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SOME PROPERTIES OF ALGEBRAS OF REAL-VALUED MEASURABLE FUNCTIONS

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Abstract. Let $M(X,\mathscr{A})$ ($M^*(X,\mathscr{A})$) be the f-ring of all (bounded) real-measurable functions on a T-measurable space $(X,\mathscr{A})$, let $M_K(X,\mathscr{A})$ be the family of all $f\in M(X,\mathscr{A})$ such that $\operatorname{coz}(f)$ is compact, and let $M_\infty(X,\mathscr{A})$ be all $f\in M(X,\mathscr{A})$ that $\{x\in X : |f(x)|\geq \frac{1}{n}\}$ is compact for any $n\in\mathbb{N}$. We introduce realcompact subrings of $M(X,\mathscr{A})$, we show that $M^*(X,\mathscr{A})$ is a realcompact subring of $M(X,\mathscr{A})$, and also $M(X,\mathscr{A})$ is a realcompact if and only if $(X,\mathscr{A})$ is a compact measurable space. For every nonzero real Riesz map $\varphi : M(X,\mathscr{A})\to\mathbb{R}$, we prove that there is an element $x_0\in X$ such that $\varphi(f)=f(x_0)$ for every $f\in M(X,\mathscr{A})$ if $(X,\mathscr{A})$ is a compact measurable space. We confirm that $M_\infty(X,\mathscr{A})$ is equal to the intersection of all free maximal ideals of $M^*(X,\mathscr{A})$, and also $M_K(X,\mathscr{A})$ is equal to the intersection of all free ideals of $M(X,\mathscr{A})$ (or $M^*(X,\mathscr{A})$). We show that $M_\infty(X,\mathscr{A})$ and $M_K(X,\mathscr{A})$ do not have free ideal.

1. Introduction

For any nonempty completely regular Hausdorff space X, $C(X)$ ($C^*(X)$) stands for the set of all (bounded) real-valued continuous functions defined on X, with pointwise operations of addition and multiplication (see [14, 12]). Recall that a *real bounded Riesz map* $\phi\colon C(X)\to\mathbb{R}$ is a linear map preserving lattice operations with $\phi(\mathbf{1})=1$ (see [6]). By a classical representation theorem, for every such ϕ there is an $x\in X$ such that $\phi(f)=f(x)$ for every $f\in C(X)$, whose proof is in [6]. Karimi Feizabadi and Ebrahimi represent the pointfree version of this representation see [4]. In this paper, we present representation of *bounded Riesz map* for the f-ring of all real-measurable functions on a T-measurable space $(X,\mathscr{A})$, i.e., $M(X,\mathscr{A})$. We replace a realcompact Hausdorff space X by realcompact T-measurable space. We show that if T-measurable space $(X,\mathscr{A})$ is compact if and only if $M(X,\mathscr{A})$ is realcompact (see Proposition 4.7). Also, if $(X,\mathscr{A})$ is a compact T-measurable space, we prove that for any nonzero f-ring homomorphism $\phi\colon M(X,\mathscr{A})\to\mathbb{R}$, there is a unique $x\in X$ such that $\phi(f)=f(x)$ for every $f\in M(X,\mathscr{A})$ (see Proposition 4.10). Therefore, there is a one-to-one correspondence between *bounded*

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Riesz maps (nonzero f-ring homomorphism) from $M(X,\mathscr{A})$ to $\mathbb{R}$ with elements of X if T-measurable space $(X,\mathscr{A})$ is compact (finite).

In [15] Kohls introduced the subring $C_\infty(X)$ of all functions $C(X)$ which vanish at infinity, and showed that

$$C_\infty(X)=\bigcap\{I\colon I \text{ is a free maximal ideal of } C^*(X)\}\,.$$

Also he showed

$$C_K(X)=\bigcap\{I\colon I \text{ is a free ideal of } C^*(X) \text{ or } C(X)\}$$

see [12, 7.E and 7.F].

Let L be a completely regular frame, let $\mathcal{R}L$ be the ring of real-valued continuous functions on L, and let $\mathcal{R}_\infty L$ be the family of all functions $\varphi\in\mathcal{R}L$ for which the set $\uparrow\varphi(\dfrac{-1}{n},\dfrac{1}{n})$, ordered by relation of L, is a compact frame for any $n\in\mathbb{N}$. $\mathcal{R}_\infty L$ was introduced by Dube in [3]. Estaji and Mahmoudi Darghadam in [8] proved that $\mathcal{R}_\infty L$ is precisely the intersection of all the free maximal ideals of $\mathcal{R}^*L$ (also, [10, 9]).

In Section 5, we introduce $M_\infty(X,\mathscr{A})$ and $M_K(X,\mathscr{A})$ for every T-measurable space $(X,\mathscr{A})$, and we give an answer to a question which was posted by Acharyya et al. [1, Question 4.11]. In fact, we show that

$$M_\infty(X,\mathscr{A})=\bigcap\{M\colon M \text{ is a free maximal ideal of } M^*(X,\mathscr{A})\}\,,$$

and

$$M_K(X,\mathscr{A})=\bigcap\{I\colon I \text{ is a free ideal of } M^*(X,\mathscr{A}) \text{ or } M(X,\mathscr{A})\}.$$

In [2] it was showed that $C_\infty(X)$ and $C_K(X)$ do not have free ideal. In this paper, we show that $M_\infty(X,\mathscr{A})$ and $M_K(X,\mathscr{A})$ do not have free ideal (see Corollary 5.9 and Corollary 5.10).

2. Preliminaries

In this section, we introduce the concepts of measurable space and commutative ring which is used in this paper.

