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# A FRANKEL TYPE THEOREM FOR CR SUBMANIFOLDS OF SASAKIAN MANIFOLDS

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Abstract. We prove a Frankel type theorem for $CR$ submanifolds of Sasakian manifolds, under suitable hypotheses on the index of the scalar Levi forms determined by normal directions. From this theorem we derive some topological information about $CR$ submanifolds of Sasakian space forms.

## 1. Introduction

In this paper we deal with $CR$ submanifolds of a Sasakian manifold, and we establish a sufficient condition for two of them to have non empty intersection, following Frankel's classical approach, which goes back to [7]. We shall also consider the case when one of the submanifolds is invariant. In [3] and [10] this kind of results were discussed for the case of two invariant submanifolds.

We shall refer to the standard reference [4] for the notation and basic facts concerning Sasakian geometry. Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. By a *$CR$ submanifold* we mean a submanifold $N$ of $M$ for which, at every $x \in N$, the subspace $H_xN \subset T_xN$ defined by

$$H_xN := \{X \in T_xN \mid \eta(X) = 0,\, \varphi X \in T_xN\}$$

has positive dimension $k$, which does not depend on $x$. In this case we denote by $HN$ the subbundle of $TN$ of rank $k$, whose fiber at $x \in N$ is $H_xN$. We remark that this definition generalizes the notion of *contact CR submanifold* introduced by Bejancu and Papaghiuc in [2] and studied by several authors (see also [1, 15]). It is closer to the classical concept of $CR$ submanifold of a complex manifold, see for instance [5, Ch. 7]. For a discussion of some natural examples of $CR$ submanifolds in our sense, see §3.

Our treatment will be based on the fact that such a submanifold is naturally endowed with a $CR$ structure $(HN, J)$, where $J$ denotes the restriction of $\varphi$ to the subbundle $HN$. Like the canonical $CR$ structure of the ambient Sasakian manifold $M$, this induced structure is also strongly pseudoconvex, namely the scalar Levi form $\mathfrak{L}_\eta$ determined by the restriction to $N$ of the contact form $\eta$ is, up to a

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constant factor, the restriction to $HN$ of the Riemannian metric $g$. We refer the reader to §2 for the definition of the (scalar) Levi forms of a $CR$ manifold; here we just recall that each of them is a Hermitian symmetric bilinear form on the holomorphic tangent space $H_xN$ at a point $x \in N$, intrinsically attached to a cotangent vector $\omega \in T_x^*N$ annihilating $H_xN$.

We shall focus on the Levi forms $\mathfrak{L}_\nu$ determined by the normal directions $\nu$ to $N$; by definition, $\mathfrak{L}_\nu$ is attached to the covector:

$$\omega(X) = g(\varphi\nu, X).$$

Each of these Levi forms will be called *characteristic*; we shall denote its index by $i(\mathfrak{L}_\nu)$ and its nullity by $n(\mathfrak{L}_\nu)$.

With this terminology, our main result is the following.

**Theorem 1.1.** *Let $(M, \varphi, \xi, \eta, g)$ be a connected, complete Sasakian manifold with nonnegative $\varphi$-bisectional curvature. Let $N$ and $P$ be two $CR$ submanifolds of $M$, and assume that one of them is closed and the other is compact. Set*

$$q := \min_{\nu \in TN^\perp} i(\mathfrak{L}_\nu) \quad \text{and} \quad s := \min_{\nu' \in TP^\perp} \left( i(\mathfrak{L}_{\nu'}) + n(\mathfrak{L}_{\nu'}) \right). \tag{1.1}$$

*Then we have*

$$N \cap P \neq \emptyset \,,$$

*provided that $q > 0$, $s > 0$ and*

$$q + s \geq \dim(M) - 1 \,. \tag{1.2}$$

**Corollary 1.2.** *Let $(M, \varphi, \xi, \eta, g)$ be a connected, complete Sasakian manifold with nonnegative $\varphi$-bisectional curvature. Let $N$ and $P$ be a $CR$ and an invariant submanifold of $M$ respectively and assume one of them is closed and the other is compact. Then we have that $N \cap P \neq \emptyset$, provided that for each characteristic Levi form of $N$:*

$$i(\mathfrak{L}_\nu) \geq \dim(M) - \dim(P) > 0 \,. \tag{1.3}$$

As an application, we shall prove the following results:

**Corollary 1.3.** *Let $(M, \varphi, \xi, \eta, g)$ be a complete, connected, regular Sasakian manifold with nonnegative $\varphi$-bisectional curvature and assume that $M$ fibers onto a Kähler manifold biholomorphic to a product $S \times \mathbb{C}$, where $S$ is a complex manifold. Then every CR submanifold $N$ of $M$ whose characteristic Levi forms have all positive index is not compact.*

**Corollary 1.4.** *Let $N$ be a CR submanifold of the Sasakian space form $\mathbb{S}^{2n+1}(c)$ with $\varphi$-sectional curvature $c$, $c > -3$. If all the characteristic Levi forms of $N$ have positive index, then $N$ cannot be contained in an open hemisphere.*

## 2. Levi forms of a $CR$ manifold

Let us start by recalling the definitions of $CR$ manifold, Levi-Tanaka form and scalar Levi forms. In the following, given a vector bundle $E$ over a smooth manifold $M$, we will denote by $\Gamma(E)$ the $\mathcal{C}^\infty(M)$-module of global smooth sections of $E$.

