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# ON THE DIOPHANTINE EQUATION $x^2 + 2^a 3^b 73^c = y^n$

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Abstract. In this paper, we find all integer solutions $(x, y, n, a, b, c)$ of the equation in the title for non-negative integers $a, b$ and $c$ under the condition that the integers $x$ and $y$ are relatively prime and $n \geq 3$. The proof depends on the famous primitive divisor theorem due to Bilu, Hanrot and Voutier and the computational techniques on some elliptic curves.

## 1. Introduction

In recent years, many papers deal with the Diophantine equation

$$x^2 + p_1^{\alpha_1} \dots p_k^{\alpha_k} = y^n,\ n \geq 3,\ \gcd(x, y) = 1 \tag{1}$$

in non negative integers $(x, y, \alpha_1, \dots, \alpha_k)$, where $p_i'$s are fixed prime. With the development of modern tools such as primitive divisor theorem, modular approach, or computational techniques, many authors investigated the above equation when $k \geq 1$, see for example [1, 2, 3, 7, 11, 13, 14, 15, 16, 17, 18, 19, 20, 22]. Especially, the cases $(p_1, p_2, p_3) = (2, 3, 11), (2, 11, 19), (2, 3, 19), (2, 3, 17), (5, 13, 17)$ are considered in [6, 9, 10, 11] and [21], respectively. For more information and the rich literature on equation (1), we refer to an excellent survey [12] and the 359 references therein.

In this paper, we study the equation (1) when $k = 3$ with $(p_1, p_2, p_3) = (2, 3, 73)$ and we get the following result.

**Theorem 1.** *All integer solutions of the equation*

$$x^2 + 2^a 3^b 73^c = y^n,\ n \geq 3,\ a, b, c \geq 0,\ x, y \geq 1,\ \gcd(x, y) = 1 \tag{2}$$

*are given by*

(1) *n=3: the solutions are given in Table* (1),
(2) *n=4: the solutions are given in Table* (2),
(3) *n=6:* $(x, y, a, b, c) = (2485, 19, 8, 7, 1), (15479, 25, 8, 5, 1), (42389, 35, 5, 5, 2)$,
(4) *n=8:* $(x, y, a, b, c) = (65, 3, 5, 0, 1)$,
(5) *n=9:* $(x, y, a, b, c) = (95, 3, 1, 0, 2)$,
(6) *n=12:* $(x, y, a, b, c) = (15479, 5, 8, 5, 1)$.

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## 2. Preliminaries

Let $\alpha, \beta$ be algebraic integers. A pair $(\alpha, \beta)$ is called a Lucas pair if $\alpha + \beta$ and $\alpha\beta$ are non-zero coprime rational integers and $\frac{\alpha}{\beta}$ is not a root of unity. For any Lucas pair $(\alpha, \beta)$, the corresponding sequences of Lucas numbers is defined by

$$L_n(\alpha, \beta) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; n = 0, 1, 2, \ldots .$$

A prime number $p$ is a primitive divisor of $L_n(\alpha, \beta)$ if $p \mid L_n(\alpha, \beta)$ and $p \nmid (\alpha - \beta)^2 L_1(\alpha, \beta) \ldots L_{n-1}(\alpha, \beta)$ $(n > 1)$. Among other things a primitive divisor $q$ of $L_n(\alpha, \beta)$ has the property that $q \equiv \left(\frac{(\alpha-\beta)^2}{q}\right) \pmod{n}$ where $\left(\frac{*}{*}\right)$ stands for Legendre symbol [8].

If $n > 4$ and $n \neq 6$ then every $n-$th term of any Lucas sequences $L_n(\alpha, \beta)$ has a primitive divisors except for an explicit finite list of parameters $\alpha, \beta$ and $n$ [4].

Let $S$ be any finite set of prime numbers. By an $S-$integer we mean a rational number $\frac{r}{s}$ with relatively prime integers $r$ and $s > 0$ such that any prime factor of $s$ belongs to $S$.