Let us recall some general notation from [16]. Let $\mathscr{A}$ be a collection of subsets of a nonempty set X. It is well known that $(X,\mathscr{A})$ is called a *measurable space* if $\mathscr{A}$ has the following three properties:

(i) $X\in\mathscr{A}$.

(ii) If $A\in\mathscr{A}$, then $A^c\in\mathscr{A}$, where A^c is the complement of A relative to X.

(iii) If $\{A_n\}_{n\in\mathbb{N}}\subseteq\mathscr{A}$, then $\bigcup_{n\in\mathbb{N}}A_n\in\mathscr{A}$.

Also, the members of $\mathscr{A}$ are called the measurable sets in X. If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y. If X is a measurable space, then the set of all measurable maps from X into $\mathbb{R}$ is denoted $M(X,\mathscr{A})$, and the members of $M(X,\mathscr{A})$ are called the *real* *measurable functions* on X, where $\mathbb{R}$ denotes the set of all real numbers with the ordinary topology.

Addition, multiplication, joint, and meet in $\mathbb{R}^X$ are defined by the formulas $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$, $(f \vee g)(x) = \max\{f(x), g(x)\}$, and $(f \wedge g)(x) = \min\{f(x), g(x)\}$. $(\mathbb{R}^X; +, \cdot, \vee, \wedge)$ is proved an f-ring, this conclusion is the immediate consequence of the corresponding statements about the field $\mathbb{R}$. Also, if $(X, \mathscr{A})$ is a measurable space, then $(M(X, \mathscr{A}); +, \cdot, \vee, \wedge)$ is a sub-f-ring of $\mathbb{R}^X$. The subset $M^*(X, \mathscr{A})$ of $M(X, \mathscr{A})$, consisting of all bounded functions in $M(X, \mathscr{A})$, is also closed under the algebraic and order operations. A measurable space $(X, \mathscr{A})$ is said to be *pseudocompact* if $M^*(X, \mathscr{A}) = M(X, \mathscr{A})$.

An element a of a lattice L is said to be *compact* if $a = \bigvee S$, $S \subseteq L$, implies $a = \bigvee F$ for some finite subset F of S. A bounded lattice L is said to be *compact* whenever its top element $\top$ is compact (see [5]). A measurable space $(X, \mathscr{A})$ is called a *compact measurable space* if $\mathscr{A}$ is a compact lattice (see [7, 11]). A measurable space $(X, \mathscr{A})$ is said to be *T-measurable* if whenever x and y are distinct points in X, there is a measurable set containing one and not the other (see [11]). In [11] proved that if $(X, \mathscr{A})$ is not a T-measurable space, we can find a T-measurable space $(Y, \mathscr{A}')$ for which $M(X, \mathscr{A}) \cong M(Y, \mathscr{A}')$. Therefore, throughout this paper, $(X, \mathscr{A})$ denotes a T-measurable space.

We recall from [11] that an ideal I of $M(X, \mathscr{A})$ is called fixed if the set $\bigcap_{f \in I} Z(f)$ is nonempty; otherwise, I is called free. In [11] it showed that a compact measurable $(X, \mathscr{A})$ is determined by fixed maximal ideals of $M(X, \mathscr{A})$. Also, the following proposition that was proved in [7] is needed in this paper.

Proposition 2.1. *The following statements are equivalent.*

(1) *The measurable space $(X, \mathscr{A})$ is a compact measurable space.*

(2) *The set X is a finite set and $\mathscr{A} = \mathcal{P}(X)$.*

(3) *The measurable space $(X, \mathscr{A})$ is a pseudocompact measurable space.*

Throughout this paper, we put

$$M_x := \{f \in M(X, \mathscr{A})\colon f(x) = 0\},$$

for every $x \in X$. It is evident that M_x is a fixed maximal ideal of $M(X, \mathscr{A})$ for every $x \in X$.

Recall that a totally ordered field F is said to be archimedean if for every element $a \in F$, there exists an element n in $\mathbb{N}$ such that $n \geq a$. Thus, a nonarchimedean field is characterized (among all totally ordered fields) by the presence of infinitely large elements, that is, elements a such that $a > n$ for every $n \in \mathbb{N}$. An element b is infinitely small if it is positive but smaller than $\frac{1}{n}$ for every $n \in \mathbb{N}$. Hence b is infinitely small if and only if $\frac{1}{b}$ is infinitely large. Therefore, the presence of infinitely small elements also characterizes the nonarchimedean fields. Every archimedean field is embeddable in $\mathbb{R}$ (see [12, page 70]).

3. Quotient lattice-ordered and totally-ordered rings of $M(X, \mathscr{A})$

In this section, we show that every quotient ring of $M(X, \mathscr{A})$ and $M^*(X, \mathscr{A})$ is a lattice-ordered ring and we obtain several equivalent conditions for ideal I

of $M(X, \mathscr{A})$ such that the quotient ring $\frac{M(X,\mathscr{A})}{I}$ is a totally-ordered ring. The concepts as well as the results of this section are counterparts of the results in the text book [12].

Definition 3.1. Let $(X, \mathscr{A})$ be a measurable space. A subset U of X is called *relatively pseudocompact* if $f(U)$ is a bounded subset of $\mathbb{R}$ for all $f \in M(X, \mathscr{A})$.

As usual, if $f \in M(X, \mathscr{A})$, then $z(f) := \{ x \in X \colon f(x) = 0 \}$ and $\operatorname{coz} := X \setminus z(f)$ are called zero set and cozero set of f, respectively. Also, for every $f, g \in M(X, \mathscr{A})$, we have $z(fg) = z(f) \cup z(g)$, $z(f^2 + g^2) = z(|f| + |g|) = z(f) \cap z(g)$, and $z(f) = z(|f|) = z(f^n)$ for every $n \in \mathbb{N}$. Also, for every subset H of $M(X, \mathscr{A})$, we put $Z[H] := \{ z(f) \colon f \in H \}$.