Let $M$ be a real smooth manifold of dimension $n$, and let $m$, $k \in \mathbb{N}$ such that $2m + k = n$. If $HM$ is a real vector subbundle of rank $2m$ of the tangent bundle $TM$ and $J : HM \to HM$ is a bundle isomorphism such that $J^2 = -Id$, the couple $(HM, J)$ is called a *CR structure* on $M$ if the following properties hold for all $X, Y \in \Gamma(HM)$:

(i) $[JX, JY] - [X, Y] \in \Gamma(HM)$;

(ii) $N_J(X, Y) := [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$.

In this case $(M, HM, J)$ is called a *CR manifold of type (m,k)* and $m, k$ are the *CR dimension* and the *CR codimension* of the $CR$ structure, respectively.

**Definition 2.1.** Let $(M, HM, J)$ be a $CR$ manifold. Given a point $x \in M$, the *Levi-Tanaka form of $M$ at $x$* is the bilinear map

$$L_x \colon H_xM \times H_xM \to T_xM/H_xM$$

defined by

$$L_x(X, Y) := \pi_x([\tilde{X}, J\tilde{Y}]_x) \quad \forall X, Y \in H_xM\,, \tag{2.1}$$

where $\tilde{X}, \tilde{Y} \in \Gamma(HM)$ are two arbitrary extensions of $X$, $Y$ and $\pi \colon TM \to TM/HM$ is the canonical projection on the quotient bundle $TM/HM$.

It is known that $L_x$ is well defined because the value $\pi_x([\tilde{X}, J\tilde{Y}]_x)$ only depends on the values of $\tilde{X}, \tilde{Y}$ at $x$, that is on $X$ and $Y$.
Moreover, according to (i) above, $L_x$ turns out to be a vector valued symmetric Hermitian form on the holomorphic tangent space $H_xM$ with respect to the complex structure $J := J_x$, that is

$$L_x(X, Y) = L_x(JX, JY), \quad L_x(X, Y) = L_x(Y, X) \tag{2.2}$$

for all $X, Y \in H_xM$.

Given a point $x$ on the $CR$ manifold $(M, HM, J)$, we will denote by

$$H^0_xM := \{\omega \in T^*_xM \mid \omega(X) = 0 \quad \forall X \in H_xM\}$$

the annihilator of $H_xM \subset T_xM$. Then we have the following definition.

**Definition 2.2.** Let $(M, HM, J)$ be a $CR$ manifold, $x \in M$ and $\omega \in H^0_xM$. The Hermitian form

$$\mathfrak{L}_\omega \colon H_xM \times H_xM \to \mathbb{R} \quad \text{s.t.} \quad \mathfrak{L}_\omega(X, Y) := \omega L_x(X, Y) \tag{2.3}$$

is called the *scalar Levi form determined by $\omega$ at $x$*.

**Remark 2.3.** Since the scalar Levi forms $\mathfrak{L}_\omega$ are symmetric, it makes sense to consider their index $i(\mathfrak{L}_\omega)$, defined as the minimum between the number of positive and negative eigenvalues of $\mathfrak{L}_\omega$.

More specifically we recall the following terminology from $CR$ geometry; see for instance [9].

**Definition 2.4.** Let $(M, HM, J)$ be a $CR$ manifold of type $(m, k)$ and let $x \in M$. $M$ is called *pseudoconvex* at $x$ if $\mathfrak{L}_\omega$ is positive definite for some $\omega \in H^0_x M$. If there exists a global section $\omega \in \Gamma(H^0 M)$ such that $\mathfrak{L}_\omega$ is positive definite at each point $x \in M$, $M$ is called *strongly pseudoconvex.*
$M$ is called *pseudoconcave* at $x$ if $i(\mathfrak{L}_\omega) > 0$ for every $\omega \in H^0_x M$, $\omega \neq 0$.

We close this section by recalling that a Sasakian manifold $(M, \varphi, \xi, \eta, g)$, as defined in [4], is a particular kind of strongly pseudoconvex $CR$ manifold of hypersurface type, i.e. of $CR$ codimension 1. We only remark that in this case the $CR$ structure is $(HM, J)$, where $HM = \ker \eta = \langle \xi \rangle^\perp$ is the contact distribution and the almost complex structure is $J = \varphi|_{HM}$. Therefore, for any $x \in M$, $H^0_x M$ is spanned by $\eta_x$ and, up to scaling, we have only one scalar Levi form $\mathfrak{L}_{\eta_x}$. Moreover, since $M$ is a contact metric manifold, the identity

$$d\eta(X, Y) = g(X, \varphi Y)$$

yields that

$$\mathfrak{L}_{\eta_x} = 2g_{x|H_x M \times H_x M}\,.$$

In all that follows, the contact distribution of a Sasakian manifold will be always denoted by $\mathcal{D}$.

## 3. $CR$ submanifolds of Sasakian manifolds

We begin by discussing some classes and examples of $CR$ submanifolds according to our definition, that we reformulate here, for the sake of convenience.

**Definition 3.1.** Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold and let $N \subset M$ be a real submanifold. For every $x \in N$ set

$$H_x N := \{X \in T_x N \mid \eta(X) = 0,\ \varphi X \in T_x N\}\,. \tag{3.1}$$

If the dimension of $H_x N$ is a positive constant, $N$ is called a *CR submanifold.* In this case we denote by $HN$ the subbundle of $TN$ whose fiber at $x \in N$ is $H_x N$.