## 3. Proof of the Theorem 1

Our first approach to equation (2) is to factorize it in $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$ as

$$(x + e\sqrt{-d})(x - e\sqrt{-d}) = y^n$$

where $d \in \{1, 2, 3, 6, 73, 146, 219, 438)\}$ and $e = 2^\alpha 3^\beta 73^\gamma$ for some non negative integers $\alpha$, $\beta$ and $\gamma$. Assume that $y$ is even. Then $a = 0$ and $x$ is odd, and hence $x^2 \equiv 1 \pmod{8}$. Since $3^b 73^c \equiv 1, 3 \pmod{8}$ depending on the parity of $b$, we get by considering equation (2) modulo 8 that either $2 \equiv 0 \pmod{8}$ or $4 \equiv 0 \pmod{8}$, which is a contradiction in each case. So $y$ is always odd and hence the ideals generated by $x + e\sqrt{-d}$ and $x - e\sqrt{-d}$ are coprime in $\mathbb{K}$. For any choice of $d$, the class number $h(\mathbb{K})$ are only 1, 2, 4, 8 or 16. So we have that $\gcd(n, h(\mathbb{K})) = 1$ when $n$ is odd. Thus, in the case $n$ is odd, we write

$$\begin{cases} x + e\sqrt{-d} = u_1 \varepsilon^n \\ x - e\sqrt{-d} = u_2 \bar{\varepsilon}^n \,, \end{cases} \tag{3}$$

for an algebraic integer $\varepsilon$ in $\mathbb{K}$ and units $u_1$, $u_2$ in the ring of algebraic integers of $\mathbb{K}$.

Let $n$ be odd. Since, for all values of $d$, except for $d = 3$, the orders of multiplicative group of units in the ring of algebraic integers of $\mathbb{K}$ are either 2 or 4 which are relatively prime to $n$, the units $u_1$ and $u_2$ can be absorbed in the factors $\varepsilon^n$ and $\bar{\varepsilon}^n$. So we may omit the factors $u_1$ and $u_2$ in the equation (3) when $d \neq 3$. If $d = 3$ then the orders of multiplicative group of units in the ring of algebraic integers of $\mathbb{K}$ is 6. Therefore, in the case $5 \leq n$ is an odd prime, similar argument also valid for $d = 3$.

If $d \in \{1, 2, 6, 73, 146, 438\}$, then $-d \not\equiv 1 \pmod 4$ and hence we take $\{1, \sqrt{-d}\}$ as an integral basis of $\mathbb{K}$, whereas we take the set $\{1, \dfrac{1+\sqrt{-d}}{2}\}$ as an integral basis of $\mathbb{K}$ when $d \in \{3, 219\}$ since $-d \equiv 1 \pmod 4$ in this case. Thus we may take $\varepsilon = u + v\sqrt{-d}$ or $\varepsilon = \dfrac{u + v\sqrt{-d}}{2}$ for some integers $u$ and $v$. More precisely we write

$$\begin{cases} x + e\sqrt{-d} = \varepsilon^n = (u + v\sqrt{-d})^n \\ x - e\sqrt{-d} = \overline{\varepsilon}^n = (u - v\sqrt{-d})^n \end{cases}, \ y = u^2 + dv^2 \tag{4}$$

if $d \in \{1, 2, 6, 73, 146, 438)\}$ and

$$\begin{cases} x + e\sqrt{-d} = \varepsilon^n = \left(\dfrac{u + v\sqrt{-d}}{2}\right)^n \\ x - e\sqrt{-d} = \overline{\varepsilon}^n = \left(\dfrac{u - v\sqrt{-d}}{2}\right)^n \end{cases}, \ u \equiv v \pmod 2, \ y = \frac{u^2 + dv^2}{4} \tag{5}$$

if $d \in \{3, 219\}$ and $n \geq 5$ is an odd prime.

Now we continue the proof of Theorem 1 for the cases $n = 3$, $n = 4$ and $n \geq 5$ separately.

