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COAXIAL FILTERS OF DISTRIBUTIVE LATTICES

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Abstract. Coaxial filters and strongly coaxial filters are introduced in distributive lattices and some characterization theorems of *pm*-lattices are given in terms of co-annihilators. Some properties of coaxial filters of distributive lattices are studied. The concept of normal prime filters is introduced and certain properties of coaxial filters are investigated. Some equivalent conditions are derived for the class of all strongly coaxial filters to become a sublattice of the filter lattice.

Introduction

In 1968, the theory of relative annihilators was introduced in lattices by Mark Mandelker [5] and he characterized distributive lattices in terms of their relative annihilators. Later many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. T.P. Speed [9] and W.H. Cornish [4] made an extensive study of annihilators in distributive lattices. The class of annulets played a vital role in characterizing many algebraic structures like normal lattices [3] and quasi-complemented lattices [4]. In [6], Y.S. Pawar and N.K. Thakare introduced the class of *pm*-lattices and characterized the *pm*-latices in topological terms. In [8], the author thoroughly investigated certain significant properties of co-annihilators, co-annihilator filters and μ-filters of distributive lattices. The main aim of this paper is to study some properties of coaxial filters of distributive lattices.

In this note, the concepts of coaxial filters and strongly coaxial filters are introduced in terms of co-annihilators of distributive lattices. *pm*-lattices are once again characterized in terms of co-annihilators and maximal ideals of distributive lattices. A set of equivalent conditions is derived for every filter of a distributive lattice to become a coaxial filter. The notion of normal prime filters is introduced and proved that every normal prime filter is a coaxial filter as well as a minimal prime filter. Some properties of coaxial filters are derived with respect to inverse homomorphic images and cartesian products. The notion of weakly *pm*-lattices is introduced. Some equivalent conditions are derived for every weakly *pm*-lattice to become a *pm*-lattice. A set of equivalent conditions is derived for every filter of a

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distributive lattice to become a strongly coaxial filter. Finally, a set of equivalent conditions is deduced for the class of all strongly coaxial filters of a distributive lattice to become a sublattice of the filter lattice.

1. Preliminaries

The reader is referred to [1] and [2] for the elementary notions and notations of distributive lattices. However some of the preliminary definitions and results are presented for the ready reference of the reader.

A non-empty subset A of a lattice L is called an ideal (filter) of L if $a \vee b \in A$ ($a \wedge b \in A$) and $a \wedge x \in A$ ($a \vee x \in A$) whenever $a, b \in A$ and $x \in L$. The set $(a] = \{x \in L \mid x \leq a\}$ (resp. $[a) = \{x \in L \mid a \leq x\}$) is called a principal ideal (resp. principal filter) generated by a. The set $\mathcal{I}(L)$ of all ideals of a distributive lattice L with 0 forms a complete distributive lattice. The set $\mathcal{F}(L)$ of all filters of a distributive lattice L with 1 forms a complete distributive lattice. A proper ideal P of a lattice L is called *prime* if for any $x, y \in L$, $x \wedge y \in P$ implies $x \in P$ or $y \in P$. A proper ideal M of a lattice is called maximal if there exists no proper ideal N such that $M \subset N$.

A bounded distributive lattice L is called a *pm*-lattice if every prime ideal of L is contained in a unique maximal ideal of L. Pawar and Thakare [6] have proved that if L is a *pm*-lattice then the space $\max(L)$ of all maximal ideals of the lattice L is a compact T_2-space (and hence it is normal). A proper filter P of L is said to be prime if for any $x, y \in L$, $x \vee y \in P$ implies that $x \in P$ or $y \in P$. A prime filter P of a lattice L is called minimal if it is the minimal element in the class of all prime filters.

Theorem 1.1 ([7]). *A prime filter P of a distributive lattice L is minimal if and only if to each $x \in P$ there exists $y \notin P$ such that $x \vee y = 1$.*

For any subset A of a distributive lattice L with 1, the co-annihilator of A is defined as the set $A^+ = \{x \in L \mid x \vee a = 1 \text{ for all } a \in A\}$. For any subset A of L, A^+ is a filter of L with $A \cap A^+ \subseteq \{1\}$.

Lemma 1.2 ([8]). *Let L be a distributive lattice with 1. For any subsets A and B of L, the following properties hold:*

(1) $A \subseteq B$ *implies* $B^+ \subseteq A^+$,

(2) $A \subseteq A^{++}$,

(3) $A^{+++} = A^+$,

(4) $A^+ = L$ *if and only if* $A \subseteq \{1\}$.

In case of filters, we have the following result.

Proposition 1.3 ([8]). *Let L be a distributive lattice with 1. For any filters F, G and H of L, the following properties hold:*

(1) $F^+ \cap F^{++} = \{1\}$,

(2) $F \cap G = \{1\}$ *implies* $F \subseteq G^+$,

(3) $(F \vee G)^+ = F^+ \cap G^+$,

(4) $(F \cap G)^{++} = F^{++} \cap G^{++}$.

It is clear that $([x))^+ = (x)^+$. Then clearly $(0)^+ = \{1\}$. The following corollary is a direct consequence of the above results.

Corollary 1.4 ([8]). *Let L be a distributive lattice with 1. For any a, b, $c \in L$,*

(1) *$a \leq b$ implies $(a)^+ \subseteq (b)^+$,*

(2) *$(a \wedge b)^+ = (a)^+ \cap (b)^+$,*

(3) *$(a \vee b)^{++} = (a)^{++} \cap (b)^{++}$,*

(4) *$(a)^+ = L$ if and only if $a = 1$.*

A filter F of a distributive lattice L with 1 is called a *co-annihilator filter* [8] if $F = F^{++}$. A filter F of a distributive lattice L with 1 is called a μ-filter of L if $x \in F$ implies $(x)^{++} \subseteq F$ for all $x \in L$. Every co-annihilator filter of a distributive lattice is a μ-filter. In [8], it is notices that the poset of all co-annihilator filters forms a complete Boolean algebra.