**Example 3.2.** A *contact CR submanifold* of $M$ is a submanifold $N$ tangent to the Reeb vector field $\xi$ and endowed with a differentiable distribution $E$ such that $\varphi(E) \subset E$ and $\varphi(E^\perp) \subset TN^\perp$, $E^\perp$ being the complementary orthogonal distribution in $TN$.
In this case, it is known that the tangent bundle of $N$ decomposes orthogonally as

$$TN = \langle \xi \rangle \oplus HN \oplus W\,,$$

where $HN = E \cap \mathcal{D}$ and $W = E^\perp \cap \mathcal{D}$ are two distributions of $TN$ such that $\varphi(HN) \subset HN$ and $\varphi W \subset TN^\perp$.
Clearly, $HN$ tuns to be as in (3.1) and then, if it is non trivial, i.e. if $N$ is not anti-invariant, then $N$ is a $CR$ submanifold according to our definition.

As particular cases of contact $CR$ submanifolds we have the following widely studied classes.

**Example 3.3.** An *invariant submanifold* of $M$ is a real submanifold $N$ of $M$ such that $\dim N < \dim M$ and $\varphi(TN) \subset TN$.
It is known that invariant submanifolds are always tangent to $\xi$ and so they are contact $CR$ submanifolds with trivial distribution $W$. Moreover they inherit a Sasakian structure from that of the ambient manifold by restriction ([4]). It follows that $HN = TN \cap \mathcal{D}$ is the contact distribution related to the induced structure on $N$ and hence, at each point $x \in N$, $\dim H_x N = \dim N - 1 > 0$ .

**Example 3.4.** A submanifold $N$ of a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ is called *generic* provided that the Reeb vector field $\xi$ is tangent to $N$ and $\varphi(TN^\perp) \subset TN$ (see [14, 16]).
In this case, at each point $x \in N$, we have the following orthogonal decomposition of the tangent space of $N$ at $x$:

$$T_x N = \langle \xi_x \rangle \oplus \varphi(T_x N^\perp) \oplus H_x N,$$

with $H_x N$ as in (3.1). From this decomposition it follows that $H_x N$ has constant dimension $2p - n - 1$, being $n = \dim M$ and $p = \dim N$.
In particular, $N$ is a contact $CR$ submanifold with $W = \varphi(TN^\perp)$. So, if $N$ is not anti-invariant, then it is a $CR$ submanifold of $M$ in our sense.

**Example 3.5.** If $(M, \varphi, \xi, \eta, g)$ is a Sasakian manifold with $\dim M \geq 5$, then every hypersurface $N$ tangent to $\xi$ is a $CR$ submanifold of $M$.
Indeed, at each point $x \in N$, $T_x N^\perp = \langle \nu \rangle$ and $g(\varphi\nu, \nu) = 0$ implies that $\varphi(TN^\perp) \subset TN$, namely $N$ is generic. Moreover, with respect to the notations of the previous example, $p = n - 1$ and hence $\dim H_x N = 2p - n - 1 = n - 3 > 0$.

Next we discuss two natural ways to construct examples.

**Example 3.6.** Let $(M, \varphi, \xi, \eta, g)$ be a regular Sasakian manifold which fibers onto a Kähler manifold $(N, J, g')$. Then it is given a Riemannian submersion $\pi : M \to N$ whose fibers are 1-dimensional submanifolds of $M$ tangent to $\xi$ and whose differential $d\pi$ commutes with the tensor field $\varphi$ and the complex structure $J$:

$$d\pi \circ \varphi = J \circ d\pi. \tag{3.2}$$

We show that for every $CR$ submanifold $S'$ of $N$, the preimage $S := \pi^{-1}(S')$ is a $CR$ submanifold of $M$ tangent to $\xi$.
According to the definition, $S'$ carries the $CR$ structure $(HS', J)$, where $HS'$ is the subbundle of $TS'$ given by:

$$HS' := TS' \cap J(TS')\,. \tag{3.3}$$

Moreover, since $\pi$ is a surjective submersion, $S$ is a submanifold of $M$ with the same codimension of $S'$, whose tangent space at $x \in S$ is

$$T_x S = (d\pi)_x^{-1}(T_{\pi(x)} S')\,.$$

It follows that $S$ is tangent to $\xi$ and, by using (3.2) and (3.3), a straightforward verification yields that

$$(d\pi)_x(H_x S) = H_{\pi(x)} S'. \tag{3.4}$$

Thus, since $(d\pi)_x\colon \mathcal{D}_x \to T_{\pi(x)}N$ is a linear isomorphism for every $x \in S$, and since the dimension of $H_{\pi(x)}S'$ is constant, we conclude that the same holds for $H_xS$, so that $S$ is a $CR$ submanifold.

**Example 3.7.** Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold with $\dim M = 2n+1$ and let $G$ be a Lie group of automorphisms of the Sasakian structure, acting smoothly on $M$. Then each orbit $N := G \cdot x$ $(x \in M)$ is a $CR$ submanifold of $M$, provided that at $x$ we have $H_xN \neq \{0\}$. This is an immediate consequence of the homogeneity of $N$. In particular, we point out that this is true if $\dim N \geq n+2$.

Indeed, set $p := \dim N \geq n+2$ and $E := T_xN \cap \mathcal{D}_x$. We note that, if $\xi_x$ were normal to $N$, $N$ would be an integral submanifold of the contact distribution $\mathcal{D}$ by the homogeneity, thus contradicting the fact that integral submanifolds of $\mathcal{D}$ have dimension no greater than $n$ (see [4], Theorem 5.1). Hence $\xi$ has to be nowhere normal to $N$ and then $\dim E = \dim \varphi E = p-1$. It follows that $E \cap \varphi E \subset H_xN$ is non trivial since $2(p-1) > 2n+1$ by assumption.