### 3.1. **The Case n=3.**

**Lemma 2.** *If $n = 3$ then all solutions $(x, y, a, b, c)$ of equation (2) are given in Table 1.*

**Proof.** Let $n = 3$. We write $a = 6a_1 + i$, $b = 6b_1 + j$, $c = 6c_1 + k$ where $i, j, k \in \{0, 1, \ldots, 5\}$. Thus, we may view equation (2) as an elliptic curve by writing it as $M^2 = N^3 - 2^i 3^j 73^k$ where $M = \left(\dfrac{x}{2^{3a_1} 3^{3b_1} 73^{3c_1}}\right)$ and $N = \left(\dfrac{y}{2^{2a_1} 3^{2b_1} 73^{2c_1}}\right)$ and therefore the problem of finding positive integer solutions of (2) is reduced to finding all $\{2, 3, 73\}-$integer points of the above 216 elliptic curves for each $i$ , $j$ and $k$. To find all $S-$integral points on the above curves we use the Magma function `SIntegralPoints` for $S = \{2, 3, 73\}$ [5]. Except for the seven triples $(i, j, k)$ given below, Magma could determine all $S-$integral points of the above curves and taking into account that $x$ and $y$ are coprime positive integers, exactly those which are given in Table (1) lead to solutions of the equation (2).

Those seven triples are $(i, j, k) = (1, 0, 5)$, $(1, 3, 5)$, $(1, 5, 5)$, $(2, 4, 5)$, $(5, 2, 5)$, $(5, 5, 3)$, $(5, 5, 5)$. We consider those values separately. Note that since

$$de^2 = 2^{6a_1+i} 3^{6b_1+j} 73^{6c_1+k}$$

and $d$ is square free, $d$ can be only 73, 146 or 438 and hence for all possibilities of $d$, we have that $-d \not\equiv 1 \pmod 4$. Thus, from (4) we get that

$$2e\sqrt{-d} = \varepsilon^3 - \overline{\varepsilon}^3 \,.$$

Therefore, by expanding the right hand side of above equation and equating the coefficients of $\sqrt{-d}$, we get that $e = v(3u^2 - dv^2)$, in other words

$$3u^2 = dv^2 \pm \frac{e}{v}\,.$$

Note that $\gcd(x, y) = 1$ implies that $\gcd(u, v) = 1$. We will use this fact without further mention in the following cases.

**Case 1:** $(i, j, k) = (1, 0, 5)$. Then $de^2 = 146(2^{3a_1}3^{3b_1}73^{3c_1+2})^2$, and therefore

$$3u^2 = 146v^2 \pm \frac{2^{3a_1}3^{3b_1}73^{3c_1+2}}{v}\,.$$

We have that either $v = \pm 2^{3a_1}3^{3b_1}73^{3c_1+2}$ or $v = \pm 2^{3a_1}3^{3b_1-1}73^{3c_1+2}$ when $b_1 > 0$. In the first case, we have that $3u^2 = 2^{6a_1+1}3^{6b_1}73^{6c_1+5} \pm 1$. From the congruence $0 \equiv 2 \cdot 3^{6b_1} \cdot 1 \pm 1 \pmod 3$, we deduce that $b_1 = 0$ and only the positive sign can be happen. So, reducing modulo 7, we get that $3u^2 \equiv 4 \pmod 7$, which is not possible in integers. Now assume that $b_1 > 0$ and $v = \pm 2^{3a_1}3^{3b_1-1}73^{3c_1+2}$. Then we have that $u^2 = 2^{6a_1+1}3^{6b_1-3}73^{6c_1+5} \pm 1$. By modulo 3, only the positive sign can occur. Thus reducing modulo 7, we find that $u^2 \equiv 5 \pmod 7$, a contradiction.