2. Coaxial filters

In this section, the concept of coaxial filters is introduced in lattices. The class of *pm*-lattices is characterized in terms of co-annihilators. A set of equivalent conditions is derived for every filter of a lattice to become a coaxial filter.

Definition 2.1. For any subset A of a bounded distributive lattice L, define

$$A^\diamond = \{x \in L \mid (a)^+ \vee (x)^+ = L \text{ for all } a \in A\}.$$

Clearly $\{1\}^\diamond = L$ and $L^\diamond = \{1\}$. For any $a \in L$, we denote $(\{a\})^\diamond$ by $(a)^\diamond$. Then it is obvious that $(0)^\diamond = \{1\}$ and $(1)^\diamond = L$. Clearly $A \cap A^\diamond = \{1\}$.

Proposition 2.2. *For any subset A of a bounded distributive lattice L, $A^\diamond$ is a filter of L.*

Proof. Clearly $1 \in A^\diamond$. Let $x, y \in A^\diamond$. For any $a \in A$, we get $(x \wedge y)^+ \vee (a)^+ = \{(x)^+ \cap (y)^+\} \vee (a)^+ = \{(x)^+ \vee (a)^+\} \cap \{(y)^+ \vee (a)^+\} = L \cap L = L$. Hence $x \wedge y \in A^\diamond$. Again, let $x \in A^\diamond$ and $x \leq y$. Then we get $(x)^+ \vee (a)^+ = L$ for any $a \in A$ and $(x)^+ \subseteq (y)^+$. For any $c \in A$, we then get $L = (x)^+ \vee (a)^+ \subseteq (y)^+ \vee (a)^+$. Hence $y \in A^\diamond$. Therefore $A^\diamond$ is a filter of L. $\square$

Note that the operation $^\diamond$ is an antitone Galois connection on the complete lattice of all filters of a bounded distributive lattice. Keeping in view of this fact, it can be concluded that the following couple of results are direct consequences.

Lemma 2.3. *For any two subsets A and B of a bounded distributive lattice L, the following properties hold:*

(1) *$A \subseteq B$ implies $B^\diamond \subseteq A^\diamond$,*

(2) *$A \subseteq A^{\diamond\diamond}$,*

(3) *$A^{\diamond\diamond\diamond} = A^\diamond$,*

(4) *$A^\diamond = L$ if and only if $A \subseteq \{1\}$.*

In case of filters, we have the following result.

Proposition 2.4. *For any two filters F and G of a bounded distributive lattice L, $(F \vee G)^\diamond = F^\diamond \cap G^\diamond$.*

The following corollary is a direct consequence of the above results.

Corollary 2.5. *Let L be a bounded distributive lattice. For any a, $b \in L$, the following properties hold:*

(1) *$a \leq b$ implies $(a)^\diamond \subseteq (b)^\diamond$,*

(2) *$(a \wedge b)^\diamond = (a)^\diamond \cap (b)^\diamond$,*

(3) *$(a)^\diamond = L$ if and only if $a = 1$.*

For any filter F of a bounded distributive lattice L, it is easy to see that $F^\diamond \subseteq F^+$. However, a set of equivalent conditions is given for every filter to satisfy the reverse inclusion which is not true in general. This result leads to another characterization of *pm*-lattices.

Theorem 2.6. *Let L be a bounded distributive lattice. Then the following assertions are equivalent:*

(1) *L is a pm-lattice;*

(2) *for any $a, b \in L$ with $a \vee b = 1$, $(a)^+ \vee (b)^+ = L$;*

(3) *for any filters F, G of L, $F \cap G = \{1\}$ if and only if $F \subseteq G^\diamond$;*

(4) *for any filter F of L, $F^\diamond = F^+$;*

(5) *for any $a \in L$, $(a)^\diamond = (a)^+$;*

(6) *for any two maximal ideals M and N of L, there exist $a \notin M$ and $b \notin N$ such that $a \wedge b = 0$.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a *pm*-lattice. Then every prime ideal of L is contained in a unique maximal ideal of L. Let $a, b \in L$ with $a \vee b = 1$. Suppose $(a)^+ \vee (b)^+ \neq L$. Then there exists a prime ideal P such that $\{(a)^+ \vee (b)^+\} \cap P = \emptyset$. Then $P \vee (a]$ is an ideal of L such that $P \subseteq P \vee (a]$. Suppose $b \in P \vee (a]$. Then $b = t \vee a$ for some $t \in P$. Hence $1 = a \vee b = a \vee (t \vee a) = t \vee a$, which implies $t \in (a)^+ \subseteq (a)^+ \vee (b)^+$. Thus $t \in \{(a)^+ \vee (b)^+\} \cap P$, which is a contradiction. Therefore $b \notin P \vee (a]$, which means that $P \vee (a]$ is a proper ideal of L. Then there exists a maximal ideal M_1 such that $P \vee (a] \subseteq M_1$. Similarly, there exists a maximal ideal M_2 such that $P \vee (b] \subseteq M_2$. Since $a \vee b = 1$, we get $b \notin M_1$ and $a \notin M_2$. Therefore $M_1 \neq M_2$. Thus the prime ideal P is contained in two distinct maximal ideals, which is a contradiction to the hypothesis. Therefore $(a)^+ \vee (b)^+ = L$.