## 4. Characteristic Levi forms

Let $N$ be a $CR$ submanifold of a Sasakian manifold $(M, \varphi, \xi, \eta, g)$ and let us consider the restriction $J := \varphi_{|HN}\colon HN \to HN$ of the structure tensor field $\varphi$ to $HN$. $J$ is well defined because $\varphi(HN) \subset HN$ and, since $HN \subset \mathcal{D}$, it turns out to be an almost complex structure on $HN$, i.e. $J^2 = -Id$. Moreover we have the following result which justifies the name we adopted for this class of submanifolds.

**Proposition 4.1.** *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold and let $N \subset M$ be a $CR$ submanifold. Then the couple $(HN, J)$ defines a $CR$ structure on $N$.*

**Proof.** It follows in a straightforward manner according to the definition of $(HN, J)$ and the fact that $(\mathcal{D}, \varphi_{|\mathcal{D}})$ is a $CR$ structure on $M$. □

According to this result, for each point $x \in N$, it makes sense to consider the Levi-Tanaka form at $x$ and the scalar Levi forms $\mathfrak{L}_\omega$, where $\omega$ varies in $H^0_xN$. In particular, given a non zero normal direction $\nu \in T_xN^\perp$, consider the 1-form $\omega : T_xN \to \mathbb{R}$ such that

$$\omega(X) := g(\varphi\nu, X) \quad \forall X \in T_xN. \tag{4.1}$$

We note that for every $X \in H_xN$, $\omega(X) = -g(\varphi X, \nu) = 0$ since $\varphi X \in T_xN$; hence $\omega \in H^0_xN$.

The scalar Levi form $\mathfrak{L}_\omega$ determined by $\omega$ will be denoted by $\mathfrak{L}_\nu$; we shall adopt the following terminology:

**Definition 4.2.** For any $x \in N$ the scalar Levi forms $\mathfrak{L}_\nu$ determined by $\nu \in T_xN^\perp$, $\nu \neq 0$, will be called *characteristic Levi forms* of $N$ at $x$.

**Remark 4.3.** In the particular case of invariant submanifolds, all the characteristic Levi forms are identically zero, because so are the corresponding covectors $\omega$. Indeed, for every $X \in T_xN$, $\varphi X$ is still tangent to $N$ and hence $\omega(X) = -g(\nu, \varphi X) = 0$.

**Example 4.4.** Consider a regular Sasakian manifold $(M, \varphi, \xi, \eta, g)$ which fibers on a Kähler manifold $N$ and let $\pi : M \to N$ be the Riemannian submersion as in

Example 3.6. We claim that if $S'$ is a pseudoconcave $CR$ submanifold of $N$, then all the characteristic Levi forms of $S = \pi^{-1}(S')$ have positive index.
Fix $x \in S$, $y = \pi(x) \in S'$, $X \in H_xS$ and set $X' := (d\pi)_x X \in H_yS'$ by identity (3.4). For any non zero normal vector $\nu \in T_xS^\perp$, from the definition of the characteristic Levi form $\mathfrak{L}_\nu$ and using basic properties of Riemannian submersions (see for instance Proposition 1.1 in [6]), we get:

$$\begin{aligned} \mathfrak{L}_\nu(X,X) &= g_x([X,\varphi X],\varphi\nu) \\ &= g_x(h[X,\varphi X],\varphi\nu) \\ &= g'_y([X',JX'],J\nu') \\ &= \mathfrak{L}'_{\nu'}(X',X')\,, \end{aligned}$$

where $h[X,\varphi X]$ is the horizontal component of $[X,\varphi X]$, $\nu' = (d\pi)_x\nu \in (T_yS')^\perp$ and $\mathfrak{L}'_{\nu'}$ is the scalar Levi form on $H_yS'$ determined by the covector

$$\omega'(X') = g'_y(X',J\nu') \quad \forall X' \in T_yS'.$$

In conclusion we have proved that, for every $CR$ submanifold $S'$ of $N$, one has

$$\mathfrak{L}_\nu(X,X) = \mathfrak{L}_{\nu'}(X',X') \quad \forall X \in H_xS\,. \tag{4.2}$$

From this equality, our claim follows immediately.

The following proposition establishes a relationship between the second fundamental form of a $CR$ submanifold $N$ and its characteristic Levi forms.

**Proposition 4.5.** *Let $(M,\varphi,\xi,\eta,g)$ be a Sasakian manifold and let $N \subset M$ be a $CR$ submanifold with second fundamental form $\alpha$. Given $x \in N$ and $0 \neq \nu \in T_xN^\perp$, one has*

$$\mathfrak{L}_\nu(X,X) = g_x(\alpha(X,X) + \alpha(\varphi X,\varphi X),\nu) \tag{4.3}$$

*for every $X \in H_xN$.*

**Proof.** First we recall that Sasakian manifolds are characterized by means of the following identity, involving the covariant derivative of $\varphi$ with respect to the Levi-Civita connection (see [4]):