**Case 2:** $(i, j, k) = (1, 3, 5)$. Then $de^2 = 438(2^{3a_1}3^{3b_1+1}73^{3c_1+2})^2$, and therefore

$$3u^2 = 438v^2 \pm \frac{2^{3a_1}3^{3b_1+1}73^{3c_1+2}}{v}\,.$$

We have that either $v = \pm 2^{3a_1}3^{3b_1+1}73^{3c_1+2}$ or $v = \pm 2^{3a_1}3^{3b_1}73^{3c_1+2}$. The case $v = \pm 2^{3a_1}3^{3b_1+1}73^{3c_1+2}$ implies that

$$3u^2 = 438(2^{3a_1}3^{3b_1+1}73^{3c_1+2})^2 \pm 1\,.$$

But this is clearly false since it gives $0 \equiv \pm 1 \pmod 3$. If $v = \pm 2^{3a_1}3^{3b_1}73^{3c_1+2}$ then we get that

$$u^2 = 2^{6a_1+1}3^{6b_1}73^{6c_1+5} \pm 1\,.$$

We could not get the result using small values of congruence consideration to solve this equation. So we write

$$A^2 = B^3 \pm 2^2 73^4\,,$$

where $A = 2 \cdot 73^2 u$ and $B = 2^{2a_1+1}3^{2b_1}73^{2c_1+3}$. Computation in Magma shows that these elliptic curves have no integral points when $uv \neq 0$.

**Case 3:** $(i, j, k) = (1, 5, 5)$. Then

$$3u^2 = 438v^2 \pm \frac{2^{3a_1}3^{3b_1+2}73^{3c_1+2}}{v}\,.$$

So we have that either $v = \pm 2^{3a_1}3^{3b_1+2}73^{3c_1+2}$ or $v = \pm 2^{3a_1}3^{3b_1+1}73^{3c_1+2}$. We see that the case $v = \pm 2^{3a_1}3^{3b_1+2}73^{3c_1+2}$ is not possible just reducing the resulting equation modulo 3. If $v = \pm 2^{3a_1}3^{3b_1+1}73^{3c_1+2}$ then

$$u^2 = 2^{6a_1+1}3^{6b_1+2}73^{6c_1+4} \pm 1\,.$$

By modulo 3, we see that only the the positive sign occurs. By modulo 13, we find that $u^2 \equiv 6 \pmod{13}$, which is not possible.

**Case 4:** $(i, j, k) = (2, 4, 5)$. Then $d = 73$ and

$$3u^2 = 73v^2 \pm \frac{2^{3a_1+1}3^{3b_1+2}73^{3c_1+2}}{v}.$$

We easily eliminate the cases $v = \pm 73^{3c_1+2}$, $v = \pm 3^{3b_1+2}73^{3c_1+2}$, $v = \pm 2^{3a_1+1}73^{3c_1+2}$, $v = \pm 2^{3a_1+1}3^{3b_1+2}73^{3c_1+2}$ by congruence consideration modulo 3. We consider two more cases $v = \pm 3^{3b_1+1}73^{3c_1+2}$ and $v = \pm 2^{3a_1+1}3^{3b_1+1}73^{3c_1+2}$. If $v = \pm 3^{3b_1+1}73^{3c_1+2}$ then

$$u^2 = 3^{6b_1+1}73^{6c_1+5} \pm 2^{3a_1+2}.$$

The congruence $1 \equiv 3 \pm 2 \pmod 8$, shows that only the negative sign can occur in the right hand side. So we get that $u^2 \equiv -1 \pmod 7$, a contradiction. If $v = \pm 2^{3a_1+1}3^{3b_1+1}73^{3c_1+2}$ then we get

$$u^2 = 2^{6a_1+2}3^{6b_1+1}73^{6c_1+5} \pm 1.$$

Reducing modulo 4, we see that only the positive sign occurs and hence by modulo 7, we find $u^2 \equiv 5 \pmod 7$, a contradiction.