Lemma 3.2. *Let $(X, \mathscr{A})$ be a measurable space. For every $f, g \in M(X, \mathscr{A})$, the following statements hold.*

(1) *If $\operatorname{coz}(f) \leq \operatorname{coz}(g)$, then there is an element h of $M(X, \mathscr{A})$ such that $f = gh$.*

(2) *If $0 \leq f \leq g$, then there is an element h of $M(X, \mathscr{A})$ such that $f = gh$.*

(3) *If $0 \leq f \leq g$ and $g \in M^*(X, \mathscr{A})$, then there is an element h of $M^*(X, \mathscr{A})$ such that $f = gh$.*

Proof. Consider the $h \colon X \to \mathbb{R}$ given by

$$h(x) = \begin{cases} \frac{f(x)}{g(x)} & \text{if} \quad x \in \operatorname{coz}(g) \\ 0 & \text{if} \quad x \notin \operatorname{coz}(g). \end{cases}$$

□

We recall that an ideal I of an f-ring A is called an ℓ-ideal or an absolutely convex ideal if $|x| \leq |y|$, and $y \in I$ imply $x \in I$. Also, I is called a convex ideal if whenever $0 \leq x \leq y$, and $y \in I$, then $x \in I$.

As an immediate consequence of Lemma 3.2, we now have the following proposition:

Proposition 3.3. *The following statements hold.*

(1) *For every ideal I of $M(X, \mathscr{A})$, $f \in I$ if and only if $|f| \in I$.*

(2) *Every ideal of $M(X, \mathscr{A})$ is a convex ideal of $M(X, \mathscr{A})$.*

(3) *Every ideal of $M(X, \mathscr{A})$ is an absolutely convex ideal of $M(X, \mathscr{A})$.*

Remark 3.4. For every ideal I of $M(X, \mathscr{A})$, by Theorem 5.2 in [12], $\frac{M(X,\mathscr{A})}{I}$ is a partially ordered ring, according to the definition:

$f + I \geq 0$ if there exists an element g in $M(X, \mathscr{A})$ such that $g \geq 0$ and $f - g \in I$.

Throughout this paper, this notation will be used. Also, by Theorem 5.3 in [12], the following statements hold for every ideal I of $M(X, \mathscr{A})$ and every $f, g \in M(X, \mathscr{A})$.

(1) $f, g \in I$ implies $f \vee g \in I$.

(2) $(f \vee g) + I = f + I \vee g + I$.

(3) $f + I \geq 0$ if and only if $f - |f| \in I$.

The above results are true for $M^*(X)$.

Proposition 3.5. *For every proper ideal I of $M(X,\mathscr{A})$ ($M^*(X,\mathscr{A})$), the following statements hold.*

(1) *For every $f \in M(X,\mathscr{A})$, $f + I \geq 0$ if and only if f is nonnegative on an element Z of $Z[I]$.*

(2) *For every $f \in M(X,\mathscr{A})$, if f is positive on at least one Z of $Z[I]$, then $f + I > 0$.*

(3) *Let I be a maximal ideal of $M(X,\mathscr{A})$ ($M^*(X,\mathscr{A})$). For every $f \in M(X,\mathscr{A})$, f is positive on at least one Z of $Z[I]$ if and only if $f + I > 0$.*

Proof. (1) *Necessity.* Let $f \in M(X,\mathscr{A})$ with $f + I \geq 0$ be given. By Remark 3.4, $f - |f| \in I$. Since f and $|f|$ have the same sing on $z(f - |f|)$, we conclude that f is nonnegative on $z(f - |f|)$.

Sufficiency. Let $f \in M(X,\mathscr{A})$ and $g \in I$ with $f|_{z(g)} \geq 0$ be given. Since $z(g) \leq f^{-1}([0,+\infty)) = z(f - |f|)$, we conclude from Lemma 3.2 that there is an element h of $M(X,\mathscr{A})$ such that $f - |f| = gh \in I$, which implies that $f + I = |f| + I \geq 0$.

(2) Let $f \in M(X,\mathscr{A})$ and $g \in I$ with $f|_{z(g)} > 0$ be given. Since $z(f) \cap z(g) = \emptyset$, we conclude that $f \notin I$. Hence, by the first statement, $f + I > 0$.

(3) Let $f \in M(X,\mathscr{A})$ with $f + I > 0$ be given. Hence, by the first statement, if $g = f - |f|$, then $f|_{z(g)} \geq 0$ and $g \in I$. By [11, Proposition 3.7], there is an element h of I such that $z(h) \cap z(f) = \emptyset$, and so $f|_{z(g^2+h^2)} > 0$ which $g^2 + h^2 \in I$. □

The following example shows that the maximal condition on I in Proposition 3.5 is necessary.

Example 3.6. Let I and J be proper ideals of $M(X,\mathscr{A})$ such that $I \subsetneq J$ and $f \in J \setminus I$. By Lemma 3.2, $f^2 \notin I$. Since $z(f^2) \in Z[J]$, we infer that $z(f^2) \cap Z \neq \emptyset$ for any $Z \in Z[I] \subseteq Z[J]$. Hence $f^2 + I > 0$ and $f^2|_Z \not> 0$ for any $Z \in Z[I]$.

Proposition 3.7. *Let I be a proper ideal of $M(X,\mathscr{A})$. Then the following statements are equivalent.*

(1) *$\frac{M(X,\mathscr{A})}{I}$ is a totally ordered ring.*

(2) *For every $f \in M(X,\mathscr{A})$, there is a zero set of $Z[I]$ on which f does not change sign.*

(3) *The ideal I is a prime ideal of $M(X,\mathscr{A})$.*

Proof. (1) $\Rightarrow$ (2) For a given element f of $M(X,\mathscr{A})$, since $\frac{M(X,\mathscr{A})}{I}$ is a totally ordered ring, we infer that $f + I \leq 0$ or $f + I \geq 0$, which from Proposition 3.5 implies that there is a zero set of $Z[I]$ on which f does not change sign.