(2) $\Rightarrow$ (3): Assume condition (2). Let F and G be two filters of L. Suppose $F \cap G = \{1\}$. Let $x \in F$. For any $a \in G$, we get $x \vee a \in F \cap G = \{1\}$. Hence $x \vee a = 1$. By condition (2), we get $(x)^+ \vee (a)^+ = L$. Thus $x \in G^\diamond$. Therefore $F \subseteq G^\diamond$. Conversely, suppose that $F \subseteq G^\diamond$. Let $x \in F \cap G$. Then $x \in F \subseteq G^\diamond$. Hence $x \in G \cap G^\diamond = \{1\}$, which means $x = 1$. Therefore $F \cap G = \{1\}$.

(3) $\Rightarrow$ (4): Assume condition (3). Let F be a filter of L. Clearly $F^{\diamond} \subseteq F^{+}$. Conversely, let $x \in F^{+}$. Hence, for any $a \in F$, we have

$$\begin{aligned} x \vee a = 1 \quad &\Rightarrow \quad [x) \cap [a) = \{1\} \\ &\Rightarrow \quad [x) \subseteq (a)^{\diamond} \qquad \text{by (3)} \\ &\Rightarrow \quad [x) \subseteq (a)^{\diamond} \text{ for all } a \in F \\ &\Rightarrow \quad x \in F^{\diamond} \end{aligned}$$

which gives that $F^{+} \subseteq F^{\diamond}$. Therefore $F^{+} = F^{\diamond}$.

(4) $\Rightarrow$ (5): It is obvious.

(5) $\Rightarrow$ (6): Assume condition (5). Let M and N be two distinct maximal ideals of L. Choose $x \in M - N$. Since $x \notin N$, we get $N \vee (x] = L$. Hence, $a \vee x = 1$ for some $a \in N$. Since $a \vee x = 1$, by (5), we get $x \in (a)^{+} = (a)^{\diamond}$. Hence $(x)^{+} \vee (a)^{+} = L$. Then $0 \in (a)^{+} \vee (x)^{+}$. Then there exist two elements $s \in (a)^{+}$ and $t \in (x)^{+}$ such that $s \wedge t = 0$. If $s \in N$, then $1 = s \vee a \in N$, which is a contradiction. If $t \in M$, then $1 = t \vee x \in M$, which is also a contradiction. Therefore there exist $t \notin M$ and $s \notin N$ such that $s \wedge t = 0$.

(6) $\Rightarrow$ (1): Assume condition (6). Let P be a prime ideal of L. Let M_1 and M_2 be two maximal ideals of L such that $P \subseteq M_1$ and $P \subseteq M_2$. Suppose $M_1 \neq M_2$. By (6), there exists two elements $x, y \in L$ such that $x \notin M_1$ and $y \notin M_2$ such that $x \wedge y = 0$. Since $x \notin M_1$ and $y \notin M_2$, we get that $x \notin P$ and $y \notin P$. Therefore, we get $0 = x \wedge y \notin P$, which is a contradiction. Hence, P should be contained in a unique maximal ideal. Therefore L is a *pm*-lattice. $\square$

Definition 2.7. A filter F of a bounded distributive lattice L is called a *coaxial filter* if for all x, $y \in L$, $(x)^{\diamond} = (y)^{\diamond}$ and $x \in F$ imply that $y \in F$.

Clearly each $(x)^{\diamond}$, $x \in L$ is a coaxial filter of L. It is evident that any filter F of a lattice L is a coaxial filter if it satisfies $(x)^{\diamond\diamond} \subseteq F$ for all $x \in F$.

Theorem 2.8. *The following assertions are equivalent in a bounded distributive lattice L:*

(1) *every filter is a coaxial filter;*

(2) *every principal filter is a coaxial filter;*

(3) *every prime filter is a coaxial filter;*

(4) *for a, $b \in L$, $(a)^{\diamond} = (b)^{\diamond}$ implies $[a) = [b)$.*

Proof. (1) $\Rightarrow$ (2): It is clear.

(2) $\Rightarrow$ (3): Assume that every principal filter is a coaxial filter. Let P be a prime filter of L. Suppose $(a)^{\diamond} = (b)^{\diamond}$ and $a \in P$. Then clearly $[a) \subseteq P$. Since $(a)^{\diamond} = (b)^{\diamond}$ and $[a)$ is a coaxial filter, we get that $b \in [a) \subseteq P$. Therefore P is a coaxial filter.

(3) $\Rightarrow$ (4): Assume that every prime filter of L is a coaxial filter. Let a, $b \in L$ such that $(a)^{\diamond} = (b)^{\diamond}$. Suppose $[a) \neq [b)$. Without loss of generality assume that $[a) \nsubseteq [b)$. Consider $\Sigma = \{F \in \mathcal{F}(L) \mid a \vee b \in F \text{ and } a \notin F\}$. Then clearly $[a \vee b) \in \Sigma$. Let $\{F_i\}_{i \in \Delta}$ be a chain in Σ. Then clearly $\bigcup\limits_{i \in \Delta} F_i$ is a filter, $a \vee b \in \bigcup\limits_{i \in \Delta} F_i$ and

$a \notin \bigcup_{i\in\Delta} F_i$. Hence $\bigcup_{i\in\Delta} F_i$ is an upper bound for $\{F_i\}_{i\in\Delta}$ in Σ. Therefore, by the Zorn's Lemma, Σ has a maximal element, say P. We now prove that P is a prime filter in L. Let x, $y \in L$ be such that $x \notin P$ and $y \notin P$. Hence $P \subset P \vee [x)$ and $P \subset P \vee [y)$. Therefore by the maximality of P, $P \vee [x)$ and $P \vee [y)$ are not in Σ. Hence $a \in P \vee [x)$ and $a \in P \vee [y)$. Therefore, we have

$$\begin{aligned} a &\in \{\, P \vee [x) \,\} \cap \{\, P \vee [y) \,\} \\ &= P \vee \{\, [x) \cap [y) \,\} \\ &= P \vee [x \vee y)\,. \end{aligned}$$