**Case 5:** $(i, j, k) = (5, 2, 5)$. In this case $d = 146$ and

$$3u^2 = 146v^2 \pm \frac{2^{3a_1+2}3^{3b_1+1}73^{3c_1+2}}{v}.$$

It is enough to reduce modulo 3 to eliminate the cases $v = \pm 2^{3a_1+2}73^{3c_1+2}$, $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+2}$. When $b_1 > 0$, we need to check one more possible case $v = \pm 2^{3a_1+2}3^{3b_1}73^{3c_1+2}$. This case implies that

$$u^2 = 2^{6a_1+5}3^{6b_1-1}73^{6c_1+5} \pm 1.$$

By reducing modulo 4, we see that only the positive sign occurs. In this case, congruence consideration modulo 5, 7 or 13 like the previous cases does not give the desired result. So, we multiply both side of the above equation by $2^2 3^4 73^4$, to get the elliptic curve

$$A^2 = B^3 + 2^2 3^4 73^4,$$

where $A = 2 \cdot 3^2 73^2 u$ and $B = 2^{2a_1+2}3^{2b_1+3}73^{2c_1+3}$. A quick computation with Magma shows that this curve has no integral points when $B \neq 0$, and therefore we conclude that this case also does not lead to a solution.

**Case 6:** $(i, j, k) = (5, 5, 3)$. Then $d = 438$, and

$$3u^2 = 438v^2 \pm \frac{2^{3a_1+2}3^{3b_1+2}73^{3c_1+1}}{v}.$$

In this case we have either $v = \pm 2^{3a_1+2}3^{3b_1+2}73^{3c_1+1}$ or $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+1}$. If $v = \pm 2^{3a_1+2}3^{3b_1+2}73^{3c_1+1}$ then $3u^2 = 2^{6a_1+5}3^{6b_1+5}73^{6c_1+3} \pm 1$, which is clearly false because of modulo 3. Let $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+1}$. Then we get that $u^2 = 2^{6a_1+5}3^{6b_1+4}73^{6c_1+2} \pm 1$. Reducing modulo 4, we see that only the positive sign can be occur. Reducing modulo 7, we get that $u^2 \equiv 5 \pmod 7$, a contradiction.

**Case 7:** $(i, j, k) = (5, 5, 5)$. Then again $d = 438$ and therefore we have that

$$3u^2 = 438v^2 \pm \frac{2^{3a_1+2}3^{3b_1+2}73^{3c_1+2}}{v}.$$

Then either $v = \pm 2^{3a_1+2}3^{3b_1+2}73^{3c_1+2}$ or $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+2}$. The case $v = \pm 2^{3a_1+2}3^{3b_1+2}73^{3c_1+2}$ is not possible because of modulo 3.

If $v = \pm 2^{3a_1+2}3^{3b_1+1}73^{3c_1+2}$ then $u^2 = 2^{6a_1+5}3^{6b_1+2}73^{6c_1+5} \pm 1$. As in the previous case, we see that this equation also has no solution in integers.

So, we conclude that these exceptional seven cases do not lead to a solution. □

### 3.2. The Case n=4.

**Lemma 3.** *If $n = 4$ then all solutions of (2) are given in Table 4.*

**Proof.** Let $n = 4$. Write $a = 4a_1 + i$, $b = 4b_1 + j$ and $c = 4c_1 + k$ where $i$, $j$, $k \in \{0, 1, 2, 3\}$. Thus, the equation (2) is of the form

$$M^2 = N^4 - 2^i 3^j 73^k$$

where $M = \left(\frac{x}{2^{2a_1}3^{2b_1}73^{2c_1}}\right)$ and $N = \left(\frac{y}{2^{a_1}3^{b_1}73^{c_1}}\right)$. To find all integer solutions of (2) corresponds to find all $S = \{2, 3, 73\}$- integral points of the above 64 quartic curves. We used the subroutine `SIntegralLjunggrenPoints` of Magma to determine all S-Integral Points of the above curves.

