}}\theta^{\rho} \wedge \theta^{\overline{\sigma}} + i\overline{A_{\alpha\gamma}}\theta^{\bar{\beta}} \wedge \theta^{\overline{\gamma}} \\ + W_{\beta\overline{\alpha}\rho}\theta^{\rho} \wedge \theta - W_{\overline{\alpha}\beta\overline{\sigma}}\theta^{\overline{\sigma}} \wedge \theta\,. \end{aligned} \tag{11}$$

We also put

$$\begin{aligned} \Omega^{\alpha} :&= d\tau^{\alpha} - \tau^{\beta} \wedge \omega_{\beta}^{\ \alpha}\,, \\ &= W_{\overline{\alpha}\rho\overline{\sigma}}\theta^{\rho} \wedge \theta^{\overline{\sigma}} - A_{\overline{\alpha}\overline{\gamma}}\tau^{\overline{\gamma}} \wedge \theta + B_{\overline{\alpha}\overline{\sigma}}\theta^{\overline{\sigma}} \wedge \theta\,, \end{aligned} \tag{12}$$

where

$$B_{\overline{\alpha}\overline{\sigma}} = B_{\overline{\sigma}\overline{\alpha}}\,. \tag{13}$$

Let $(\xi, X_{\alpha}, X_{\overline{\alpha}})$ be the dual frame to $(\theta, \theta^{\alpha}, \theta^{\overline{\alpha}})$. Define an operator D locally by

$$DX_{\alpha} = \omega_{\alpha}^{\ \beta} X_{\beta}\,, \quad D\colon \Gamma\big(H(M)\big) \to \Gamma\big((T^{*}(M) \otimes H(M)\big)\,. \tag{14}$$

D defines a connection on $H(M)$, see [6, p. 32]. We can define an hermitian metric $(\ \ ,\overline{\ \ })$ in the fibres of $H(M)$ by

$$(X_{\alpha}, \overline{X}_{\beta}) = \delta_{\alpha}^{\ \beta}\,. \tag{15}$$

Next, we turn to a formulation of the Webster connection by N. Tanaka [5]. We have $T^{1,0}M = \{X - iJX \mid X \in HM\}$ and using the decomposition $\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M \oplus \mathbb{C}\xi$, we extend J to $\mathbb{C}TM$ with $J\xi = 0$. Then we have

$$J^{2}X = -X + \theta(X)\xi\,, \quad X \in TM_{x}\,. \tag{16}$$

For, let $\mathrm{pr}\colon \mathbb{C}TM \to \mathbb{C}HM$ be the natural projection. Any $Y \in \mathbb{C}TM$ can be written as $Y = \mathrm{pr}(Y) + \theta(Y)\xi$ Then $J^{2}Y = -\mathrm{pr}(Y) = -Y + \theta(Y)\xi$. We put

$$\Omega = -d\theta\,. \tag{17}$$

We define a tensor field on M by

$$g(X,Y) = \Omega(JX, Y)\,. \tag{18}$$

Then $g(X,Y) = g(Y,X)$, $g(JX,JY) = g(X,Y)$ and g is positive definite on HM. Recall $T(X,Y) = \nabla_{X}Y - \nabla_{Y}X - [X,Y]$.

Theorem 2.1 (N. Tanaka [5, p. 29])**.** *There exists a unique affine connection*

$$\nabla\colon \Gamma(TM) \to \Gamma(TM \otimes TM^{*})$$

on M such that

(1) *The contact structure HM is parallel, i.e.,*

$$\nabla_{X}\Gamma(HM) \subset \Gamma(HM) \quad \textit{for any} \quad X \in \Gamma(TM)\,. \tag{19}$$

(2) *The tensor field* ξ, J, Ω *are all parallel, i.e.,* $\nabla\xi = \nabla J = \nabla\Omega = 0$. *(It follows that* $\nabla\theta = \nabla g = 0$.*)*

(3) *The torsion* T *of* ∇ *satisfies:*

$$\begin{aligned} T(X,Y) &= -\Omega(X,Y)\xi\,, \\ T(\xi, JY) &= -JT(\xi,Y)\,, \quad X,\, Y \in HM_x\,. \end{aligned}$$

Let $X,\, Y \in \Gamma(\mathbb{C}HM)$. Denote by $[X,Y]_{HM}$ the $\mathbb{C}HM$-component of $[X,Y]$ in the decomposition:

$$\mathbb{C}TM = \mathbb{C}HM \oplus \mathbb{C} \otimes (TM/HM)\,.$$

Also denote by $[X,Y]_{1,0}$ (resp. by $[X,Y]_{0,1}$) the $TM^{1,0}$ component (resp. the $\overline{TM^{1,0}}$-component) of $[X,Y]_{HM}$ in the decomposition $\mathbb{C}HM = TM^{1,0} \oplus TM^{0,1}$. ∇ can be extended to a differential operator of $\Gamma(\mathbb{C}TM)$ to $\Gamma(\mathbb{C}TM) \otimes \mathbb{C}TM^*$ in a natural manner. By (19), $\nabla J = 0$ and $T^{1,0}M = \{X - iJX \mid X \in HM\}$, we have

$$\begin{aligned} &\nabla_X \Gamma(TM^{1,0}) \subset \Gamma(TM^{1,0})\,, \\ &\nabla_X \Gamma(TM^{0,1}) \subset \Gamma(TM^{0,1})\,, \quad X \in \Gamma(\mathbb{C}TM)\,. \end{aligned}$$

Then we have

Proposition 2.2 ([5, p. 31])**.** *The extension* $\nabla\colon \Gamma(\mathbb{C}TM) \to \Gamma(\mathbb{C}TM \otimes \mathbb{C}TM^*)$ *is given as follows. For* $X,\, Y \in \Gamma(TM^{1,0})$,

(20) $\quad \nabla_{\overline{X}} Y = [\overline{X}, Y]_{1,0}\,,$

(21) $\quad \nabla_X Y$ *is given by* $\Omega(\nabla_X Y, \overline{Z}) = X\Omega(Y, \overline{Z}) - \Omega(Y, \overline{\nabla_{\overline{x}} Z})\ \forall Z \in \Gamma(TM^{1,0})\,,$

(22) $\quad \nabla_\xi Y = [\xi, Y] - \dfrac{1}{2} J([\xi, JY] - J[\xi, Y]) = [\xi, Y]_{1,0}\,.$

$\nabla_X \overline{Y}$, $\nabla_{\overline{X}} \overline{Y}$, $\nabla_\xi \overline{Y}$ *are given by conjugations, and* $\nabla_X \xi$, $\nabla_{\overline{X}} \xi$, $\nabla_\xi \xi$ *are all zero.*