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X\,. \tag{4.4}$$

Now, fix $x \in N$, $X \in H_xN$ and consider a smooth section of $HN$ which extends $X$. Then $\varphi X$ is again tangent to $N$. Using the fact that $X$, $\varphi X$ and $\varphi\nu$ are all orthogonal to $\xi$ and identity (4.4), we get:

$$\begin{aligned} \mathfrak{L}_\nu(X,X) &= g_x([X,\varphi X],\varphi\nu) \\ &= g_x(\nabla_X\varphi X,\varphi\nu) - g_x(\nabla_{\varphi X}X,\varphi\nu) \\ &= g_x(\varphi\nabla_X X,\varphi\nu) + g_x(\varphi\nabla_{\varphi X}X,\nu) \\ &= g_x(\nabla_X X,\nu) + g_x(\nabla_{\varphi X}\varphi X,\nu) \\ &= g_x(\alpha(X,X) + \alpha(\varphi X,\varphi X),\nu)\,. \end{aligned}$$

□

## 5. Some remarks about $\varphi$-bisectional curvature

In the following we shall deal with Sasakian manifolds with nonnegative $\varphi$-bisectional curvature. So, for the convenience of the reader, we recall the definition of this kind of curvature which was introduced by Tanno and Baik in [13], and used in [3] and [10] to obtain Frankel type theorems about the intersection of two invariant submanifolds. It is an adaptation to the Sasakian case of the notion of holomorphic bisectional curvature introduced by Goldberg and Kobayashi in [8] for Kähler manifolds.

**Definition 5.1.** Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold. We say that $M$ has *nonnegative $\varphi$-bisectional curvature* if

$$H(X, Y) := K(X, Y) + K(X, \varphi Y) \geq 0 \tag{5.1}$$

for every $x \in M$ and $X, Y \in T_xM$ such that $X, Y, \varphi Y, \xi_x$ are mutually orthonormal, where

$$K(X, Y) := R(X, Y, X, Y) = g(R(X, Y)Y, X)$$

denotes the sectional curvature at $x$ of the 2-plane $\langle X, Y \rangle \subset T_xM$, and similarly for $K(X, \varphi Y)$.

**Example 5.2.** Let $(M, \varphi, \xi, \eta, g)$ be a regular Sasakian manifold which fibers onto a Kähler manifold $(N, J, g')$ and let $\pi : M \to N = M/\xi$ be the Riemannian submersion as in Examples 3.6 and 4.4. In [13] it is shown that, given a point $x \in M$ and two tangent vectors $X, Y \in T_xM$ such that $X, Y, \varphi Y, \xi_x$ are mutually orthonormal, the $\varphi$-bisectional curvature $H(X, Y)$ is given by

$$H(X, Y) = H'(X', Y'),$$

where $H'(X', Y')$ is the holomorphic bisectional curvature of $N$ at $p = \pi(x)$, related to the vectors $X' := (d\pi)_x X$ and $Y' := (d\pi)_x Y$. Hence, if $H'$ is nonnegative, so is the $\varphi$-bisectional curvature $H$ of $M$.
For instance, Takahashi's globally $\varphi$-symmetric spaces are examples of regular Sasakian manifolds and from Theorem 6.4 in [12] it follows that those of compact type have nonnegative $\varphi$-bisectional curvature.

**Proposition 5.3.** *Let $(M, \varphi, \xi, \eta, g)$ be a Sasakian manifold with nonnegative $\varphi$-bisectional curvature. Let $x \in M$ and $X, W \in T_xM$ such that $\eta(X) = 0$ and $X, \varphi X, W$ are mutually orthonormal. Then one has:*

$$R(X, W, X, W) + R(\varphi X, W, \varphi X, W) \geq 0.$$

**Proof.** Here $W$ might not be normal to $\xi_x$. However, decomposing $W$ as $W = Y + Z$ with $Y \in \mathcal{D}_x$ and $Z \in \langle \xi_x \rangle$, we get:

$$\begin{aligned} R(X, W, X, W) &= R(X, Y, X, Y) + R(X, Z, X, Z) + 2g(R(X, Y)Z, X) \\ &= R(X, Y, X, Y) + R(X, Z, X, Z), \end{aligned} \tag{5.2}$$

where we have used the following curvature characterization of Sasakian manifolds (see [4]):

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y \quad \forall X, Y \in \Gamma(TM).$$

From this formula we also get that $R(X,Z,X,Z)$ is nonnegative. Similarly,

$$R(\varphi X,W,\varphi X,W)=R(\varphi X,Y,\varphi X,Y)+R(\varphi X,Z,\varphi X,Z)\,, \tag{5.3}$$

where $R(\varphi X,Z,\varphi X,Z)\geq 0$. Thus, by adding the identities (5.2) and (5.3), we have:

$$R(X,W,X,W)+R(\varphi X,W,\varphi X,W)\geq 0\,,$$

since, up to scaling, the left-hand side is the sum of a $\varphi$-bisectional curvature and two nonnegative terms. □

## 6. Proofs of the results

In this section we will give the proof of our Frankel type theorem. Firstly we recall some basic facts about the second variation formula for the arc length functional (see for instance [11]).