If $x \vee y \in P$, then $a \in P \vee (x \vee y] = P$, which is a contradiction to that $a \notin P$. Thus we get $x \vee y \notin P$. Hence P is a prime filter. Therefore by hypothesis (3), we can get that P is a coaxial filter of L. Since $P \in \Sigma$, we get that $a \vee b \in P$ and $a \notin P$. Since P is prime, we get $b \in P$. Since $b \in P$ and P is coaxial, we get $a \in P$, which is a contradiction to $a \notin P$. Therefore $[a) = [b)$.

(4) $\Rightarrow$ (1): Assume condition (4). Let F be a filter of L. Suppose a, $b \in L$ be such that $(a)^{\diamond} = (b)^{\diamond}$. Then by (4), we get that $[a) = [b)$. Suppose $a \in F$. Then we get $b \in [b) = [a) \subseteq F$. Therefore F is a coaxial filter of L. $\square$

The notion of normal prime filters is now introduced.

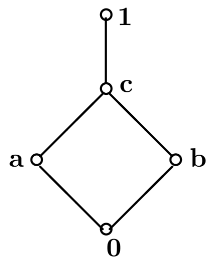
Definition 2.9. A prime filter P of a bounded distributive lattice L is called a *normal prime filter* if to each $x \in P$, there exists $x' \notin P$ such that $(x)^{\diamond} \vee (x')^{\diamond} = L$.

Proposition 2.10. *Every normal prime filter is a minimal prime filter.*

Proof. Let P be a normal prime filter of a bounded distributive lattice L. Suppose $x \in P$. Since P is normal, there exists $x' \notin P$ such that $(x)^{\diamond} \vee (x')^{\diamond} = L$. Hence we get $L = (x)^{\diamond} \vee (x')^{\diamond} \subseteq (x \vee x')^{\circ}$. Thus by Corollary 2.5(3), we get that $x \vee x' = 1$. Therefore P is a minimal prime filter of L. $\square$

In general, the converse of the above proposition is not true, i.e. every minimal prime filter need not be a normal filter. It can be seen in the following example.

Example 2.11. Consider the following bounded distributive lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given by:



Consider the prime filter $P = \{1, a, c\}$. It can be easily observed that P is a minimal prime filter but not a normal prime filter.

However, in the following proposition, we derive a sufficient condition for every minimal prime filter to become a normal prime filter.

Proposition 2.12. *If L is a pm-lattice, then every minimal prime filter of L is a normal prime filter.*

Proof. Assume that L is a *pm*-lattice and P a minimal prime filter of L. Let $x \in P$. Then there exists $x' \notin P$ such that $x \vee x' = 1$. Since L is a *pm*-lattice, we get $(x)^\diamond \vee (x')^\diamond = (x)^+ \vee (x')^+ = L$. Therefore P is a normal prime filter in L. $\square$

Proposition 2.13. *Let P be a normal prime filter of a bounded distributive lattice L. Then for each $x \in L$, we have the following property:*

$$x \notin P \quad \textit{if and only if} \quad (x)^\diamond \subseteq P\,.$$

Proof. Let P be a normal prime filter of L and $x \in L$. Suppose $x \notin P$. Suppose $t \in (x)^\diamond$. Then $L = (t)^+ \vee (x)^+ \subseteq (t \vee x)^+$. Hence $t \vee x = 1$. Since P is prime and $x \notin P$, we must have $t \in P$. Therefore $(x)^\diamond \subseteq P$. Conversely, assume that $(x)^\diamond \subseteq P$. Suppose $x \in P$. Since P is normal prime, there exists $x' \notin P$ such that $(x)^\diamond \vee (x')^\diamond = L$. Hence $L = (x)^\diamond \vee (x')^\diamond \subseteq (x)^+ \vee (x')^+$. Hence $x' \in (x)^\diamond \subseteq P$, which is a contradiction. Therefore $x \notin P$. $\square$

Theorem 2.14. *Every normal prime filter of a bounded distributive lattice is a coaxial filter.*

Proof. Let P be a normal prime filter of L. Suppose $x, y \in L$ such that $(x)^\diamond = (y)^\diamond$ and $x \in P$. Since P is normal, there exists $x' \notin P$ such that $(x)^\diamond \vee (x')^\diamond = L$. Hence $L = (x)^\diamond \vee (x')^\diamond = (y)^\diamond \vee (x')^\diamond \subseteq (y \vee x')^\diamond$. Hence by Corollary 2.5(3), we get $y \vee x' = 1 \in P$. Since P is prime and $x' \notin P$, it yields that $y \in P$. Therefore P is a coaxial filter. $\square$

In the following result, we prove a necessary and sufficient condition for the inverse image of a coaxial filter to become again a coaxial filter.

Theorem 2.15. *Let f be a homomorphism of bounded distributive lattices from $(L, \vee, \wedge, 0, 1)$ onto $(L', \vee, \wedge, 0, 1)$. Then the following assertions are equivalent:*

(1) *if G is a coaxial filter of L', then $f^{-1}(G)$ is a coaxial filter in L,*

(2) *for each $x \in L'$, $f^{-1}((x)^\diamond)$ is a coaxial filter in L.*

Proof. (1) $\Rightarrow$ (2): Assume that $f^{-1}(G)$ is a coaxial filter in L for each coaxial filter G of L'. Since $(x)^\diamond$ is a coaxial filter in L' for each $x \in L'$, we get from (1) that $f^{-1}((x)^\diamond)$ is a coaxial filter in L.