(2) $\Rightarrow$ (3) Given $gh \in I$, consider the function $|g| - |h|$. By hypothesis, there is an element f of I such that $z(f) \cap (|g| - |h|)^{-1}(-\infty, 0) = \emptyset$. Hence $Z(f) \cap Z(g) \subseteq Z(h)$. Since

$$Z((hg)^2 + f^2) = Z(hg) \cap Z(f) = [Z(h) \cap Z(f)] \cup [Z(g) \cap Z(f)] \subseteq Z(h)$$

and $(hg)^2 + f^2 \in I$, we conclude from Lemma 3.2 that $h \in I$. Thus, I is prime.

(3) $\Rightarrow$ (1) Observe that for every $f \in M(X, \mathscr{A})$, $(f \vee \mathbf{0})(f \wedge \mathbf{0}) = \mathbf{0} \in I$. Since, by hypothesis, either $f \vee \mathbf{0} \in I$ or $f \wedge \mathbf{0} \in I$, we conclude that $Z(f \vee \mathbf{0}) \in Z[I]$ or $Z(f \wedge \mathbf{0}) \in Z[I]$, which implies that f does not change sign on them. Therefore, by Proposition 3.5, $\frac{M(X,\mathscr{A})}{I}$ is a totally ordered ring. $\square$

Similar to the proof of Proposition 3.7, $M^*(X, \mathscr{A})/I$ is a totally ordered ring if and only if I is a prime ideal of $M^*(X)$ for every proper ideal I of $M^*(X, \mathscr{A})$.

Definition 3.8. A subring R' of a partial order ring R is called absolutely convex, if $f \in R'$ and $g \in R$ such that $|g| \leq |f|$, then $g \in R'$.

It is clear that $M^*(X, \mathscr{A})$ is an absolutely convex subring of $M(X, \mathscr{A})$.

Proposition 3.9. *Let R be an absolutely convex subring of $M(X, \mathscr{A})$. If P is a prime ideal of R, then P is an absolutely convex ideal of R.*

Proof. Let $g \in P$ and $f \in R$ such that $|f| \leq |g|$. Then the function $h\colon X \to \mathbb{R}$ given by

$$h(x) = \begin{cases} \frac{f^2(x)}{g(x)} & \text{if} \quad x \notin z(g) \\ 0 & \text{if} \quad x \in z(g) \end{cases}$$

belongs to $M(X, \mathscr{A})$, and, by hypothesis, $h \in R$, because $|h| \leq |f|$. Since $f^2 = gh \in P$, we infer that $f \in P$. $\square$

We recall from [13] that $M(X, \mathscr{A})$ is a regular reduced ring, which implies that every prime ideal of $M(X, \mathscr{A})$ is a maximal ideal of $M(X, \mathscr{A})$. Then for every prime ideal P of $M(X, \mathscr{A})$, $\frac{M(X,\mathscr{A})}{P}$ is a totally ordered filed.

4. Real compact T-measurable space and Real Riesz map on $M(X, \mathscr{A})$.

In this section, we introduce realcompact T-measurable space and prove that realcompact T-measurable spaces are the same compact T-measurable spaces. Also we show that for every realcompact T-measurable space $(X, \mathscr{A})$ and nonzero homomorphism $\varphi\colon M(X, \mathscr{A}) \to \mathbb{R}$ there exists an element x_0 in X such that $\varphi(f) = f(x_0)$ for every $f \in M(X, \mathscr{A})$.

For every proper ideal P of $M(X, \mathscr{A})$, it is clear that $\theta\colon \mathbb{R} \to \frac{M(X,\mathscr{A})}{P}$ given by $r \mapsto \mathbf{r} + P$ is a monomorphism, which implies that $\frac{M(X,\mathscr{A})}{P}$ has a copy of $\mathbb{R}$. This fact leads to the following definition. Except for Proposition 4.10, the concepts as well as the other results of this section are counterparts of the results in the text book [12].

Definition 4.1. Let R be a subring of $M(X, \mathscr{A})$. A maximal ideal M of R is called real if $\frac{R}{M} \cong \mathbb{R}$, otherwise it is called hyper-real.

Recall that a totally ordered field F is said to be *archimedean* if for every element $a \in F$, there exists an element n in $\mathbb{N}$ such that $n \geq a$. Hence, by [12, Theorem 0.21], we have:

Proposition 4.2. *A maximal ideal M of $M(X, \mathscr{A})$ (resp., $M^*(X, \mathscr{A})$) is real if and only if $\frac{M(X,\mathscr{A})}{M}$ (resp., $\frac{M^*(X,\mathscr{A})}{M}$) is archimedean.*

Proposition 4.3. *Let M be a maximal ideal of $M(X,\mathscr{A})$ and $f \in M(X,\mathscr{A})$. Then the following statements are equivalent.*

(1) *$|f+M|$ is an infinitely large element of $\frac{M(X,\mathscr{A})}{M}$.*

(2) *$f|_Z$ is unbounded for every $Z \in Z[M]$.*

(3) *The zero set $Z_n := \{x \in X : |f(x)| \geq n\}$ belongs to $Z[M]$ for every $n \in \mathbb{N}$.*

Proof. (1) $\Leftrightarrow$ (2) Let $n \in \mathbb{N}$ be given. Then, by Proposition 3.5, $|f+M| \leq n$ if and only if $|f||_Z \leq n$ for some $Z \in Z[M]$.