In the following, we shall identify ∇ with Webster's D. We have

$$\begin{aligned} D_{\overline{X}_\beta} X_\alpha &= \omega_\alpha^\gamma(\overline{X}_\beta) X_\gamma \overset{(2)}{=} d\theta^\gamma(X_\alpha, \overline{X}_\beta) X_\gamma \\ &= -\theta^\gamma([X_\alpha, \overline{X}_\beta]) X_\gamma = [\overline{X}_\beta, X_\alpha]_{1,0} = \nabla_{\overline{X}_\beta} X_\alpha\,. \end{aligned}$$

And we check that

$$\begin{aligned} -d\theta(D_{X_\beta} X_\alpha, \overline{X}_\gamma) &= -i\theta^\rho \wedge \theta^{\overline{\rho}}\big(\omega_\alpha^\sigma(X_\beta) X_\sigma, \overline{X}_\gamma\big) = -i\omega_\alpha^\gamma(X_\beta) = i\overline{\omega_\gamma^\alpha}(X_\beta) \\ &= X_\beta\big(-i\theta^\rho \wedge \theta^{\overline{\rho}}(X_\alpha, \overline{X}_\gamma)\big) + i\theta^\rho \wedge \theta^{\overline{\rho}}\big(X_\alpha, \overline{\omega_\gamma^\sigma}(X_\beta)\overline{X}_\sigma\big) \\ &= X_\beta\big(-d\theta(X_\alpha, \overline{X}_\gamma)\big) - (-d\theta)(X_\alpha, \overline{\nabla_{\overline{X}_\beta} X_\gamma}) \quad \text{for all } X_\gamma\,. \end{aligned}$$

Hence, $D_{X_\beta} X_\alpha = \nabla_{X_\beta} X_\alpha$. We also have

$$D_\xi X_\alpha = \omega_\alpha^\gamma(\xi) X_\gamma \overset{(2)}{=} -d\theta^\gamma(\xi, X_\alpha) X_\gamma = \theta^\gamma([\xi, X_\alpha]) X_\gamma = [\xi, X_\alpha]_{1,0} = \nabla_\xi X_\alpha\,.$$

Then we identify the torsion terms. We have

$$\begin{aligned}
T(X_\alpha, \overline{X}_\beta) &= \nabla_{X_\alpha}\overline{X}_\beta - \nabla_{\overline{X}_\beta} X_\alpha - [X_\alpha, \overline{X}_\beta] \\
&= [X_\alpha, \overline{X}_\beta]_{0,1} + [X_\alpha, \overline{X}_\beta]_{1,0} - [X_\alpha, \overline{X}_\beta] \\
&= -\theta([X_\alpha, \overline{X}_\beta])\xi \\
&= d\theta(X_\alpha, \overline{X}_\beta)\xi \\
&= i\delta_\alpha^{\ \beta}\xi = -\Omega(X_\alpha, \overline{X}_\beta)\xi,
\end{aligned}$$

and

$$\begin{aligned}
T(X_\alpha, X_\beta) &= (\omega_\beta^{\ \gamma}(X_\alpha) - \omega_\alpha^{\ \gamma}(X_\beta) - \theta^\gamma([X_\alpha, X_\beta]))X_\gamma \\
&= (\omega_\beta^{\ \gamma}(X_\alpha) - \omega_\alpha^{\ \gamma}(X_\beta) + d\theta^\gamma(X_\alpha, X_\beta))X_\gamma = 0\,,
\end{aligned}$$

and

$$\begin{aligned}
T(\xi, X_\alpha) &= \nabla_\xi X_\alpha - \nabla_{X_\alpha}\xi - [\xi, X_\alpha] \\
&= [\xi, X_\alpha]_{1,0} - [\xi, X_\alpha] \\
&= -\theta^{\bar\beta}([\xi, X_\alpha])\overline{X}_\beta - \theta([\xi, X_\alpha])\xi \\
&= (\theta^{\overline{\gamma}} \wedge \overline{\omega_\gamma^{\ \beta}} + \theta \wedge \tau^{\bar\beta})(\xi, X_\alpha)\overline{X}_\beta \\
&= \tau^{\bar\beta}(X_\alpha)\overline{X}_\beta \\
&= A_{\alpha\beta}\overline{X}_\beta\,.
\end{aligned}$$

Finally, we identify the curvatures terms. We have

$$\begin{aligned}
R(Y,Z)X_\beta &= \nabla_Y\nabla_Z X_\beta - \nabla_Z\nabla_Y X_\beta - \nabla_{[Y,Z]}X_\beta \\
&= ((Y\omega_\beta^{\ \alpha}(Z) + \omega_\beta^{\ \gamma}(Z)\omega_\gamma^{\ \alpha}(Y)) - \big(Z\omega_\beta^{\ \alpha}(Y) + \omega_\beta^{\ \gamma}(Y)\omega_\gamma^{\ \alpha}(Z)\big) \\
&\quad - \omega_\beta^{\ \alpha}([Y,Z]))X_\alpha \overset{(11)}{=} \big((d\omega_\beta^{\ \alpha} - \omega_\beta^{\ \gamma} \wedge \omega_\gamma^{\ \alpha})(Y,Z)\big)X_\alpha\,,
\end{aligned}$$

and

$$\begin{aligned}
R(X_\rho, X_\sigma)X_\beta &= (-iA_{\beta\gamma}\theta^\gamma \wedge \theta^\alpha)(X_\rho, X_\sigma)X_\alpha \\
&= -iA_{\beta\gamma}(\delta_\rho^\gamma\delta_\sigma^\alpha - \delta_\sigma^\gamma\delta_\rho^\alpha)X_\alpha \\
&= -i(A_{\beta\rho}X_\sigma - A_{\beta\sigma}X_\rho)\,, \\
R(X_\rho, \overline{X}_\sigma)X_\beta &= R_{\beta\overline{\alpha}\rho\overline{\sigma}}X_\alpha\,, \\
R(\overline{X}_\rho, \overline{X}_\sigma)X_\beta &= (i\overline{A_{\alpha\gamma}}\theta^{\bar\beta} \wedge \theta^{\overline{\gamma}})(\overline{X}_\rho, \overline{X}_\sigma)X_\alpha \\
&= i\overline{A_{\alpha\gamma}}(\delta_\beta^\rho\delta_\gamma^\sigma - \delta_\beta^\sigma\delta_\gamma^\rho)X_\alpha \\
&= i(\delta_\beta^\rho\overline{A_{\alpha\sigma}} - \delta_\beta^\sigma\overline{A_{\alpha\rho}})X_\alpha\,, \\
R(X_\rho, \xi)X_\beta &= W_{\beta\overline{\alpha}\rho}X_\alpha\,, \\
R(\overline{X}_\sigma, \xi)X_\beta &= -W_{\overline{\alpha}\beta\overline{\sigma}}X_\alpha\,.
\end{aligned}$$

3. A key identity for Webster pseudo-torsion computation

In this section, we obtain a key identity (53) for Webster pseudo-torsion computation in Section 5.