(2) $\Rightarrow$ (1): Assume that $f^{-1}((x)^\diamond)$ is a coaxial filter in L for each $x \in L'$. Let G be a coaxial filter of L'. Then clearly $f^{-1}(G)$ is a filter in L. Let x, $y \in L$ be such that $(x)^\diamond = (y)^\diamond$ and $x \in f^{-1}(G)$. Then $f(x) \in G$. For any $a \in L'$, we get

$$\begin{aligned}
a \in (f(x))^\diamond \quad &\Leftrightarrow \quad f(x) \in (a)^\diamond \\
&\Leftrightarrow \quad x \in f^{-1}((a)^\diamond) \\
&\Leftrightarrow \quad y \in f^{-1}((a)^\diamond) \qquad \text{since } f^{-1}((a)^\diamond) \text{ is coaxial in } L \\
&\Leftrightarrow \quad f(y) \in (a)^\diamond \\
&\Leftrightarrow \quad a \in (f(y))^\diamond
\end{aligned}$$

Hence $(f(x))^\diamond = (f(y))^\diamond$. Since $f(x) \in G$ and G is a coaxial filter, we get $f(y) \in G$. Hence $y \in f^{-1}(G)$. Therefore $f^{-1}(G)$ is a coaxial filter in L. $\square$

We now discuss the properties of direct products of coaxial filters of bounded distributive lattices. First we need the following lemma whose proof is routine.

Lemma 2.16. *Let L_1 and L_2 be two bounded distributive lattices. For any $a \in L_1$, $b \in L_2$ and $(a,b) \in L_1 \times L_2$, we have the following properties:*

(1) $(a,b)^+ = (a)^+ \times (b)^+$,

(2) $(a,b)^+ \vee (c,d)^+ = (a \vee c, b \vee d)^+$,

(3) $(a,b)^\diamond = (a)^\diamond \times (b)^\diamond$.

Theorem 2.17. *Let $L = L_1 \times L_2$ be the product of lattices L_1 and L_2. If F_1 and F_2 are coaxial filters of L_1 and L_2 respectively, then $F_1 \times F_2$ is a coaxial filter of the product lattice $L_1 \times L_2$. Conversely, every coaxial filter of $L_1 \times L_2$ can be expressed as $F = F_1 \times F_2$ where F_1 and F_2 are coaxial filters of L_1 and L_2, respectively.*

Proof. Let F_1 and F_2 be the coaxial filters of L_1 and L_2 respectively. Then clearly $F_1 \times F_2$ is a filter of $L_1 \times L_2$. Let $a, c \in L_1$ and $b, d \in L_2$ be such that $(a,b)^\diamond = (c,d)^\diamond$ and $(a,b) \in F_1 \times F_2$. Then $a \in F_1$ and $b \in F_2$. Since $(a,b)^\diamond = (c,d)^\diamond$, we get $(a)^\diamond \times (b)^\diamond = (c)^\diamond \times (d)^\diamond$ and hence $(a)^\diamond = (c)^\diamond$ and $(b)^\diamond = (d)^\diamond$. Since F_1 is a coaxial filter and $a \in F_1$, we get that $c \in F_1$. Similarly, we get $d \in F_2$. Hence $(c,d) \in F_1 \times F_2$. Therefore $F_1 \times F_2$ is a coaxial filter in $L_1 \times L_2$.

Conversely, let F be a coaxial filter of $L_1 \times L_2$. Consider $F_1 = \{a \in L_1 \mid (a,1) \in F\}$ and $F_2 = \{a \in L_2 \mid (1,a) \in F\}$. Clearly, F_1 is a filter in L_1. Let x, $y \in L_1$ be such that $(x)^\diamond = (y)^\diamond$ and $x \in F_1$. Then $(x,1) \in F$. Since $(x)^\diamond = (y)^\diamond$, we get $(x,1)^\diamond = (x)^\diamond \times (1)^\diamond = (y)^\diamond \times (1)^\diamond = (y,1)^\diamond$. Since F is a coaxial filter in $L_1 \times L_2$, we get $(y,1) \in F$. Hence $y \in F_1$. Therefore F_1 is a coaxial filter in L_1. Similarly, we can obtain that F_2 is a coaxial filter in L_2.

We now prove that $F = F_1 \times F_2$. Clearly $F \subseteq F_1 \times F_2$. Conversely, let $(a_1, a_2) \in F_1 \times F_2$. Then $a_1 \in F_1$ and $a_2 \in F_2$. Hence $(a_1, 1) \in F$ and $(1, a_2) \in F$. Hence $(a_1, 0) = (1,0) \wedge (a_1, 1) \in F$ and also $(0, a_2) = (0,1) \wedge (1, a_2) \in F$. Thus $(a_1, a_2) = (a_1, 0) \vee (0, a_2) \in F$. Therefore $F_1 \times F_2 \subseteq F$. □

We now introduce the concept of weakly *pm*-lattices.

Definition 2.18. A bounded distributive lattice L is called a *weakly pm-lattice* if it satisfies the property: $(x)^+ \vee (y)^+ = (x)^\diamond \vee (y)^\diamond$ for all $x, y \in L$.

It is evident that every *pm*-lattice is a weakly *pm*-lattice. In general, the converse is not true. However, in the following, a set of equivalent conditions is derived for every weakly *pm*-lattice to become a *pm*-lattice.