(1) $\Leftrightarrow$ (3) Let $n \in \mathbb{N}$ be given. Then, by Proposition 3.5, $|f+M| \geq n$ if and only if $|f||_Z \geq n$ for some $Z \in Z[M]$, and since $Z \subseteq Z_n$, we conclude that $Z_n \in Z[M]$. $\square$

The following corollary relates unbounded functions on X with infinitely large elements modulo maximal ideals.

Corollary 4.4. *Let $f \in M(X,\mathscr{A})$ be given. Then $f \in M(X,\mathscr{A}) \setminus M^*(X,\mathscr{A})$ if and only if there exists a maximal ideal M of $M(X,\mathscr{A})$ such that $|f+M|$ is an infinitely large element of $\frac{M(X,\mathscr{A})}{M}$.*

Proof. *Necessity.* We put $Z_n := \{x \in X : |f(x)| \geq n\}$ for any $n \in \mathbb{N}$. Because $\{Z_n : n \in \mathbb{N}\}$ has the finite intersection property, we conclude that there is an ultrafilter $\mathcal{F}$ of $\mathscr{A}$ such that $\{Z_n : n \in \mathbb{N}\} \subseteq \mathcal{F}$. Since, by [11, Proposition 3.6.], $M := Z^{-1}[\mathcal{F}]$ is a maximal ideal of $M(X,\mathscr{A})$, we conclude from Proposition 4.3 that $|f+M|$ is an infinitely large element of $\frac{M(X,\mathscr{A})}{M}$.

Sufficiency. It is obvious. $\square$

Lemma 4.5. *Let $Z \in \mathscr{A}$ be given. Z is a compact element of $\mathscr{A}$ if and only if $Z \notin \mathcal{F}$ for every free ultrafilter $\mathcal{F}$ of $\mathscr{A}$.*

Proof. *Necessity.* Let $Z \in \mathcal{F}$ for some free ultrafilter $\mathcal{F}$ of $\mathscr{A}$. Then

$$Z = \operatorname{coz}(\chi_Z) \cap X = \operatorname{coz}(\chi_Z) \cap \bigcup_{F \in \mathcal{F}} (X \setminus F) = \bigcup_{F \in \mathcal{F}} \left(\operatorname{coz}(\chi_Z) \cap \operatorname{coz}(\chi_{X \setminus F})\right),$$

which implies that there are $F_1, F_2, \cdots, F_n \in \mathcal{F}$ such that

$$Z = \bigcup_{i=1}^{n} \left(\operatorname{coz}(\chi_Z) \cap \operatorname{coz}(\chi_{X \setminus F_i})\right) = Z \cap \bigcup_{i=1}^{n} (X \setminus F_i) \in \mathcal{F},$$

and so $\emptyset = Z \cap \bigcap_{i=1}^{n} F_i \in \mathcal{F}$, which is a contradiction.

Sufficiency. Let Z be not compact. Since $(Z, Z_{\mathscr{A}})$ is not a compact measurable space, we conclude from Proposition 2.1 that there is an element f of $M(Z)$ such that $f \notin M^*(Z)$. Then the function $g\colon X \to \mathbb{R}$ given by

$$g(x) = \begin{cases} |f(x)| + 1 & \text{if} \quad x \in Z \\ 0 & \text{if} \quad x \in X \setminus Z \end{cases}$$

belongs to $M(X,\mathscr{A}) \setminus M^*(X,\mathscr{A})$. We put $Z_n := \{x \in X : |g(x)| \geq n\}$ for every $n \in \mathbb{N}$. Then there exists a free ultrafilter $\mathcal{F}$ of $\mathscr{A}$ such that $\{Z_n : n \in \mathbb{N}\} \subseteq \mathcal{F}$, because $\{Z_n : n \in \mathbb{N}\}$ has the finite intersection property. Since $Z \in \mathcal{F}$, we obtain a contradiction, by Lemma 4.5. $\square$

Proposition 4.6. *Let $f \in M(X, \mathscr{A})$ be given. $|f + M|$ is an infinitely large element of $\frac{M(X,\mathscr{A})}{M}$ for every free maximal ideal M of $M(X, \mathscr{A})$ if and only if $f|_Z$ is unbounded for every noncompact measurable set $Z \in \mathscr{A}$.*

Proof. *Necessity.* Let $Z \in \mathscr{A}$ be not compact, and let $f|_Z$ be bounded. By Lemma 4.5, $Z \in \mathcal{F}$ for some free $Z_{\mathscr{A}}$-ultrafilter $\mathcal{F}$ of $\mathscr{A}$. If we put $M := Z^{-1}(\mathcal{F})$, then $|f + M|$ is bounded, which is a contradiction.

Sufficiency. Let M be a free maximal ideal of $M(X, \mathscr{A})$, then, by [11, Proposition 3.6], $\mathcal{F} := Z(M)$ is a free ultrafilter of $\mathscr{A}$. Since, by Lemma 4.5, no element of $Z(M)$ is compact, we conclude from our hypothesis and Proposition 4.3 that $|f + M|$ is an infinitely large element of $\frac{M(X,\mathscr{A})}{M}$. □

The following proposition relates compactness of X with the real maximal ideals of $M(X, \mathscr{A})$.

Proposition 4.7. *The following statements hold.*

(1) *Every maximal ideal of $M^*(X, \mathscr{A})$ is real.*

(2) *Every maximal ideal of $M(X, \mathscr{A})$ is real if and only if X is compact.*

Proof. (1) Let M be a maximal ideal of $M^*(X)$. If $f \in M^*(X, \mathscr{A})$, then $|f| \leq n$ for some $n \in \mathbb{N}$, and hence $|f + M| \leq n$. Therefore, $\frac{M^*(X,\mathscr{A})}{M}$ is archimedean, and so, by Corollary 4.2, M is a real maximal ideal of $M^*(X, \mathscr{A})$.