Let M be the boundary of a strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Let r be a smooth real-valued defining function of M i.e. $M = \{r = 0\}$ and $dr \neq 0$. Throughout this section, the range of indices are: $0 \leq i, j, k \cdots \leq n+1$, $0 \leq \alpha, \beta, \gamma \cdots \leq n$. Coordinates for $\mathbb{C}^{n+1}$ will be given by $(z_1, z_2, \ldots, z_{n+1})$. We will use the conventions: $r_j = \frac{\partial r}{\partial z^j}$, $r_{j\overline{k}} = \frac{\partial^2 r}{\partial z^j \partial \overline{z}^k}$. The CR structure is on M is given by

$$T^{1,0}M = \{X = x_j \frac{\partial}{\partial z^j} : dr(X) = x^j r_j = 0\} . \tag{23}$$

We define a $2n$ dimensional subbundle of TM by

$$\mathbb{C}HM = T^{1,0}M \oplus T^{0,1}M \quad \text{where} \quad T^{0,1}M := \overline{T^{1,0}M} , \tag{24}$$

and $HM := \text{Re } (T^{1,0}M \oplus T^{0,1}M)$. HM carries a complex structure map

$$J \colon HM \to HM, \quad J^2 = -Id , \tag{25}$$

and we denote its extension to $\mathbb{C}TM$ by J,

$$J \colon \mathbb{C}HM \to \mathbb{C}HM, \; J^2 = -Id \text{ and } J|_{T^{1,0}M} = \text{ multiplication by } i = \sqrt{-1} . \tag{26}$$

Define a one form θ on $\mathbb{C}^{n+1}$ by

$$\theta = -i\partial r = -ir_j dz^j . \tag{27}$$

On CTM, θ is a real one form annihilating $T^{1,0}M \oplus T^{0,1}M$,

$$\theta = i\partial r = i\overline{\partial} r = \frac{i}{2}(\overline{\partial} r - \partial r) . \tag{28}$$

For X, $Y \in T^{1,0}M$,

$$\begin{aligned} &\theta([X,Y]) = 0 , \quad \theta([\overline{X},\overline{Y}]) = 0 , \\ \text{and } \; &\theta([X,\overline{Y}]) = -d\theta(X,\overline{Y}) = -i\partial\overline{\partial} r(X,\overline{Y}) . \end{aligned} \tag{29}$$

For X, $Y \in T^{1,0}M$, the Levi form is given by

$$L_\theta(X,\overline{Y}) = \theta([JX,\overline{Y}]) = -d\theta(JX,\overline{Y}) = \partial\overline{\partial} r(X,\overline{Y}) . \tag{30}$$

M is said to be *strongly pseudoconvex* if $L_\theta(X,\overline{Y})$ is positive definite as a Hermitian form on $T^{1,0}M$. In other words,

$$\forall \; w^j \frac{\partial}{\partial z^j} \neq 0 , \quad w^j r_j = 0 \Rightarrow r_{j\overline{k}} w^j w^{\overline{k}} > 0 . \tag{31}$$

Note that the matrix $r_{j\overline{k}}$ is not necessary invertible though (31) is satisfied.

Example 3.1. The real hyperquadric in $\mathbb{C}^2$ given by

$$M := \{(z_1, z_2) \in \mathbb{C}^2 \mid r(z_1, z_2) = 0, \; r = z_1\overline{z}_1 - \frac{z_2 - \overline{z}_2}{2i}\} \quad \text{which is s.p.c.}$$

$T^{1,0}M$ is spanned by $\frac{\partial}{\partial z_1} + 2i\overline{z}_1 \frac{\partial}{\partial z_2}$. We see that

$$\begin{pmatrix} r_{1\overline{1}} & r_{1\overline{2}} \\ r_{2\overline{1}} & r_{2\overline{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{while} \quad \begin{pmatrix} 1 & 2i\overline{z}_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2iz_1 \end{pmatrix} = 1 .$$

Neither does s.p.c., (31) imply the positive definiteness of $r_{j\overline{k}}$, as we see from

Example 3.2. $M := \{(z_1, z_2) \in \mathbb{C}^2 \mid r(z_1, z_2) = 0,\ r = 1 + z_1\overline{z}_1 - z_2\overline{z}_2\}$ which is s.p.c.
$T^{1,0}M$ is spanned by $\overline{z}_2\frac{\partial}{\partial z_1} + \overline{z}_1\frac{\partial}{\partial z_2}$. We see that

$$\begin{pmatrix} r_{1\overline{1}} & r_{1\overline{2}} \\ r_{2\overline{1}} & r_{2\overline{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{while} \quad \begin{pmatrix} \overline{z}_2 & \overline{z}_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_2 \\ z_1 \end{pmatrix} = 1\,.$$

Let ξ be the unique real vector field on M such that

$$\theta(\xi) = 1\,, \tag{32}$$

$$\xi \rfloor d\theta = 0\,. \tag{33}$$

Let

$$\xi = \xi^j \frac{\partial}{\partial z^j} + \overline{\xi^j} \frac{\partial}{\partial \overline{z}^j}\,. \tag{34}$$

We have

$$\theta(\xi) = 1 \quad \text{means} \quad i r_{\overline{k}} \xi^{\overline{k}} = 1 \quad \text{or} \quad r_j \xi^j = i\,, \tag{35}$$

$$\xi \rfloor d\theta = 0 \quad \text{means} \quad x^j r_j = 0 \Rightarrow x^j r_{j\overline{k}} \xi^{\overline{k}} = 0\,. \tag{36}$$

Let $TM = HM \oplus \mathbb{R}\xi$, we extend (25),

$$J \colon TM \to TM \quad \text{by} \quad J\xi = 0\,. \tag{37}$$

Then, J as a $\binom{1}{1}$ tensor satisfies

$$J^2 X = -X + \theta(X)\xi \tag{38}$$

for all $X \in TM$. With J as a $\binom{1}{1}$ tensor, we regard $g(X, Y) := -d\theta(JX, Y) = L_\theta(X, Y)$ as $\binom{0}{2}$ tensor on TM. Note that , for $X, Y \in TM$, $\theta([JX, Y]) \neq -d\theta(JX, Y)$ since $\theta([X, Y])$ is not a tensor, for instance, we have $\theta([f\xi, \xi]) = \theta(\xi(f)\xi) = \xi(f)$. In the following, we write $\langle X, Y\rangle := g(X, \overline{Y})$. Choose $X_1, \ldots, X_n$ in $T^{1,0}_p M$ for some point p in M. Let

$$X_\alpha = x^j_\alpha \frac{\partial}{\partial z^j} \tag{39}$$

satisfying

$$x^j_\alpha r_j = 0, \tag{40}$$

$$x^j_\alpha r_{j\overline{k}} \overline{x^k_\beta} = \delta^\beta_\alpha\,. \tag{41}$$