Theorem 2.19. *Let L be a weakly pm-lattice. Then the following are equivalent:*

(1) *L is a pm-lattice;*

(2) *for x, $y \in L, (x)^\diamond \vee (y)^\diamond = (x \vee y)^\diamond$;*

(3) *for x, $y \in L$, $x \vee y = 1$ implies $(x)^\diamond \vee (y)^\diamond = L$.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a *pm*-lattice. Let x, $y \in L$. Since L is a *pm*-lattice, by Theorem 2.6, we get $(x)^\diamond \vee (y)^\diamond = (x)^+ \vee (y)^+ = (x \vee y)^+ = (x \vee y)^\diamond$.
(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (1): Assume that condition (3) is satisfied. Let $x, y \in L$ be such that $x \vee y = 1$. Since L is weakly *pm*-lattice, we get $L = (x)^\diamond \vee (y)^\diamond = (x)^+ \vee (y)^+$. By Theorem 2.6, it yields that L is a *pm*-lattice. □

Corollary 2.20. *A weakly pm-lattice in which every prime filter is normal is a pm-lattice.*

Proof. Let L be a weakly *pm*-lattice. Let $a, b \in L$ be such that $a \vee b = 1$. Suppose $(x)^\diamond \vee (y)^\diamond \neq L$. Then there exists a prime filter P such that $(x)^\diamond \vee (y)^\diamond \subseteq P$. Then $(x)^\diamond \subseteq P$ and $(y)^\diamond \subseteq P$. Since P is normal, by Proposition 2.13, we get $x \notin P$ and $y \notin P$. Hence $1 = x \vee y \notin P$ which is a contradiction. Thus $(x)^\diamond \vee (y)^\diamond = L$. By the main theorem, L is a *pm*-lattice. □

3. Strongly coaxial filters

In this section, the concept of strongly coaxial filters is introduced in bounded distributive lattices. A set of equivalent conditions is derived for the class of all strongly coaxial filters to become a sublattice to the filter lattice.

Definition 3.1. For any filter F of a bounded distributive lattice L, define

$$\eta(F) = \{x \in L \mid (x)^\diamond \vee F = L\}.$$

The following lemma is an immediate consequence from the above definition.

Lemma 3.2. *For any two filters F, G of a bounded distributive lattice L, we have*

(1) $\eta(F) \subseteq F$,

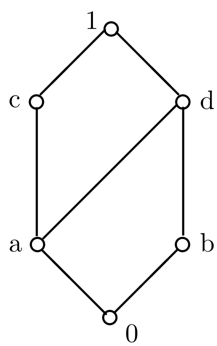
(2) $F \subseteq G$ *implies* $\eta(F) \subseteq \eta(G)$,

(3) $\eta(F \cap G) = \eta(F) \cap \eta(G)$.

Proof. (1) Let $x \in \eta(F)$. Then $(x)^\diamond \vee F = L$. Hence $x = a \wedge b$ for some $a \in (x)^\diamond \subseteq (x)^+$ and $b \in F$. Then $x \vee a = 1$ and $x \vee b \in F$. Thus $x = x \vee x = x \vee (a \wedge b) = (x \vee a) \wedge (x \vee b) = 1 \wedge (x \vee b) = x \vee b \in F$. Therefore $\eta(F) \subseteq F$.
(2) and (3) can be routinely verified. □

However, the closure property of the operation η does not hold in a bounded distributive lattice. That is $\eta(\eta(F))$ and F need not be the same for any filter F of a bounded distributive lattice. It can be seen in the following example:

Example 3.3. Consider the following bounded distributive lattice $L = \{0, a, b, c, d, 1\}$ whose Hasse diagram is given by:



Observe that $(a)^+ = \{1\}$, $(b)^+ = (d)^+ = \{1, c\}$ and $(c)^+ = \{1, b, d\}$. Hence $(a)^\diamond = \{1\}$, $(b)^\diamond = (d)^\diamond = \{1, c\}$ and $(c)^\diamond = \{1, b, d\}$. Consider the filter $F = \{1, c\}$. Clearly $\eta(F) = \{1, c\}$ and hence $\eta(\eta(F)) = \{1\}$. Therefore $\eta(\eta(F)) \neq F$.

Proposition 3.4. *For any filter F of a bounded distributive lattice L, $\eta(F)$ is a filter of L.*

Proof. Clearly $1 \in \eta(F)$. Let x, $y \in \eta(F)$. Then $(x)^\diamond \vee F = L$ and $(y)^\diamond \vee F = L$. Hence $(x \wedge y)^\diamond \vee F = \{(x)^\diamond \cap (y)^\diamond\} \vee F = \{(x)^\diamond \vee F\} \cap \{(y)^\diamond \vee F\} = L$. Hence $x \wedge y \in \eta(F)$. Again let $x \in \eta(F)$ and $x \leq y$. Since $x \leq y$, we get $(x)^\diamond \subseteq (y)^\diamond$. Then $L = (x)^\diamond \vee F \subseteq (y)^\diamond \vee F$. Thus $y \in \eta(F)$. Therefore $\eta(F)$ is a filter of L. □

Definition 3.5. A filter F of a bounded distributive lattice L is called *strongly coaxial* if $F = \eta(F)$.

Proposition 3.6. *Every strongly coaxial filter is a coaxial filter.*

Proof. Let F be a strongly coaxial filter of a lattice L. Then $F = \eta(F)$. Let x, $y \in L$ be such that $(x)^\diamond = (y)^\diamond$ and $x \in F = \eta(F)$. Then clearly $(x)^\diamond \vee F = L$. Hence $(y)^\diamond \vee F = L$ and so $y \in \eta(F) = F$. Thus F is a coaxial filter of L. □

In general, the converse of the above proposition is not true. However, in the following theorem, we derive a set of equivalent conditions for every filter of a bounded distributive lattice to become strongly coaxial.