(2) *Necessity.* We argue by contradiction. Let us assume that X is not compact. By Proposition 2.1, there exists an element f of $M(X, \mathscr{A})$ such that $f \notin M^*(X, \mathscr{A})$. Hence, by Corollary 4.4, a maximal ideal M of $M(X, \mathscr{A})$ such that $|f + M|$ is an infinitely large element of $\frac{M(X,\mathscr{A})}{M}$, which implies that there is a maximal ideal M of $M(X, \mathscr{A})$ which is not real. This is a contradiction to the fact that every maximal ideal of $M(X, \mathscr{A})$ is real.

Sufficiency. Since, by Proposition 2.1, $M(X, \mathscr{A}) = M^*(X, \mathscr{A})$, we conclude from the first statement that every maximal ideal of $M(X, \mathscr{A})$ is real. □

Definition 4.8. Let A be a $\mathbb{Q}$-algebra (or a $\mathbb{R}$-algebra). A function $\phi \colon A \to \mathbb{R}$ is called a *real Riesz map* if $\phi(ra + bc) = r\phi(a) + \phi(b)\phi(c)$ for every $a, b, c \in A$ and $r \in \mathbb{Q}$ ($r \in \mathbb{R}$). Also, a nonzero real Riesz map is called a *real bounded Riesz map.*

Remark 4.9. Let $\varphi \colon M(X, \mathscr{A}) \to \mathbb{R}$ be a ring homomorphism, i.e., $\varphi(f + gh) = \varphi(f) + \varphi(g)\varphi(h)$ for every f, g, $h \in M(X, \mathscr{A})$, then the following statements hold.

(1) If $f \geq g$, then $\varphi(f) \geq \varphi(g)$ for every $f, g \in M(X, \mathscr{A})$.

(2) If $\varphi \neq \mathbf{0}$, then $\varphi(\mathbf{r}) = r$ for every $r \in \mathbb{R}$.

(3) If $\varphi \neq \mathbf{0}$, then φ is a real bounded Riesz map.

The next proposition contains a complete description of the real bounded Riesz map on $M(X, \mathscr{A})$.

Proposition 4.10. *Let $\varphi \colon M(X, \mathscr{A}) \to \mathbb{R}$ be a nonzero homomorphism. If every maximal ideal of $M(X, \mathscr{A})$ is real, then there exists an element x_0 of X such that $\varphi(f) = f(x_0)$ for every $f \in M(X, \mathscr{A})$.*

Proof. By Proposition 4.7, $(X, \mathscr{A})$ is compact and so, by [11, Proposition 4.11], every proper ideal of $M(X, \mathscr{A})$ is fixed. Since, by Remark 4.9, φ is an f-ring epimorphism, we infer that the canonic map $\overline{\varphi}\colon \frac{M(X,\mathscr{A})}{ker(\varphi)} \to \mathbb{R}$ given by $f+ker(\varphi) \mapsto \varphi(f)$ is an isomorphism. Therefore, $ker(\varphi)$ is a maximal ideal of $M(X, \mathscr{A})$, then there exists an element x_0 of X such that $ker(\varphi) = M_{x_0}$. We claim that $\varphi(f) = f(x_0)$ for every $f \in M(X, \mathscr{A})$. We have $\varphi(g) = 0 = g(x_0)$ for every $g \in ker(\varphi)$. If $g \in M(X, \mathscr{A}) \setminus ker(\varphi)$ with $\varphi(g) = r \neq g(x_0)$ for some $r \in \mathbb{R} \setminus \{0\}$, then $\overline{\varphi}(\mathbf{1} + ker(\varphi)) = 1 = \overline{\varphi}(\frac{\mathbf{1}}{\mathbf{r}} g + ker(\varphi))$, and this is a contradiction, which proves the claim. □

5. The intersection of free ideals

In this section, we show that $M_\infty(X, \mathscr{A})$ is equal to the intersection of free maximal ideals of $M^*(X, \mathscr{A})$ and $M_K(X, \mathscr{A})$ is equal to the intersection of free ideals of $M^*(X, \mathscr{A})$ ($M(X, \mathscr{A})$), also we prove that $M_\infty(X, \mathscr{A})$ and $M_K(X, \mathscr{A})$ do not have free ideal.

Lemma 5.1. *Let $\{f_n\}_{n\in\mathbb{N}} \subseteq M(X, \mathscr{A})$ such that $0 \leq f_n(x) \leq f_{n+1}(x)$ for every $n \in \mathbb{N}$ and every $x \in X$. If the sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f pointwise on X, then $f \in M(X, \mathscr{A})$.*

Proof. Consider $r \in \mathbb{R}$. Since $0 \leq f_n(x) \leq f_{n+1}(x)$ for every $(x, n) \in X \times \mathbb{N}$, we conclude from $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in X$ that

$$\{x \in X\colon f(x) > r\} = \bigcup_{n\in\mathbb{N}} \{x \in X\colon f_n(x) > r\} \in \mathscr{A}$$

for every $r \in \mathbb{R}$, which implies that $f \in M(X, \mathscr{A})$. □

We can now state the counterpart of [12, Theorem 5.14] for $M(X, \mathscr{A})$. Also, the next proposition provides a complete description of the real maximal ideals of $M(X, \mathscr{A})$.