Note that we use all Euclidean coordinates $z^1, \ldots, z^{n+1}$ in the description of the CR structure of M. In this way, we dispense with distinguishing one coordinate, say z^{n+1}, such that $\frac{\partial r}{\partial z^{n+1}} \neq 0$, as is required in Chern-Moser and subsequent works. Our computation is therefore symmetric in all $z^1, \ldots, z^{n+1}$. Write

$$J(u) := (-1)^{n+1} \det \begin{pmatrix} u & u_{\overline{1}} & \cdots & u_{\overline{n+1}} \\ u_1 & u_{1\overline{1}} & \cdots & u_{1\overline{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ u_{n+1} & u_{n+1\overline{1}} & \cdots & u_{n+1\overline{n+1}} \end{pmatrix}, \tag{42}$$

$$F := \begin{pmatrix} r & r_{\overline{1}} & \cdots & r_{\overline{n+1}} \\ r_1 & r_{1\overline{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ r_{n+1} & r_{n+1\overline{1}} & \cdots & r_{n+1\overline{n+1}} \end{pmatrix}, \tag{43}$$

and

$$\langle\langle\, \xi, \xi \,\rangle\rangle := \xi^j r_{j\overline{k}} \xi^{\overline{k}} \,. \tag{44}$$

Then, we have

$$-\langle\langle\, \xi, \xi \,\rangle\rangle r_j + i r_{j\overline{k}} \xi^{\overline{k}} = 0\,. \tag{45}$$

Proof of (45)**.** $(r_j dz^j)(X_\alpha) = x^j_\alpha r_j \overset{(40)}{=} 0$ and $(r_{j\overline{k}} \xi^{\overline{k}} dz^j)(X_\alpha) = x^j_\alpha r_{j\overline{k}} \xi^{\overline{k}} = 0$ for all α, implies that, since $dr \neq 0$, $r_{j\overline{k}} \xi^{\overline{k}} = b r_j$ for some b. By contraction with ξ^j, $\langle\langle\, \xi, \xi \,\rangle\rangle = bi$. Thus, we obtain (45). Write

$$a^{\overline{j}k} := \overline{x^j_\alpha} x^k_\alpha \,. \tag{46}$$

Then

$$r_{\overline{j}} a^{\overline{j}k} = 0\,. \tag{47}$$

Write

$$X_{n+1} := \xi^j \frac{\partial}{\partial z^j} \quad \text{and} \quad x^j_{n+1} = \xi^j\,. \tag{48}$$

Then

(49)

$$\begin{pmatrix} x^1_1 & \cdots & x^{n+1}_1 \\ \vdots & & \vdots \\ x^1_{n+1} & \cdots & x^{n+1}_{n+1} \end{pmatrix} \begin{pmatrix} r_{1\overline{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & & \vdots \\ r_{n+1\overline{1}} & \cdots & r_{n+1\overline{n+1}} \end{pmatrix} \begin{pmatrix} \overline{x^1_1} & \cdots & \overline{x^{n+1}_1} \\ \vdots & & \vdots \\ \overline{x^1_{n+1}} & \cdots & \overline{x^{n+1}_{n+1}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \langle\langle\, \xi, \xi \,\rangle\rangle \end{pmatrix}.$$

Write

$$\begin{pmatrix} y_1^1 & \cdots & y_1^{n+1} \\ \vdots & & \vdots \\ y_{n+1}^1 & \cdots & y_{n+1}^{n+1} \end{pmatrix} := \begin{pmatrix} x_1^1 & \cdots & x_1^{n+1} \\ \vdots & & \vdots \\ x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{pmatrix}^{-1}$$

$$\stackrel{(48)}{=} \begin{pmatrix} y_1^1 & \cdots & y_1^n & -ir_1 \\ \vdots & & \vdots & \vdots \\ y_{n+1}^1 & \cdots & y_{n+1}^n & -ir_{n+1} \end{pmatrix}. \tag{50}$$

Then

(51)

$$\begin{pmatrix} r_{1\bar{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & & \vdots \\ r_{n+1\bar{1}} & \cdots & r_{n+1\overline{n+1}} \end{pmatrix} \begin{pmatrix} \overline{x_1^1} & \cdots & \overline{x_1^{n+1}} \\ \vdots & & \vdots \\ \overline{x_{n+1}^1} & \cdots & \overline{x_{n+1}^{n+1}} \end{pmatrix} \begin{pmatrix} x_1^1 & \cdots & x_1^{n+1} \\ \vdots & & \vdots \\ x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} y_1^1 & \cdots & y_1^{n+1} \\ \vdots & & \vdots \\ y_{n+1}^1 & \cdots & y_{n+1}^{n+1} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & \langle\langle \xi, \xi \rangle\rangle \end{pmatrix} \begin{pmatrix} x_1^1 & \cdots & x_1^{n+1} \\ \vdots & & \vdots \\ x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} y_1^1 & \cdots & y_1^n & \langle\langle \xi, \xi \rangle\rangle y_1^{n+1} \\ \vdots & & \vdots & \vdots \\ y_{n+1}^1 & \cdots & y_{n+1}^n & \langle\langle \xi, \xi \rangle\rangle y_{n+1}^{n+1} \end{pmatrix} \begin{pmatrix} x_1^1 & \cdots & x_1^{n+1} \\ \vdots & & \vdots \\ x_{n+1}^1 & \cdots & x_{n+1}^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1-(1-\langle\langle \xi, \xi \rangle\rangle) y_1^{n+1} x_{n+1}^1 & \cdots & -(1-\langle\langle \xi, \xi \rangle\rangle) y_1^{n+1} x_{n+1}^{n+1} \\ \vdots & & \vdots \\ -(1-\langle\langle \xi, \xi \rangle\rangle) y_{n+1}^{n+1} x_{n+1}^1 & \cdots & 1-(1-\langle\langle \xi, \xi \rangle\rangle) y_{n+1}^{n+1} x_{n+1}^{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} 1+(1-\langle\langle \xi, \xi \rangle\rangle) ir_1 \xi^1 & \cdots & (1-\langle\langle \xi, \xi \rangle\rangle) ir_1 \xi^{n+1} \\ \vdots & & \vdots \\ (1-\langle\langle \xi, \xi \rangle\rangle) ir_{n+1} \xi^1 & \cdots & 1+(1-\langle\langle \xi, \xi \rangle\rangle) ir_{n+1} \xi^{n+1} \end{pmatrix}$$

i.e.

$$r_{i\bar{k}} \overline{x_l^k} x_l^j = \delta_i^j + (1-\langle\langle \xi, \xi \rangle\rangle) ir_i \xi^j \,.$$

By (46), (48),

$$r_{i\bar{k}} (a^{\bar{k}j} + \xi^{\bar{k}} \xi^j) = \delta_i^j + (1-\langle\langle \xi, \xi \rangle\rangle) ir_i \xi^j \,.$$

By (45),

$$r_{i\bar{k}} a^{\bar{k}j} - i\langle\langle \xi, \xi \rangle\rangle r_i \xi^j = \delta_i^j + ir_i \xi^j - i\langle\langle \xi, \xi \rangle\rangle r_i \xi^j \,.$$