Let $(M,g)$ be a Riemannian manifold, let $N$ and $P$ be two submanifolds of $M$ and let $\gamma\colon[0,l]\to M$ be a geodesic parametrized by arc length and intersecting orthogonally $N$ at $x:=\gamma(0)$ and $P$ at $y:=\gamma(l)$. Taken a variation $\Gamma\colon(-\epsilon,\epsilon)\times[0,l]\to M$ of $\gamma$ such that the longitudinal curves $\Gamma_s$ are curves from $N$ to $P$, the second variation formula for the arc length $L_\Gamma$ is given by

$$\begin{aligned} L''_\Gamma(0):=\frac{d^2}{ds^2}L(\Gamma_s)\bigg|_{s=0}&=\int_0^l\left[\|\nabla_{\dot\gamma}X^\perp\|^2-g(R(X^\perp,\dot\gamma)\dot\gamma,X^\perp)\right]\,dt\\ &\quad+g(\alpha(X,X),\dot\gamma)|_0^l\,, \end{aligned} \tag{6.1}$$

where $X$ is the variation vector field of $\Gamma$, $X^\perp$ is its normal component with respect to $\dot\gamma$ and $\alpha$ denotes the second fundamental form of $N$ or $P$ with an abuse of notation.
In particular, if $X$ is normal to $\dot\gamma$ and parallel along $\gamma$, the previous formula reduces to the following:

$$L''_\Gamma(0)=-\int_0^l R(X,\dot\gamma,X,\dot\gamma)\,dt+g(\alpha(X,X),\dot\gamma)|_0^l\,. \tag{6.2}$$

We also recall that given a geodesic $\gamma$ as before and a vector field $X$ along $\gamma$ such that $X(0)\in T_xN$ and $X(l)\in T_yP$, there always exists a variation $\Gamma$ of $\gamma$ made up by curves from $N$ to $P$ and having $X$ as the variation vector field. Such a variatiation of $\gamma$ will be denoted by $\Gamma_X$ and for the related arc length functional we will write $L_X$ instead of $L_{\Gamma_X}$.
Furthermore, we shall denote by $\mathfrak{X}(\gamma)$ the module of smooth vector fields along $\gamma$.

Now, coming back to our purpose, we prove a lemma which provides a way to construct an orthonormal set $\{E,\varphi E\}$ consisting of parallel vector fields along a geodesic $\gamma$.

**Lemma 6.1.** *Let $N$ be a CR submanifold of a Sasakian manifold $(M,\varphi,\xi,\eta,g)$ and let $\gamma\colon[0,l]\to M$ be a geodesic starting from $x\in N$ and orthogonal to $N$ at $x$. If $e\in H_xN$ and $E$, $\tilde E\in\mathfrak{X}(\gamma)$ are obtained by parallel translation of $e$, $\varphi e$ respectively along $\gamma$, then $E$ is orthogonal to $\xi$ along $\gamma$ and $\tilde E=\varphi E$.*

**Proof.** To prove that $\tilde{E} = \varphi E$ we need to show that $\varphi E$ is parallel along $\gamma$ and to this aim we follow the same idea of a proof in [3].
Since $e \in H_x N$ is tangent to $N$, while $\dot\gamma(0)$ is normal to $N$, we have that $e \perp \dot\gamma(0)$. Moreover, since $E$ and $\dot\gamma$ are parallel along $\gamma$, we have that $E \perp \dot\gamma$ along $\gamma$, that is $g(E, \dot\gamma)(t) = 0$ for every $t \in [0, l]$. Hence:

$$(\nabla_{\dot\gamma}\varphi)E = g(\dot\gamma, E)\xi - \eta(E)\dot\gamma = -\eta(E)\dot\gamma\,. \tag{6.3}$$

On the other hand,

$$(\nabla_{\dot\gamma}\varphi)E = \nabla_{\dot\gamma}\varphi E - \varphi(\nabla_{\dot\gamma}E) = \nabla_{\dot\gamma}\varphi E\,, \tag{6.4}$$

since $E$ is parallel along $\gamma$. Therefore $\nabla_{\dot\gamma}\varphi E = -\eta(E)\dot\gamma$ and to prove our claim, we just have to prove that $\eta(E)(t) = 0$ for every $t \in [0, l]$. From this will also follow that $E$ is normal to $\xi$ along $\gamma$. So we consider the function

$$f : [0, l] \to \mathbb{R} \quad \text{s.t.} \quad f(t) := \eta(E)(t) = g_{\gamma(t)}(E(t), \xi_{\gamma(t)})$$

and we prove that $f$ is identically zero. We note that

$$f' = g(\nabla_{\dot\gamma}E, \xi) + g(E, \nabla_{\dot\gamma}\xi) = -g(E, \varphi\dot\gamma) = g(\varphi E, \dot\gamma)\,; \tag{6.5}$$

$$f'' = g(\nabla_{\dot\gamma}\varphi E, \dot\gamma) + g(\varphi E, \nabla_{\dot\gamma}\dot\gamma) = -\eta(E)g(\dot\gamma, \dot\gamma) = -\|\dot\gamma\|^2 f\,, \tag{6.6}$$

where $c := \|\dot\gamma\|^2 \in \mathbb{R}$ is constant. Moreover, $f(0) = g_x(e, \xi_x) = 0$ by the definition of $H_x N$ and $f'(0) = g_x(\varphi e, \dot\gamma(0)) = 0$ because of $\varphi e \in T_x N$ and $\dot\gamma(0) \in T_x N^\perp$.
In conclusion, we proved that $f$ is a solution of the following Cauchy problem