Proposition 5.2. *Let M be a maximal ideal of $M(X, \mathscr{A})$. Then the following statements are equivalent.*

1. *M is real.*
2. *$Z[M]$ closed under countable intersection.*
3. *$Z[M]$ has countable intersection property.*

Proof. (1) ⇒ (2) Let $\{z(f_n)\colon n \in \mathbb{N}\} \subseteq Z[M]$ with $\bigcap_{n\in\mathbb{N}} z(f_n) \notin Z[M]$. We define $g_n = |f_n| \wedge \frac{1}{2^n}$ for every $n \in \mathbb{N}$ and $g = \Sigma_{n\in\mathbb{N}} g_n$. By Lemma 5.1, $g \in M(X, \mathscr{A})$. Since $z(g) = \bigcap_{n\in\mathbb{N}} z(f_n) \notin Z[M]$, we conclude from [11, Proposition 3.13] that $g \notin M$. For every $m \in \mathbb{N}$ and every $x \in \bigcap_{i=1}^m z(f_i)$, we have $g(x) \leq \Sigma_{m<n\in\mathbb{N}} 2^{-n} = 2^{-m}$. Hence, by Proposition 3.5, $g+M \leq 2^{-m}$ for each $m \in \mathbb{N}$, i.e, $g+M$ is infinitely small, and hence $\frac{M(X,\mathscr{A})}{M}$ is nonarchimedean and M is not real, which is a contradiction.

(2) ⇒ (3) It is clear, because $\emptyset \notin Z[M]$.

(3) ⇒ (1) By way of contradiction assume that M is not a real ideal of $M(X, \mathscr{A})$, then there is an element f of $M(X, \mathscr{A})$ such that $f + M$ is an infinitely large

element of $\frac{M(X,\mathscr{A})}{M}$, and by Proposition 4.3, $Z_n := \{x \in X : |f(x)| \geq n\} \in Z[M]$ for every $n \in \mathbb{N}$. It is clear that $\bigcap_{n\in\mathbb{N}} Z_n = \emptyset$, which is a contradiction. $\square$

Definition 5.3. Let $(X, \mathscr{A})$ be a measurable space. An ultrafilter $\mathcal{F}$ of $\mathscr{A}$ is called real if $Z^{-1}(\mathcal{F})$ is a real maximal ideal of $M(X, \mathscr{A})$.

Proposition 5.4. *Let $\mathcal{F}$ be an ultrafilter of $\mathscr{A}$, then the following statements hold.*

(1) *$\mathcal{F}$ is a real ultrafilter of $\mathscr{A}$ if and only if $\mathcal{F}$ is closed under countable intersection.*

(2) *If $\mathcal{F}$ is a real ultrafilter of $\mathscr{A}$ and $\{f_n : n \in \mathbb{N}\} \subseteq M(X, \mathscr{A})$ such that $\bigcap_{n\in\mathbb{N}} z(f_n) \in \mathcal{F}$, then $z(f_n) \in \mathcal{F}$ for some $n \in \mathbb{N}$.*

Proof. (1) It follows from Proposition 5.2.

(2) We argue by contradiction. Let us assume that $z(f_n) \notin \mathcal{F}$ for every $n \in \mathbb{N}$. Then, by [11, Proposition 3.7], there is an element $z(g_n)$ of $\mathcal{F}$ such that $z(f_n) \cap z(g_n) = \emptyset$ for every $n \in \mathbb{N}$. By the first statement, $\bigcap_{n\in\mathbb{N}} z(g_n) \in \mathcal{F}$ and by hypothesis

$$\emptyset = \Big(\bigcap_{n\in\mathbb{N}} z(g_n)\Big) \cap \Big(\bigcap_{n\in\mathbb{N}} z(f_n)\Big) \in \mathcal{F}\,,$$

which is a contradiction. $\square$

We say that $f \in M(X, \mathscr{A})$ *vanish at infinity* if the set $\{\, x \in X : |f(x)| \geq \frac{1}{n} \,\}$ is compact for every $n \in \mathbb{N}$. Let $M_\infty(X, \mathscr{A})$ denote the family of all functions of $M(X, \mathscr{A})$ that vanish at infinity. It is clear that $M_\infty(X, \mathscr{A})$ is an absolutely convex subring of $M(X, \mathscr{A})$.

We can now give an answer to a question which was posted by Acharyya et al. [1, Question 4.11].

Theorem 5.5. *The subset $M_\infty(X, \mathscr{A})$ of $M^*(X, \mathscr{A})$ is equal to the intersection of all free maximal ideals of $M^*(X, \mathscr{A})$.*

Proof. Let $f \in \bigcap\{M : M \text{ is a free maximal ideal of } M^*(X, \mathscr{A})\}$ be given. We argue by contradiction. Let us assume that $f \notin M_\infty(X, \mathscr{A})$. Then there exists an element $n \in \mathbb{N}$ such that $Z_n := \{x \in X : |f(x)| \geq \frac{1}{n}\}$ is not a compact element of $\mathscr{A}$. By Lemma 4.5, there exists a free ultrafilter $\mathcal{F}$ in $\mathscr{A}$ such that $Z_n \in \mathcal{F}$. On the other hand, note that $z(f) \in \mathcal{F}$, hence $\emptyset = Z_n \cap z(f) \in \mathcal{F}$, which is a contradiction. Therefore,

$$\bigcap\{M : M \text{ is a free maximal ideal of } M^*(X, \mathscr{A})\} \subseteq M_\infty(X, \mathscr{A})\,.$$

Let M be a free maximal ideal of $M^*(X, \mathscr{A})$ and $f \in M_\infty(X, \mathscr{A})$. Then, by Proposition 4.7, M is a real maximal ideal of $M^*(X, \mathscr{A})$, and so, by Proposition 5.2, $Z[M]$ is closed under countable intersection. We put $Z_n := \{x \in X : |f(x)| \geq \frac{1}{n}\}$

for every $n \in \mathbb{N}$. Then

$$\begin{aligned} f \in M_\infty(X, \mathscr{A}) &\Rightarrow Z_n \text{ is compact for every } n \in \mathbb{N} \\ &\Rightarrow Z_n \notin Z[M] \text{ for every } n \in \mathbb{N}, \text{ by Lemma 4.5} \\ &\Rightarrow X \setminus Z_n \in Z[M] \text{ for every } n \in \mathbb{N} \\ &\Rightarrow z(f) = \bigcap_{n\in\mathbb{N}} (X \setminus Z_n) \in Z[M] \\ &\Rightarrow f \in M\,. \end{aligned}$$