Hence,

$$-ir_i \xi^j + r_{i\bar{k}} a^{\bar{k}j} = \delta_i^j \,. \tag{52}$$

By (35), (45), (47), (52),

(53)

$$\begin{pmatrix} r & r_{\overline{1}} & \cdots & r_{\overline{n+1}} \\ r_1 & r_{1\overline{1}} & \cdots & r_{1\overline{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ r_{n+1} & r_{n+1\overline{1}} & \cdots & r_{n+1\overline{n+1}} \end{pmatrix} \begin{pmatrix} -\langle\langle \xi, \xi \rangle\rangle & -i\xi^1 & \cdots & -i\xi^{n+1} \\ i\overline{\xi^1} & a^{\overline{1}1} & \cdots & a^{\overline{1}n+1} \\ \vdots & \vdots & & \vdots \\ i\overline{\xi^{n+1}} & a^{\overline{n+1}1} & \cdots & a^{\overline{n+1}n+1} \end{pmatrix} = I\,.$$

□

4. An alternative proof of the Li-Luk formula for Webster pseudo-torsion for a real hypersurface in $\mathbb{C}^{n+1}$

This section gives an alternative proof of the Li-Luk formula for Webster pseudo-torsion (for definition, see (69)) for a strongly pseudoconvex pseudohermitian hypersurface in $\mathbb{C}^{n+1}$. For the convenience of readers and fixing notations, we recall some facts and definitions in the beginning. We will also use some definitions and results in Section 2. Let M be a strongly pseudoconvex pseudohermitian hypersurface given by $M = \{z \in \mathbb{C}^{n+1} \mid r = 0\}$, where r is a real valued defining function for M and r is C^3 in a neighborhood of M. Let TM be the tangent bundle on M and let $HM := TM \cap iTM$, the holomorphic tangent bundle on M. As in the previous sections, we fix the real one form θ be a pseudohermitian structure on M. Let $\theta^1, \ldots, \theta^n, \overline{\theta^1}, \ldots, \overline{\theta^n}$ be a local admissible coframe for M, $1 \leq \alpha, \beta \leq n$. As before we use the convention $\theta^{\overline{\alpha}} := \overline{\theta^\alpha}$. Webster shows that there are uniquely determined 1-forms $\omega_{\alpha}^{\ \beta}, \tau^\beta$ on M satisfying

$$d\theta = i\theta^\gamma \wedge \theta^{\overline{\gamma}}\,, \tag{54}$$

$$d\theta^\alpha = \theta^\beta \wedge \omega_{\beta\cdot}^{\ \alpha} + \theta \wedge \tau^\alpha\,, \tag{55}$$

$$\omega_{\alpha\cdot}^{\ \beta} + \overline{\omega_{\beta}^{\ \alpha}} = 0\,, \tag{56}$$

$$\overline{\tau^\alpha} = A_{\alpha\gamma}\theta^\gamma\,, \tag{57}$$

$$A_{\alpha\gamma} = A_{\gamma\alpha}\,. \tag{58}$$

Let $\xi, X_1, \ldots, X_n, \overline{X_1}, \ldots, \overline{X_n}$ be the dual frame satisfying

$$\theta(\xi) = 1\,, \quad d\theta(\xi, \cdot) = i\theta^\gamma \wedge \theta^{\overline{\gamma}}(\xi, \cdot) = 0\,. \tag{59}$$

And we have

$$-\operatorname{id}\theta(X_\alpha, \overline{X_\beta}) = i\theta^\gamma \wedge \theta^{\overline{\gamma}}(X_\alpha, \overline{X_\beta}) = \delta_\alpha^\beta\,. \tag{60}$$

The Levi form L_θ on $TM^{1,0}$ is defined by $L_\theta(\cdot, \bar{\cdot}) := -\operatorname{id}\theta(\cdot, \bar{\cdot})$. Hence,

$$L_\theta(X_\alpha, \overline{X_\beta}) = \delta_\alpha^\beta =: \langle X_\alpha, \overline{X_\beta} \rangle\,. \tag{61}$$

Covariant differentiation is given by

$$\nabla X_\alpha = \omega_{\alpha}^{\ \beta} X_\beta\,, \quad \nabla X_\alpha = \omega_{\alpha}^{\ \beta} X_\beta\,, \quad \nabla\xi = 0\,. \tag{62}$$

We also have

$$\nabla_{\overline{X}_\gamma} X_\alpha = [\overline{X}_\alpha, X_\alpha]_{TM^{1,0}}\,, \tag{63}$$

and $\nabla_{X_\gamma} X_\alpha$ is defined by

$$\langle \nabla_{X_\gamma} X_\alpha, X_\beta \rangle = X_\gamma \langle X_\alpha, X_\beta \rangle - \langle X_\alpha, \nabla_{\overline{X}_\gamma} X_\beta \rangle \,. \tag{64}$$

We have

$$\nabla_\xi X_\alpha = [\xi, X_\alpha]_{TM^{1,0}} \,. \tag{65}$$

The torsion tensor is defined by $T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ for $X, Y \in \mathbb{C}TM$. We have

$$T(X_\alpha, \overline{Y}_\beta) = i\delta_\alpha^\beta \xi \,, \tag{66}$$

$$T(X_\alpha, X_\beta) = 0 \,, \tag{67}$$

$$T(\xi, X\alpha) = A_{\alpha\beta}\overline{X}_\beta \,. \tag{68}$$

The Webster pseudo-torsion is defined as [3],

$$\operatorname{Tor}(z)(U,V) = i(A_{\overline{\alpha}\overline{\beta}}\overline{u}^\alpha \overline{v}^\beta - A_{\alpha\beta}u^\alpha v^\beta) \,, \tag{69}$$

where $U = u^j \frac{\partial}{\partial z_j}, V = v^j \frac{\partial}{\partial z_j} \in H_z M$ and $z \in M$. We will use following notations.

$$J(r) := - \begin{vmatrix} r & r_{\overline{k}} \\ r_j & r_{j\overline{k}} \end{vmatrix} \,, \tag{70}$$

$$H(r) := (r_{j\overline{k}}) \,. \tag{71}$$

We shall prove the following theorem.