Theorem 3.7. *Consider the following assertions in a bounded distributive lattice L:*

(1) *every prime filter is normal,*

(2) *every filter is strongly coaxial,*

(3) *every prime filter is strongly coaxial.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3). *If L is a weakly pm-lattice, then all the above conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2): Assume that every prime filter is normal. Let F be a filter of L. Clearly $\eta(F) \subseteq F$. Conversely, let $x \in F$. Suppose $(x)^\diamond \vee F \neq L$. Then there exists a prime filter P of L such that $(x)^\diamond \vee F \subseteq P$. Hence $(x)^\diamond \subseteq P$ and $x \in F \subseteq P$. Since P is normal and $(x)^\diamond \subseteq P$, by Proposition 2.13, we get that $x \notin P$, which is a contradiction to that $x \in P$. Hence $(x)^\diamond \vee F = L$. Thus $x \in \eta(F)$. Therefore F is strongly coaxial.
(2) $\Rightarrow$ (3): It is obvious.
Suppose that L is a weakly *pm*-lattice.
(3) $\Rightarrow$ (1): Assume that every prime filter is strongly coaxial. Let P be a prime filter of L. Then by our assumption, $\eta(P) = P$. Let $x \in P$. Then $(x)^\diamond \vee P = L$. Hence $a \wedge b = 0$ for some $a \in (x)^\diamond$ and $b \in P$. Since $a \in (x)^\diamond$ and L is a weakly *pm*-lattice, we get $(x)^\diamond \vee (a)^\diamond = (x)^+ \vee (a)^+ = L$. Suppose $a \in P$. Then $0 = a \wedge b \in P$, which is a contradiction. Thus $a \notin P$ and hence P is a normal prime filter of L. □

Theorem 3.8. *The following assertions are equivalent in a bounded distributive lattice L:*

(1) $(x)^{\diamond} \vee (x)^{\diamond\diamond} = L$ *for all* $x \in L$;

(2) *every filter of the form* $F = F^{\diamond\diamond}$ *is strongly coaxial;*

(3) *for each* $x \in L, (x)^{\diamond\diamond}$ *is strongly coaxial.*

Proof. (1) $\Rightarrow$ (2): Assume condition (1). Let F be a filter of L such that $F = F^{\diamond\diamond}$. Clearly $\eta(F) \subseteq F$. Conversely, let $x \in F$. Clearly $(x)^{\diamond\diamond} \subseteq F^{\diamond\diamond}$. Hence $L = (x)^{\diamond} \vee (x)^{\diamond\diamond} \subseteq (x)^{\diamond} \vee F^{\diamond\diamond} = (x)^{\diamond} \vee F$. Thus $x \in \eta(F)$. Therefore F is a strongly coaxial filter of L.
(2) $\Rightarrow$ (3): It is obvious.
(3) $\Rightarrow$ (1): Assume condition (3). Then we get $\eta((x)^{\diamond\diamond}) = (x)^{\diamond\diamond}$. Since $x \in (x)^{\diamond\diamond}$, we get $(x)^{\diamond} \vee (x)^{\diamond\diamond} = L$. □

Definition 3.9. For any maximal filter M of a bounded distributive lattice L, define $\Omega(M) = \{x \in L \mid (x)^{\diamond} \nsubseteq M\}$.

For any maximal filter M of a bounded distributive lattice L, it can be easily observed that $\eta(M) = \Omega(M)$. Thus it can be easily seen that the set $\Omega(M)$ is a filter of L such that $\Omega(M) \subseteq M$. Let us denote that μ is the set of all maximal filter of a bounded distributive lattice L. For any filter F of a bounded distributive lattice L, let us consider that $\mu(F) = \{M \in \mu \mid F \subseteq M\}$.

Theorem 3.10. *Suppose* $\mu(F)$ *is finite for any filter* F *of a bounded distributive lattice* L. *Then* $\eta(F) = \bigcap_{M\in\mu(F)} \Omega(M)$.

Proof. Let $x \in \eta(F)$ and $F \subseteq M$ where $M \in \mu$. Then $L = (x)^{\diamond} \vee F \subseteq (x)^{\diamond} \vee M$. Suppose $(x)^{\diamond} \subseteq M$, then $M = L$, which is a contradiction. Hence $(x)^{\diamond} \nsubseteq M$. Thus $x \in \Omega(M)$ for all $M \in \mu(F)$. Therefore $\eta(F) \subseteq \bigcap_{M\in\mu(F)} \Omega(M)$. Conversely, let $x \in \bigcap_{M\in\mu(F)} \Omega(M)$. Then $x \in \Omega(M)$ for all $M \in \mu(F)$. Suppose $(x)^{\diamond} \vee F \neq L$. Then there exists a maximal filter M_0 such that $(x)^{\diamond} \vee F \subseteq M_0$. Hence $(x)^{\diamond} \subseteq M_0$ and $F \subseteq M$. Since $F \subseteq M_0$, by hypothesis, we get $x \in \Omega(M_0)$. Hence $(x)^{\diamond} \nsubseteq M_0$, which is a contradiction. Hence $(x)^{\diamond} \vee F = L$. Thus $x \in \eta(F)$. Therefore $\bigcap_{M\in\mu(F)} \Omega(M) \subseteq \eta(F)$. □

From the above theorem, it can be easily observed that $\eta(F) \subseteq \Omega(M)$ for every $M \in \mu(F)$. In the following, we derive a set of equivalent conditions for the class of all strongly coaxial filters of a lattice to become a sublattice of the filter lattice $\mathcal{F}(L)$ of the bounded distributive lattice L.