Hence,

$$M_\infty(X, \mathscr{A}) \subseteq \bigcap \{M \colon M \text{ is a free maximal ideal of } M^*(X, \mathscr{A})\}\,.$$

Therefore, $M_\infty(X, \mathscr{A})$ is equal to the intersection of all free maximal ideals of $M^*(X, \mathscr{A})$. □

Lemma 5.6. *If I is a free ideal of $M(X, \mathscr{A})$ or $M^*(X, \mathscr{A})$, then for any compact (finite) measurable subset A of X, there exists an element f_A of I such that $A \subseteq \operatorname{coz}(f_A)$.*

Proof. Let I be a free ideal of $M(X, \mathscr{A})$ or $M^*(X, \mathscr{A})$, and let A be a compact element of $\mathscr{A}$. Since I is a free ideal of $M(X, \mathscr{A})$, we conclude that for any $x \in A$, there exists an element f_x of I such that $x \in \operatorname{coz}(f_x)$, which implies that $A \subseteq \bigcup_{x\in A} \operatorname{coz}(f_x)$, and so there exists a finite subset A' of A such that $A \subseteq \bigcup_{x\in A'} \operatorname{coz}(f_x) = \operatorname{coz}(\sum_{x\in A'} f_x^2)$ and $\sum_{x\in A'} f_x^2 \in I$, because A is compact. □

Let $M_K(X)$ denote the family of all functions in $M(X, \mathscr{A})$ having compact cozero set. It is clear that $M_K(X, \mathscr{A})$ is an absolutely convex subring of $M(X, \mathscr{A})$, and also, $M_K(X, \mathscr{A}) \subseteq M_\infty(X, \mathscr{A})$.

Proposition 5.7. *Let $(X, \mathscr{A})$ be a T- measurable space, then the following statements hold.*

(1) $M_K(X) \subseteq \bigcap\{I \colon I$ *is a free ideal of* $M^*(X, \mathscr{A})\}$.

(2) $M_K(X) \subseteq \bigcap\{I \colon I$ *is a free ideal of* $M(X, \mathscr{A})\}$.

Proof. (1) Let I be an arbitrary free ideal of $M^*(X, \mathscr{A})$ and $f \in M_K(X)$. Since $\operatorname{coz}(f)$ is compact and I is a free ideal of $M^*(X, \mathscr{A})$, we conclude from Lemma 5.6 that there exists an element g of I such that $\operatorname{coz}(f) \subseteq \operatorname{coz}(g)$, and so, by Lemma 3.2, there exists an element h of $M^*(X, \mathscr{A})$ such that $f = gh \in I$. Therefore, $M_K(X) \subseteq \bigcap\{I \colon I \text{ is a free ideal of } M^*(X, \mathscr{A})\}$.

The proof of the second statement is similar to the proof of the first statement. □

The following theorem relates the intersection of all free ideals of $M(X, \mathscr{A})$ with $M_K(X, \mathscr{A})$.

Theorem 5.8. *The following statements hold.*

(1) *The subset $M_K(X, \mathscr{A})$ of $M(X, \mathscr{A})$ is equal to the intersection of all free ideals of $M(X, \mathscr{A})$.*

(2) *The subset $M_K(X,\mathscr{A})$ of $M^*(X,\mathscr{A})$ is equal to the intersection of all free ideals of $M^*(X,\mathscr{A})$.*

Proof. (1) Let $f \notin M_K(X,\mathscr{A})$ be given. Then $\operatorname{coz}(f)$ is not compact, which from Lemma 4.5 implies that $\operatorname{coz}(f) \in \mathcal{F}$ for some free ultrafilter $\mathcal{F}$ of $\mathscr{A}$, i.e., $f \notin Z^{-1}[\mathcal{F}]$. Hence,

$$f \notin \bigcap \{I \colon I \text{ is a free ideal of } M(X,\mathscr{A})\}.$$

Therefore, by Proposition 5.7, the proof is now complete.

(2) The proof of the second statement is similar to the proof of the first statement. $\square$

In the following corollaries, we show that $M_\infty(X,\mathscr{A})$ and $M_K(X,\mathscr{A})$ do not have free ideal.

Corollary 5.9. *Every proper ideal of $M_\infty(X,\mathscr{A})$ is fixed.*

Proof. Let Q be a free maximal ideal of $M_\infty(X,\mathscr{A})$, then there exists a maximal ideal M of $M^*(X,\mathscr{A})$ such that $M_\infty(X,\mathscr{A}) \nsubseteq M$ and $Q = M \cap M_\infty(X,\mathscr{A})$, which implies that M is a free ideal of $M^*(X,\mathscr{A})$ such that $M_\infty(X,\mathscr{A}) \nsubseteq M$, but this is a contradiction to the fact that $M_\infty(X,\mathscr{A})$ is equal to the intersection of all free maximal ideals of $M^*(X,\mathscr{A})$. On the other hand, note that every proper ideal of $M_\infty(X,\mathscr{A})$ is contained in a maximal ideal of $M_\infty(X,\mathscr{A})$, and so every proper ideal of $M_\infty(X,\mathscr{A})$ is fixed. $\square$

Corollary 5.10. *Every proper ideal of $M_K(X,\mathscr{A})$ is fixed.*

Proof. The proof is similar to the proof of Corollary 5.9. $\square$

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