$$\begin{cases} f'' + cf = 0 \\ f(0) = 0 \\ f'(0) = 0 \end{cases}$$

so that $f = 0$. $\square$

**Proof of Theorem 1.1.** Assume by contradiction that $N \cap P = \emptyset$. Thanks to the topological assumptions on the submanifolds, there exist two points $x \in N$ and $y \in P$ such that $l := d(x, y) = d(N, P) > 0$. Moreover, by the completeness of $M$, there exists a length minimizing geodesic $\gamma\colon [0, l] \to M$, parametrized by arc length, joining $x$ and $y$ and intersecting orthogonally $N$ and $P$. Clearly, since $L(\gamma) = d(N, P)$, for every variation $\Gamma$ of $\gamma$ made up by curves from $N$ to $P$, we must have $L''_\Gamma(0) \geq 0$.
Now, set $\nu := \dot\gamma(0) \in T_x N^\perp$ and $\nu' := \dot\gamma(l) \in T_y P^\perp$. Since the numbers $q$ and $s$ defined in (1.1) are strictly positive, there exist two linear subspaces $V \subset H_x N$ and $W \subset H_y P$ of dimensions $q$ and $s$ respectively such that $\mathfrak{L}_\nu$ is negative definite on $V$ and $\mathfrak{L}_{\nu'}$ is positive semi-definite on $W$.
Let us denote by $V' \subset T_y M$ the image of $V$ under the parallel transport along $\gamma$: since $\dot\gamma(0)$ is normal to $N$ and $V \subset H_y N \subset \langle \xi_x \rangle^\perp$, using Lemma 6.1, we see that $V' \subset \langle \dot\gamma(l), \xi_y \rangle^\perp$. Moreover, $W \subset \langle \dot\gamma(l), \xi_y \rangle^\perp$ as well. Therefore

$$V' + W \subset \langle \dot\gamma(l), \xi_y \rangle^\perp$$

and

$$\dim(V' + W) \leq \dim M - 2\,.$$

Thus, by using (1.2), we have:

$$\dim(V' \cap W) \geq q + s - \dim M + 2 \geq 1\,.$$

Hence we can consider a non zero vector $e' \in V' \cap W$, which is the image of a vector $e \in V$ under parallel translation. In other words there exists a vector field $E \in \mathfrak{X}(\gamma)$ which is parallel along $\gamma$ and such that $E(0) = e$, $E(l) = e'$. From Lemma 6.1 it follows that $\varphi E$ is parallel along $\gamma$ and $\varphi E(l) = \varphi e' \in T_y P$ by the definition of $H_y P$.

Now, fix two variations $\Gamma_E$, $\Gamma_{\varphi E}$ of $\gamma$, having $E$ and $\varphi E$ as variation vector fields respectively. Computing the second variation formula (6.2), we have:

$$L''_E(0) = -\int_0^l R(E, \dot\gamma, E, \dot\gamma)\,dt + g(\alpha(E,E), \dot\gamma)|_0^l\,;$$

$$L''_{\varphi E}(0) = -\int_0^l R(\varphi E, \dot\gamma, \varphi E, \dot\gamma)\,dt + g(\alpha(\varphi E, \varphi E), \dot\gamma)|_0^l\,.$$

By adding these two expressions and by using Proposition 4.5, we get:

$$\begin{aligned} L''_E(0) + L''_{\varphi E}(0) = -\int_0^l &[R(E, \dot\gamma, E, \dot\gamma) + R(\varphi E, \dot\gamma, \varphi E, \dot\gamma)]\,dt \\ &+ \mathfrak{L}_\nu(e,e) - \mathfrak{L}_{\nu'}(e', e')\,. \end{aligned} \tag{6.7}$$

In view of Proposition 5.3, being $e \in V$ and $e' \in W$, we conclude that expression (6.7) is negative, thus arriving at a contradiction. □

**Proof of Corollary 1.2.** Hypothesis (1.3) implies that

$$q \geq \dim M - \dim P > 0\,.$$

Moreover, since $P$ is invariant, according to Remark 4.3 we have that $\mathfrak{L}_{\nu'} = 0$ for every $y \in P$ and $\nu' \in T_y P^\perp$. Therefore $n(\mathfrak{L}_{\nu'}) = \dim H_y P = \dim P - 1$ (cf. Example 3.3) and

$$s = \dim P - 1 > 0\,.$$

It follows that $q + s \geq \dim M - 1$ and then we get $N \cap P \neq \emptyset$ by applying the theorem. □

**Remark 6.2.** In the same setting of Theorem 1.1, if $N$ is a compact $CR$ submanifold whose characteristic Levi forms have all positive index, then $N$ intersects every closed, $CR$ and totally geodesic hypersurface $P$.

Indeed, since characteristic Levi forms are Hermitian and symmetric, $i(\mathfrak{L}_\nu) > 0$ is equivalent to $i(\mathfrak{L}_\nu) \geq 2$ and hence $q \geq 2$. Moreover, since $P$ is a totally geodesic hypersurface, it is a $CR$ submanifold whose scalar Levi forms $\mathfrak{L}_{\nu'}$ ($\nu' \in TP^\perp$) are all identically zero by Proposition 4.5. Hence

$$s = \dim H_y P = \dim P - 2 = \dim M - 3.$$

Therefore $q + s \geq \dim M - 1$ and the claim is proved by applying the theorem.

Finally we present the proofs of Corollary 1.3 and Corollary 1.4.

**Proof of Corollary 1.3.** Consider the fibration $\pi\colon M \to M/\xi$ and, by contradiction, assume that $M$ admits a compact $CR$ submanifold $N$, whose characteristic Levi forms have all positive index. Then $\pi(N)$ is a compact set. Since $M/\xi$ is biholomorphic to $S \times \mathbb{C}$, one can always find a Levi flat real hypersurface $P$ in it, such that $\pi(N) \cap P = \emptyset$ (to see this, it suffices to consider $S \times E$, where $E \subset \mathbb{C}$ is a real straight line disjoint from $p(\pi(N))$, where $p\colon S \times \mathbb{C} \to \mathbb{C}$ is the natural projection).
Then $N \cap \pi^{-1}(P) = \emptyset$, but this is in contrast with the statement of Theorem 1.1, because $\pi^{-1}(P)$ is a $CR$ hypersurface of $M$ whose characteristic Levi forms all vanish by (4.2). □

**Remark 6.3.** This last corollary can be applied to the Sasakian space form $M(-3) = \mathbb{R}^{2n+1}$. In fact it is a complete, connected Sasakian manifold which has constant $\varphi$-bisectional curvature equal to zero, fibering onto the complex Euclidean space $\mathbb{C}^n$.