Taking into account $\gcd(x, y) = 1$, we see that only the results given in Table (2) lead to the solutions of equation (2). □

### 3.3. The Case $n \geq 5$.

**Lemma 4.** *If $n \geq 5$ then the equation (2) has no positive integer solution except for $(x, y, a, b, c) = (2485, 19, 8, 7, 1)$, $(15479, 25, 8, 5, 1)$, $(42389, 35, 5, 5, 2)$ for $n = 6$, $(65, 3, 5, 0, 1)$ for $n = 8$, $(95, 3, 1, 0, 2)$ for $n = 9$, $(15479, 5, 8, 5, 1)$ for $n = 12$.*

**Proof.** Let $n \geq 5$. If there exists any solution of equation (2) for $n = 2^k, k \geq 3$ then this solution can be derived from the solutions with $n = 4$ since $y^{2^k} = \left(y^{2^{k-2}}\right)^4$. So, by picking up the solutions which contain the perfect power values of $y$ among the solutions of (2) for $n = 4$ given in Table 4, we see that there exist exactly two such solutions, namely $(65, 9, 5, 0, 1)$ and $(15479, 125, 8, 5, 1)$ for $(x, y, a, b, c)$. Therefore, from these solutions we get that two more solutions $(x, y, a, b, c) = (65, 3, 5, 0, 1)$ and $(x, y, a, b, c) = (15479, 5, 8, 5, 1)$ for $n = 8$ and $n = 12$, respectively.

Similarly, for the case $n = 3k$, $k \geq 2$, we may get all solutions of equation (2) from the solutions given in Table (1) for $n = 3$. Thus, we find that equation (2) has also solutions $(x, y, a, b, c) = (2485, 19, 8, 7, 1)$, $(15479, 25, 8, 5, 1)$ and $(42389, 35, 5, 5, 2)$ for $n = 6$ and $(95, 3, 1, 0, 2)$ for $n = 9$. Hence, without loss of generality, from now on we may assume that $n \geq 5$ is an odd prime.

From the equations (4) and (5), one can find that

$$e = L_n v = \left|\frac{\varepsilon^n - \overline{\varepsilon}^n}{\varepsilon - \overline{\varepsilon}}\right| v \quad \text{where} \quad \begin{cases} \varepsilon = u + v\sqrt{-d} \\ \overline{\varepsilon} = u - v\sqrt{-d} \end{cases}$$

and

$$2e = L_n^{'} v = \left|\frac{\varepsilon^n - \overline{\varepsilon}^n}{\varepsilon - \overline{\varepsilon}}\right| v \quad \text{where} \quad \begin{cases} \varepsilon = \dfrac{u + v\sqrt{-d}}{2} \\ \overline{\varepsilon} = \dfrac{u - v\sqrt{-d}}{2} \end{cases}$$

according to the value of $d$ belonging the sets $\{1, 2, 6, 73, 146, 438\}$ and $\{3, 219\}$, respectively.

It is easy to see that the sequence $L_n$ and $L_n^{'}$ are both Lucas sequences. All Lucas sequences which are having no primitive divisors are explicitly listed in [4] for $n \geq 5$ and neither $L_n$ nor $L_n^{'}$ does not match any of them.

So we need to consider the possibility of $L_n$ or $L_n^{'}$ has a primitive divisor. Assume that $L_n$ or $L_n^{'}$ has a primitive divisor, say $q$. Then either $q = 2$, 3 or $q = 73$. From the fact that any primitive divisor is congruent to $\pm 1$ modulo $n$ we easily eliminate the possibility $q = 2$ and $q = 3$ since $n \geq 5$. So we continue with $q = 73$. By the definition of primitive divisor, we have that $q \nmid (\varepsilon - \bar{\varepsilon})^2 = -4dv^2$ for the case $L_n$ and $q \nmid (\varepsilon - \bar{\varepsilon})^2 = -dv^2$ for the case $L_n^{'}$ which implies that $d = 1, 2, 3$ or 6. Note that for all these values of $d$, $\left(\dfrac{(\alpha - \beta)^2}{q}\right) = 1$, and hence we find that $73 \equiv 1 \pmod{n}$ which is a contradiction since $n \geq 5$ is an odd prime. Thus neither $L_n$ nor $L_n^{'}$ has a primitive divisor. This completes the proof. $\square$