Theorem 4.1 ([3]). *Let M be a C^4 strongly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$. Let r be a defining function for M which is C^3 in a neigborhood of M. Consider the pseudohermitian structure defined by $\theta = -i\partial r$ on M. Then for any $U = u^j \frac{\partial}{\partial z_j}, V = v^j \frac{\partial}{\partial z_j} \in H_z M$ and $z \in M$, we have*

$$\operatorname{Tor}(z)(U,V) = 2\operatorname{Re}\left(\frac{\overline{u^l v^k}}{J(r)} \left(N - \det H(r)\right) r_{\overline{lk}} \right) \,, \tag{72}$$

where

$$N = \sum_i (-1)^{j+i} r_{\overline{i}} \begin{vmatrix} & | & \\ - & \mathbf{r_{i\bar{j}}} & - \\ & | & \end{vmatrix} \frac{\partial}{\partial z^{\overline{j}}} \,. \tag{73}$$

We will need some preliminaries to prove this theorem. First, by (53), we have

$$1 = r(-\langle\langle\, \xi, \xi \,\rangle\rangle) + r_1(-i\xi^1) + r_2(-i\xi^2) + \cdots + r_{n+1}(-i\xi^{n+1}) \,. \tag{74}$$

Expanding $-J(r)$ by the 1st column, we have

(75)

$$\begin{vmatrix} r & r_{\bar k} \\ r_j & r_{j\bar k} \end{vmatrix} = r \begin{vmatrix} r_{1\bar 1} & \cdots & r_{1\overline{n+1}} \\ \vdots & & \vdots \\ r_{n+1\bar 1} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix} - r_1 \begin{vmatrix} r_{\bar 1} & \cdots & r_{\overline{n+1}} \\ \mathbf{r_{1\bar 1}} & \cdots & \mathbf{r_{1\overline{n+1}}} \\ \vdots & & \vdots \\ r_{n+1\bar 1} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix} + \cdots$$

$$+ (-1)^{n+1} r_{n+1} \begin{vmatrix} r_{\bar 1} & \cdots & r_{\overline{n+1}} \\ r_{1\bar 1} & \cdots & r_{1\overline{n+1}} \\ \vdots & & \vdots \\ \mathbf{r_{n+1\bar 1}} & \cdots & \mathbf{r_{n+1\overline{n+1}}} \end{vmatrix} .$$

Hence, by (74), (75), we have

$$-\langle\langle\, \xi, \xi\,\rangle\rangle = \frac{|r_{j\bar k}|}{\begin{vmatrix} r & r_{\bar k} \\ r_j & r_{j\bar k} \end{vmatrix}} = -\frac{\det H(r)}{J(r)} \tag{76}$$

and

(77)

$$-i\xi^j = \frac{(-1)^j}{\begin{vmatrix} r & r_{\bar k} \\ r_j & r_{j\bar k} \end{vmatrix}} \begin{vmatrix} r_{\bar 1} & \cdots & r_{\overline{n+1}} \\ r_{1\bar 1} & \cdots & r_{1\overline{n+1}} \\ \mathbf{r_{j\bar 1}} & \cdots & \mathbf{r_{j\overline{n+1}}} \\ r_{n+1\bar 1} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix}$$

$$= \frac{(-1)^j}{-J(r)} \left(r_{\bar 1} \begin{vmatrix} \mathbf{r_{1\bar 1}} & \cdots & r_{1\overline{n+1}} \\ \mathbf{r_{j\bar 1}} & \cdots & \mathbf{r_{j\overline{n+1}}} \\ \mathbf{r_{n+1\bar 1}} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix} - r_{\bar 2} \begin{vmatrix} r_{1\bar 1} & \mathbf{r_{1\bar 2}} & \cdots & r_{1\overline{n+1}} \\ \mathbf{r_{j\bar 1}} & \mathbf{r_{j\bar 2}} & \cdots & \mathbf{r_{j\overline{n+1}}} \\ r_{n+1\bar 1} & \mathbf{r_{n+1\bar 2}} & \cdots & r_{n+1\overline{n+1}} \end{vmatrix} \right.$$

$$\left. + (-1)^n r_{\overline{n+1}} \begin{vmatrix} r_{1\bar 1} & \cdots & \mathbf{r_{1\overline{n+1}}} \\ \mathbf{r_{j\bar 1}} & \cdots & \mathbf{r_{j\overline{n+1}}} \\ r_{n+1\bar 1} & \cdots & \mathbf{r_{n+1\overline{n+1}}} \end{vmatrix} \right)$$

$$= \sum_{k=1}^{n+1} \frac{(-1)^j(-1)^{k+1}}{-J(r)} r_{\bar k} \begin{vmatrix} r_{1\bar 1} & \vert & r_{1\overline{n+1}} \\ - & \mathbf{r_{j\bar k}} & - \\ r_{n+1\bar 1} & \vert & r_{n+1\overline{n+1}} \end{vmatrix}$$

Proof of Theorem 4.1.

Step 1. We first find a relation between the torsion tensor T and the Webster torsion Tor. Let $U = \mu^\alpha X_\alpha, V = \nu^\beta X_\beta \in T^{1,0}M$. We have

$$\begin{aligned} \mathrm{Tor}(U,V) &= \mathrm{Tor}(\mu^\alpha X_\alpha, \nu^\beta X_\beta) \\ &= 2\mathrm{Re}(i\overline{A_{\alpha\beta}}) \\ &= 2\mathrm{Re}(i\langle\overline{T(\xi, X_\alpha)}, X_\beta\rangle)\overline{\mu^\alpha}\overline{\nu^\beta} \\ &= 2\mathrm{Re}(i\langle\overline{T(\xi, \mu^\alpha X_\alpha)}, \nu^\beta X_\beta\rangle) \\ &= 2\mathrm{Re}(i\langle\overline{T(\xi, U)}, V\rangle)\,. \end{aligned} \tag{78}$$

Step 2. We compute

$$
\begin{aligned}
T(\xi, U) &= \nabla_\xi U - \nabla_U \xi - [\xi, U] \\
&= [\xi, U]_{T^{1,0}M} - [\xi, U] \\
&= -[\xi, U]_{T^{0,1}M} \\
&= -\left[\xi^j \frac{\partial}{\partial z^j} + \overline{\xi^j} \frac{\partial}{\partial \overline{z^j}}, U\right]_{T^{0,1}M} \\
&= (U\overline{\xi^j}) \frac{\partial}{\partial \overline{z^j}}. \qquad (79)
\end{aligned}
$$

We check that $(U\overline{\xi^j})\frac{\partial}{\partial \overline{z^j}} \in T^{1,0}M$ as follows. Using $U = u^j \frac{\partial}{\partial z^j}$, we have

$$
(U\overline{\xi^j}) r_{\bar j} = U(\overline{\xi^j} r_{\bar j}) - \overline{\xi^j} U r_{\bar j} = -u^k r_{k\bar j} \overline{\xi^j} = 0.
$$