**Proof of Corollary 1.4.** Recall that, as a manifold, $\mathbb{S}^{2n+1}(c)$ is the unit sphere $\mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1}$, where we adopt the following notation:

$$(z_1, \dots, z_{n+1}) = (x_1, \dots, x_{n+1}, x_{n+2}, \dots, x_{2n+2}), \quad z_k = x_k + i x_{n+1+k}\,.$$

Moreover, the Sasakian structure on $\mathbb{S}^{2n+1}(c)$ is obtained by applying a $\mathcal{D}$-homothetic deformation to the canonical Sasakian structure of $\mathbb{S}^{2n+1}$ and this deformed structure is invariant under the action of the unitary group $U(n+1)$. As a consequence, unitary transformations on $\mathbb{S}^{2n+1}(c)$ map open hemispheres in open hemispheres and $CR$ submanifolds into $CR$ ones, preserving the index of all characteristic Levi forms. Therefore, given a $CR$ submanifold $N \subset \mathbb{S}^{2n+1}(c)$ as in the statement, without loss of generality it suffices to prove that $N$ cannot be contained in the open hemisphere

$$S := \{x \in \mathbb{S}^{2n+1}(c) |\ x_{2n+2} > 0\}\,.$$

Let $\pi\colon \mathbb{S}^{2n+1}(c) \to \mathbb{C}\mathrm{P}_n$ be the canonical projection and let us consider the hyperplane $\sigma\colon z_{n+1} = 0$ of $\mathbb{C}\mathrm{P}_n$: since $\sigma$ is a holomorphic submanifold, $\pi^{-1}(\sigma)$ is an invariant submanifold of $\mathbb{S}^{2n+1}(c)$. Furthermore, since the characteristic Levi forms are Hermitian and symmetric we have that $i(\mathfrak{L}_\nu) > 0$ is equivalent to

$$i(\mathfrak{L}_\nu) \geq 2 = \dim M - \dim \pi^{-1}(\sigma)\,.$$

Finally, since $c > -3$, $\mathbb{S}^{2n+1}(c)$ has nonnegative $\varphi$-bisectional curvature and by applying Corollary 1.2, we have that $N \cap \pi^{-1}(\sigma) \neq \emptyset$. This means that there exists a point $P \in N$ with coordinates $P(z_1, \dots, z_n, 0)$; in particular $P \notin S$. □

## References


[1] Bejancu, A., *Geometry of CR-submanifolds*, Mathematics and its Applications (East European Series), vol. 23, D. Reidel Publishing Co., Dordrecht, 1986.

[2] Bejancu, A., Papaghiuc, N., *Semi-invariant submanifolds of a Sasakian manifold*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **27** (1981), 163–170.

[3] Binh, T.Q., Ornea, L., Tamássy, L., *Intersections of Riemannian submanifolds. Variations on a theme by T.J. Frankel*, Rend. Mat. Appl. (7) **19** (1999), 107–121.

[4] Blair, D.E., *Riemannian Geometry of Contact and Symplectic Manifolds*, second ed., Progress in Mathematics, vol. 203, Birkhäuser, Boston, 2010.

[5] Djorić, M., Okumura, M., *CR submanifolds of complex projective space*, Developments in Mathematics, vol. 19, Springer, New York, 2010.

[6] Falcitelli, M., Ianus, S., Pastore, A.M., *Riemannian submersions and related topics*, World Scientific Publishing Co., Singapore, 2004.

[7] Frankel, T., *Manifolds with positive curvature*, Pacific J. Math. **11** (1961), 165–171.

[8] Goldberg, S.I., Kobayashi, S., *Holomorphic bisectional curvature*, J. Differential Geom. **1** (1967), 225–233.

[9] Medori, C., Nacinovich, M., *Levi-Tanaka algebras and homogeneous CR manifolds*, Compositio Math. **109** (1997), 195–250.

[10] Pitis, G., *On the topology of Saskian manifolds*, Math. Scand. **93** (2003), 99–108.

[11] Sakai, T., *Riemannian Geometry*, Translations of Mathematical Monographs, vol. 149, Amer. Math. Soc., Providence, 1996.

[12] Takahashi, T., *Sasakian $\phi$-symmetric spaces*, Tôhoku Math. J. **29** (1977), 91–113.

[13] Tanno, S., Baik, S.-B., *$\phi$-holomorphic special bisectional curvature*, Tôhoku Math. J. **22** (1970), 184–190.

[14] Yano, K., Kon, M., *Generic submanifolds of Sasakian manifolds*, Kodai Math. J. **3** (1980), 163–196.

[15] Yano, K., Kon, M., *CR submanifolds of Kaehlerian and Sasakian manifolds*, Progress in Mathematics, vol. 30, Birkhäuser, Boston, Mass., 1983.

[16] Yano, K., Kon, M., *Structures on Manifolds*, Series in Pure Mathematics, vol. 3, World Scientific Publishing Co., Singapore, 1984.



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