Theorem 3.11. *Suppose* $\mu(F)$ *is finite for any filter* F *of a bounded distributive lattice* L. *Then the following assertions are equivalent:*

(1) *for any* $M \in \mu$, $\Omega(M)$ *is maximal;*

(2) *for any* $F, G \in \mathcal{F}(L)$, $F \vee G = L$ *implies* $\eta(F) \vee \eta(G) = L$;

(3) *for any* $F, G \in \mathcal{F}(L)$, $\eta(F) \vee \eta(G) = \eta(F \vee G)$;

(4) *for any two distinct maximal filters* M *and* N, $\Omega(M) \vee \Omega(N) = L$;

(5) *for any $M \in \mu$, M is the unique member of μ such that $\Omega(M) \subseteq M$.*

Proof. (1) $\Rightarrow$ (2) : Assume condition (1). Then clearly $\Omega(M) = M$ for all $M \in \mu$. Let $F, G \in \mathcal{F}(L)$ be such that $F \vee G = L$. Suppose $\eta(F) \vee \beta(G) \neq L$. Then there exists a maximal filter M such that $\eta(F) \vee \eta(G) \subseteq M$. Hence $\eta(F) \subseteq M$ and $\eta(G) \subseteq M$. Now

$$\begin{aligned}
\eta(F) \subseteq M \quad &\Rightarrow \quad \bigcap_{N \in \mu(F)} \Omega(N) \subseteq M \\
&\Rightarrow \quad \Omega(M_i) \subseteq M \quad \text{for some } M_i \in \mu(F) \text{ (since } M \text{ is prime)} \\
&\Rightarrow \quad M_i \subseteq M \qquad \text{by condition (1)} \\
&\Rightarrow \quad F \subseteq M \qquad \text{since } M_i \in \mu(F)
\end{aligned}$$

Similarly, we can get $G \subseteq M$. Hence $L = F \vee G \subseteq M$, which is a contradiction. Therefore $\eta(F) \vee \eta(G) = L$.

(2) $\Rightarrow$ (3) : Assume condition (2). Let $F, G \in \mathcal{F}(L)$. Clearly $\eta(F) \vee \eta(G) \subseteq \eta(F \vee G)$. Let $x \in \eta(F \vee G)$. Then $((x)^\diamond \vee F) \vee ((x)^\diamond \vee G) = (x)^\diamond \vee F \vee G = L$. Hence by condition (2), we get $\eta((x)^\diamond \vee G) \vee \eta((x)^\diamond \vee G) = L$. Thus $x \in \eta((x)^\diamond \vee F) \vee \eta((x)^\diamond \vee G)$. Hence $x = r \wedge s$ for some $r \in \eta((x)^\diamond \vee F)$ and $s \in \eta((x)^\diamond \vee G)$. Now

$$\begin{aligned}
r \in \eta((x)^\diamond \vee F) \quad &\Rightarrow \quad (r)^\diamond \vee (x)^\diamond \vee F = L \\
&\Rightarrow \quad L = ((r)^\diamond \vee (x)^\diamond) \vee F \subseteq (r \vee x)^\diamond \vee F \\
&\Rightarrow \quad (r \vee x)^\diamond \vee F = L \\
&\Rightarrow \quad r \vee x \in \eta(F)
\end{aligned}$$

Similarly, we can get $s \vee x \in \eta(G)$. Hence

$$\begin{aligned}
x &= x \vee x \\
&= x \vee (r \wedge s) \\
&= (x \vee r) \wedge (x \vee s) \in \eta(F) \vee \eta(G)
\end{aligned}$$

Hence $\eta(F \vee G) \subseteq \eta(F) \vee \eta(G)$. Therefore $\eta(F) \vee \eta(G) = \eta(F \vee G)$.

(3) $\Rightarrow$ (4) : Assume condition (3). Let M and N be two distinct maximal filters of L. Choose $x \in M - N$ and $y \in N - M$. Since $x \notin N$, there exists $x_1 \in N$ such that $x \wedge x_1 = 0$. Since $y \notin M$, there exists $y_1 \in M$ such that $y \wedge y_1 = 0$. Hence $(x \wedge y_1) \wedge (y \wedge x_1) = (x \wedge x_1) \wedge (y \wedge y_1) = 0$. Now

$$\begin{aligned}
L &= \eta(L) \\
&= \eta([0)) \\
&= \eta([(x \wedge y_1) \wedge (y \wedge x_1))) \\
&= \eta([x \wedge y_1) \vee [y \wedge x_1)) \\
&= \eta([x \wedge y_1)) \vee \eta([y \wedge x_1)) \qquad \text{by condition (4)} \\
&\subseteq \Omega(M) \vee \Omega(N) \qquad \text{since } [x \wedge y_1) \subseteq M, [y \wedge x_1) \subseteq N
\end{aligned}$$

Therefore $\Omega(M) \vee \Omega(N) = L$.

(4) $\Rightarrow$ (5) : Assume condition (4). Let $M \in \mu$. Suppose $N \in \mu$ such that $N \neq M$

and $\Omega(N) \subseteq M$. Since $\Omega(M) \subseteq M$, by hypothesis, we get $L = \Omega(M) \vee \Omega(N) = M$, which is a contradiction. Therefore M is the unique maximal filter such that $\Omega(M)$ is contained in M.

(5) $\Rightarrow$ (1) : Let $M \in \mu$. Suppose $\Omega(M)$ is not maximal. Let M_0 be a maximal filter of L such that $\Omega(M) \subseteq M_0$. We have always $\Omega(M_0) \subseteq M_0$, which is a contradiction to the hypothesis. Therefore $\Omega(M)$ is maximal. □

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