Tab. 1: Solutions for $n = 3$ and $\gcd(x, y) = 1$.

| $(M, N, i, j, k)$ | $a$ | $b$ | $c$ | $x$ | $y$ |
|---|---|---|---|---|---|
| (181/9, 1430/27, 0, 0, 2) | 0 | 6 | 2 | 1430 | 181 |
| (55/9, 82/27, 0, 1, 1) | 0 | 7 | 1 | 82 | 55 |
| (3961/324, 233875/5832, 0 ,1 ,1 ) | 6 | 13 | 1 | 233875 | 3961 |
| (283/9, 4744/27, 0,1 ,1 ) | 0 | 7 | 1 | 4744 | 283 |
| (462745/2304, 291230531/110592 ,0 , 1,3 ) | 24 | 7 | 3 | 291230531 | 462745 |
| (62281/36, 15542875/216,0,2,2) | 6 | 8 | 2 | 15542875 | 62281 |
| (127/9, 782/27, 0, 3, 1) | 0 | 9 | 1 | 782 | 127 |
| (45025/2304, 8195759/110592, 0, 3, 1) | 24 | 9 | 1 | 8195759 | 45025 |
| (319/9, 5570/27, 0, 3, 1) | 0 | 9 | 1 | 5570 | 319 |
| (93385/1296, 28462213/46656, 0, 3, 1) | 12 | 15 | 1 | 28462213 | 93385 |
| (13, 46, 0, 4, 0) | 0 | 4 | 0 | 46 | 13 |
| (7, 10, 0, 5, 0) | 0 | 5 | 0 | 10 | 7 |
| (67, 532, 0, 5, 1) | 0 | 5 | 1 | 532 | 67 |
| (3501025/16, 6550777007/64, 0, 5, 3) | 12 | 5 | 3 | 6550777007 | 3501025 |
| (3, 5, 1, 0, 0) | 1 | 0 | 0 | 5 | 3 |
| (195, 2723, 1, 0, 1) | 1 | 0 | 1 | 2723 | 195 |
| (27, 95, 1, 0, 2) | 1 | 0 | 2 | 95 | 27 |
| (7, 17, 1, 3, 0) | 1 | 3 | 0 | 17 | 7 |
| (98377/16, 30855995/64, 1, 4, 2) | 13 | 4 | 2 | 30855995 | 98377 |
| (4681/4, 320005/8, 1, 5, 2) | 7 | 5 | 2 | 320005 | 4681 |
| (5, 11, 2, 0, 0) | 2 | 0 | 0 | 11 | 5 |