Step 3. Let $U = u^j \frac{\partial}{\partial z^j}, V = v^k \frac{\partial}{\partial z^k}$ such that $u^j r_j = 0$, $v^k r_k = 0$. Using (78), (79), we have

$$
\begin{aligned}
\operatorname{Tor}(U, V) &= 2\mathrm{Re}\left(i \left\langle \overline{(U\overline{\xi^j}) \frac{\partial}{\partial \overline{z^j}}}, v^k \frac{\partial}{\partial z^k} \right\rangle\right) \\
&= 2\mathrm{Re}\left(i \left\langle \overline{u^l} \frac{\partial \xi^j}{\partial \overline{z^l}} \frac{\partial}{\partial z^j}, v^k \frac{\partial}{\partial z^k} \right\rangle\right) \\
&= 2\mathrm{Re}\left(i \overline{u^l} \frac{\partial \xi^j}{\partial \overline{z^l}} r_{j\bar k} \overline{v^k}\right) \\
&= 2\mathrm{Re}\left(i \overline{u^l v^k} \left(\frac{\partial}{\partial \overline{z^l}} (\xi^j r_{j\bar k}) - \xi^j r_{j\overline{kl}}\right)\right) \\
&= 2\mathrm{Re}\left(\overline{u^l v^k} \left(\frac{\partial}{\partial \overline{z^l}} (a r_{\bar k}) - i\xi^j \frac{\partial r_{\overline{kl}}}{\partial z^j}\right)\right) \\
&= 2\mathrm{Re}\left(\overline{u^l v^k} \left(-\langle\langle \xi, \xi \rangle\rangle r_{\overline{lk}} - i\xi^j \frac{\partial r_{\overline{kl}}}{\partial z^j}\right)\right). \qquad (80)
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\operatorname{Tor}(U, V) &= 2\mathrm{Re}\left(\frac{\overline{u^l v^k}}{J(r)} \left(-|r_{i\bar j}| r_{\overline{lk}} + \sum_i (-1)^{j+i} r_{\bar i} \begin{vmatrix} & | & \\ - & \mathbf{r_{i\bar j}} & - \\ & | & \end{vmatrix} r_{j\overline{lk}}\right)\right) \\
&= 2\mathrm{Re}\left(\frac{\overline{u^l v^k}}{J(r)} \left(\sum_i (-1)^{j+i} r_{\bar i} \begin{vmatrix} & | & \\ - & \mathbf{r_{i\bar j}} & - \\ & | & \end{vmatrix} \frac{\partial}{\partial z^j} - \det H(r)\right) r_{\overline{lk}}\right) \\
&= 2\mathrm{Re}\left(\frac{\overline{u^l v^k}}{J(r)} (N - \det H(r)) r_{\overline{lk}}\right). \qquad \square
\end{aligned}
$$

5. A formula for Webster pseudo-torsion for on the link of an isolated singularity of a n-dimensional complex subvariety in $\mathbb{C}^{n+1}$

In this section we derive a formula for the Webster pseudo-torsion on the link of an isolated singularity of a n-dimensional complex subvariety in $\mathbb{C}^{n+1}$. Let $M := \{f = 0\} \cap \{r = 0\}$ where r is a defining function of the sphere of radius ϵ, centered at the origin and f is a holomorphic function away from the origin, we assume that $\partial f \wedge dr \neq 0$ along M. Then M is a strongly pseudoconvex CR manifold of real hypersurface type, of dimension $2n-1$. We will use the result in the last section to find an explicit formula for Webster torsion of M. The key idea is to express the components of the characteristic vector field ξ in terms of the derivatives of f and r.

Let $\mathcal{N} := \{z \in \mathbb{C}^{n+1} | f = 0\}$ where $f(0) = 0$, $\overline{\partial} f = 0$, $\partial f \neq 0$. Let $S := \{z \in \mathbb{C}^{n+1} | r = |z^1|^2 + |z^2|^2 + \cdots + |z^{n+1}|^2 - \epsilon = 0\}$ for some $\epsilon > 0$. Let $M := \mathcal{N} \cap S$, we assume $\partial f \wedge dr \neq 0$ along M. The complexified tangent bundles for S and M are denoted by $\mathbb{C}TS$ and $\mathbb{C}TM$ respectively. Let the pseudohermitian structure of S be given by $\theta = i\overline{\partial} r = -i\partial r$ on $\mathbb{C}TS$. Then, the pseudohermitian structure of M is given by $\theta|_M$. We will denote $\theta|_M$ by θ. Throughout this section the ranges of indices are : $1 \leq A, B, \cdots \leq n+1$, $1 \leq j, k, \cdots \leq n$, $1 \leq \alpha, \beta, \cdots \leq n-1$, and we will use the summation convention. Let $\theta, \theta^\alpha, \theta^{\overline{\alpha}}$ be a local basis of $\mathbb{C}TM^*$ such that $d\theta = i\theta^\alpha \wedge \theta^{\overline{\alpha}}$. Let $\xi, X_\alpha, X_{\overline{\alpha}}$ be the dual basis. We may write

$$\xi = \xi^A \frac{\partial}{\partial z^A} + \overline{\xi^A} \frac{\partial}{\partial \overline{z^A}}\,, \tag{81}$$

$$X_\alpha = x_\alpha^A \frac{\partial}{\partial z^A}\,. \tag{82}$$

We have

$$\xi \rfloor \theta = 1 \Rightarrow \xi^A r_A = i\,, \tag{83}$$

$$\xi \rfloor \partial f = 0 \Rightarrow \xi^A f_A = 0\,, \tag{84}$$

$$X_\alpha \rfloor \theta = 0\,, \tag{85}$$

$$X_\alpha \rfloor \partial f = 0\,, \tag{86}$$

$$X_\alpha \rfloor \theta^\beta = \delta_\alpha^\beta\,, \tag{87}$$

$$\xi \rfloor \theta^\beta = 0\,, \tag{88}$$

$$\xi \rfloor d\theta = 0\,, \tag{89}$$

and

$$d\theta = i r_{\overline{A}B} dz^{\overline{A}} \wedge dz^B = i\delta_{\overline{A}B} dz^{\overline{A}} \wedge dz^B = i dz^{\overline{A}} \wedge dz^A\,. \tag{90}$$

Hence, we have

$$\overline{x^A_\alpha} r_{\overline{A}} = 0\,, \tag{91}$$

$$\overline{x^A_\alpha}\,\overline{f_A} = 0\,, \tag{92}$$

$$\overline{x^A_\alpha} r_{\overline{A}B}\xi^B = 0 \Rightarrow \overline{x^A_\alpha}\xi^A = 0\,. \tag{93}$$

We consider (93) as a system of linear equations in unknowns ξ^A. The matrix $(\overline{x^A_\alpha})$ has rank $n-1$. So (93) has only 2 independent solutions. On the other hand the matrix $\begin{pmatrix} \overline{f_1} & \cdots & \overline{f_{n+1}} \\ r_{\overline{1}} & \cdots & r_{\overline{n+1}} \end{pmatrix}$ has rank 2. Hence, we may write

$$\xi^A = a\overline{f_A} + br_{\overline{A}}\,, \tag{94}$$

for a, $b \in \mathbb{C}$. Contracting (94) with $\overline{\xi^A}$, using (83), (86) we obtain $\|\xi\|^2 = -ib$ where $\|\xi\|^2 := \xi^A\overline{\xi^A}$. Hence,

$$b = i\|\xi\|^2\,. \tag{95}$$

Contracting (94) with f_A, we obtain $0 = a\overline{f_A}f_A + br_{\overline{A}}f_A$. So,

$$a = -\frac{br_{\overline{A}}f_A}{\overline{f_C}f_C}\,. \tag{96}$$

By (94), (95), (96), we have

$$\xi^A = -i\|\xi\|^2\frac{r_{\overline{B}}f_B\overline{f_A}}{\overline{f_C}f_C} + i\|\xi\|^2 r_{\overline{A}}\,. \tag{97}$$

Contracting (97) with r_A, using (83),

$$i = r_A\xi^A = -i\|\xi\|^2\left(-\frac{r_{\overline{B}}f_B\overline{f_D}r_D}{\overline{f_C}f_C} + r_{\overline{D}}r_D\right)\,. \tag{98}$$

We solve for $\|\xi\|^2$ in (98) and using (97), we obtain

$$\begin{aligned} \xi^A &= \frac{i\left(-\frac{r_{\overline{B}}f_B\overline{f_A}}{\overline{f_C}f_C} + r_{\overline{A}}\right)}{\frac{r_{\overline{B}}f_B\overline{f_D}r_D}{\overline{f_C}f_C} - r_{\overline{D}}r_D} \\ &= \frac{i\left(-\frac{z^B f_B\overline{f_A}}{\overline{f_C}f_C} + z^A\right)}{\frac{z^B f_B\overline{f_D}z^{\overline{D}}}{\overline{f_C}f_C} - \epsilon}\,. \end{aligned} \tag{99}$$

Now, we are ready to show:

Theorem 5.1. *Let $\mathcal{N} := \{z \in \mathbb{C}^{n+1} \mid f = 0\}$ where $f(0) = 0, \overline{\partial} f = 0, \partial f \neq 0$. Let $S := \{z \in \mathbb{C}^{n+1} \mid r = |z^1|^2 + |z^2|^2 + \cdots + |z^{n+1}|^2 - \epsilon = 0\}$ for some $\epsilon > 0$. Let $M := \mathcal{N} \cap S$, we assume $\partial f \wedge dr \neq 0$ along M. Consider the pseudohermitian* *structure defined by* $\theta = -i\partial r$ *on* M*. Then for any* $U = u^A \frac{\partial}{\partial z_A}$, $V = v^B \frac{\partial}{\partial z_B} \in H_z M$ *and* $z \in M$*, we have*

$$\operatorname{Tor}(z)(U,V) = 2\operatorname{Re}\left(i\overline{u^B v^A} \frac{\partial \xi^A}{\partial \overline{z^B}} \right) \tag{100}$$

where

$$\xi^A = \frac{i\left(-\frac{z_B f_B \overline{f_A}}{\overline{f_C} f_C} + z_A \right)}{\frac{z_B f_B \overline{f_D} z_{\overline{D}}}{\overline{f_C} f_C} - \epsilon}.$$

Proof of Theorem 5.1.
Step 1. We first find a relation between the torsion tensor T and the Webster torsion Tor. Let $U = \mu^\alpha X_\alpha, V = \nu^\beta X_\beta \in T^{1,0}M$. By computation similar to (78), we have

$$\operatorname{Tor}(U,V) = 2\mathrm{Re}(i\langle \overline{T(\xi,U)}, V\rangle). \tag{101}$$

Step 2. By computation similar to (79), we have

$$T(\xi,U) = (U\overline{\xi^A})\frac{\partial}{\partial \overline{z^A}}. \tag{102}$$

We check that $(U\overline{\xi^A})\frac{\partial}{\partial \overline{z^A}} \in T^{1,0}M$ as follows. Using $U = u^A \frac{\partial}{\partial z^A}$, we have

$$(\overline{U}\xi^A) f_A = \overline{U}(\xi^A f_A) - \xi^A \overline{U}(f_A) = 0.$$

Step 3. Let $U = u^A \frac{\partial}{\partial z^A}, V = v^A \frac{\partial}{\partial z^A}$ such that $u^A r_A = 0, u^A f_A = 0, v^A r_A = 0, v^A f_A = 0$. Using (101), (102), we have

$$\begin{aligned}
\operatorname{Tor}(U,V) &= 2\mathrm{Re}\left(i \left\langle \overline{(U\overline{\xi^A})\frac{\partial}{\partial \overline{z^A}}}, v^A \frac{\partial}{\partial z^A} \right\rangle \right) \\
&= 2\mathrm{Re}\left(i \left\langle \overline{u^B} \frac{\partial \xi^C}{\partial \overline{z^B}} \frac{\partial}{\partial z^C}, v^A \frac{\partial}{\partial z^A} \right\rangle \right) \\
&= 2\mathrm{Re}\left(i\overline{u^B v^A} \frac{\partial \xi^A}{\partial \overline{z^B}} \right).
\end{aligned}$$

□

Example 5.2. Let $f = (z^3)^2 - z^1 z^2$. Let $M := \{f = 0\} \cap \{|z^1|^2 + |z^2|^2 + |z^3|^2 = 1\}$. We may see that the the codimension 3 real hypersurface M is spherical as follows. Using the map F given by

$$\begin{aligned}
\tilde{z^1} &= -\frac{1}{\sqrt{2}}(z^1 - iz^2), \\
\tilde{z^2} &= \frac{1}{\sqrt{2}}(z^1 + iz^2), \\
\tilde{z^3} &= z^3,
\end{aligned}$$

the CR manifold M_0 given by

$$\begin{cases} (z^1)^2 + (z^2)^2 + (z^3)^2 = 0\,, \\ |z^1|^2 + |z^2|^2 + |z^3|^2 = 1 \end{cases}$$

is mapped to

$$\begin{cases} 2\tilde{z^1}\tilde{z^2} - (\tilde{z^3})^2 = 0\,, \\ |\tilde{z^1}|^2 + |\tilde{z^2}|^2 + |\tilde{z^3}|^2 = 1\,. \end{cases}$$

Together with the map $\phi\colon S^3 \to M_0$ given by

$$(\zeta, \eta) \mapsto \left(\frac{\zeta^2 - \eta^2}{\sqrt{2}}, \frac{i(\zeta^2 + \eta^2)}{\sqrt{2}}, \frac{2\zeta\eta}{\sqrt{2}} \right) =: (z^1, z^2, z^3)$$

where $S^3 := \{(\zeta, \eta) \in \mathbb{C}^2 : |\zeta|^2 + |\eta|^2 - 1 = 0\}$. ϕ is well defined, holomorphic, onto. By [2], M_0 is CR diffeomorphic to S^3/G where $G = \{I, -I\}$, so that M_0 is locally biholomorphic to S^3. Hence, M is locally biholomorphic to S^3. Then $z^B f_B = 0$. By (100) $\mathrm{Tor}(z)(U, V) = 0$, $\forall z \in M$.

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