| $(M, N, i, j, k)$ | $a$ | $b$ | $c$ | $x$ | $y$ |
|---|---|---|---|---|---|
| (361/36, 2485/216, 2, 1, 1) | 8 | 7 | 1 | 2485 | 361 |
| (925/81, 18053/729, 2, 1, 1) | 2 | 13 | 1 | 18053 | 925 |
| (5917/9, 455147/27, 2, 1, 1) | 2 | 7 | 1 | 455147 | 5917 |
| (3145/4, 176059/8, 2, 4, 2) | 8 | 4 | 2 | 176059 | 3145 |
| (13, 35, 2, 5, 0) | 2 | 5 | 0 | 35 | 13 |
| (397/9, 3293/27, 2, 5, 1) | 2 | 11 | 1 | 3293 | 397 |
| (913/16, 21689/64, 2, 5, 1) | 14 | 5 | 1 | 21689 | 913 |
| (61, 395, 2, 5, 1) | 2 | 5 | 1 | 395 | 61 |
| (85, 737, 2, 5, 1) | 2 | 5 | 1 | 737 | 85 |
| (6217/64, 470843/512, 2, 5, 1) | 20 | 5 | 1 | 470843 | 6217 |
| (625/4, 15479/8, 2, 5, 1) | 8 | 5 | 1 | 15479 | 625 |
| (1165, 39763, 2, 5, 1) | 2 | 5 | 1 | 39763 | 1165 |
| (74833/36, 20470951/216, 2, 5, 1) | 8 | 11 | 1 | 20470951 | 74833 |
| (380269/81, 234067645/729, 2, 5, 3) | 2 | 17 | 3 | 234067645 | 380269 |
| (937/9, 28135/27, 3, 0, 2) | 3 | 6 | 2 | 28135 | 937 |
| (1493065/36, 1824391621/216, 3, 3, 2) | 9 | 9 | 2 | 1824391621 | 1493065 |
| (97, 955, 3, 4, 0) | 3 | 4 | 0 | 955 | 97 |
| (1568665/85264, 1629420733/24897088, 15, 5, 6) | 27 | 5 | 12 | 1629420733 | 1568665 |
| (73/9, 595/27, 4, 1, 0) | 4 | 7 | 0 | 595 | 73 |
| (145/9, 703/27, 4, 1, 1) | 4 | 7 | 1 | 703 | 145 |
| (4753/36, 308935/216, 4, 1, 2) | 10 | 7 | 2 | 308935 | 4753 |
| (5257/9, 380915/27, 4, 1, 2) | 4 | 7 | 2 | 380915 | 5257 |
| (193, 2681, 4, 4, 0) | 4 | 4 | 0 | 2681 | 193 |
| (265, 3421, 4, 4, 2) | 4 | 4 | 2 | 3421 | 265 |
| (265/4, 667/8, 4, 5, 1) | 10 | 5 | 1 | 667 | 265 |
| (97, 793, 4, 5, 1) | 4 | 5 | 1 | 793 | 97 |
| (649, 16525, 4, 5, 1) | 4 | 5 | 1 | 16525 | 649 |
| (5977/9, 445445/27, 4, 5, 2) | 4 | 11 | 2 | 445445 | 5977 |
| (15697/9, 1371223/27, 5, 1, 4) | 5 | 7 | 4 | 1371223 | 15697 |
| (457, 9035, 5, 4, 2) | 5 | 4 | 2 | 9035 | 457 |
| (1153, 39151, 5, 5, 0) | 5 | 5 | 0 | 39151 | 1153 |
| (84097, 24387695, 5, 5, 1) | 5 | 5 | 1 | 24387695 | 84097 |
| (1225, 42389, 5, 5, 2) | 5 | 5 | 2 | 42389 | 1225 |

TAB. 2: Solutions for $n = 4$ and $gcd(x, y) = 1$ .

| $(M, N, i, j, k)$ | $a$ | $b$ | $c$ | $x$ | $y$ |
|---|---|---|---|---|---|
| (± 125/12,15479/144, 0, 1, 1) | 8 | 5 | 1 | 15479 | 125 |
| (± 41/8,367/64, 0, 2, 1) | 12 | 2 | 1 | 367 | 41 |
| (± 3/2,7/4, 1, 0, 0) | 5 | 0 | 0 | 7 | 3 |
| (± 9/2,65/4, 1, 0, 1) | 5 | 0 | 1 | 65 | 9 |
| (± 5/2,23/4, 1, 1, 0) | 5 | 1 | 0 | 23 | 5 |
| (± 35/4,1079/16, 1, 2, 1) | 9 | 2 | 1 | 1079 | 35 |
| (± 827/16,94105/256, 1, 2, 3) | 17 | 2 | 3 | 94105 | 827 |
| (± 17/2,143/4, 1, 3, 1) | 5 | 3 | 1 | 143 | 17 |
| (± 37/6,1223/36, 2, 0, 1) | 6 | 4 | 1 | 1223 | 37 |
| (± 137/12,18607/144, 2, 0, 1) | 10 | 4 | 1 | 18607 | 137 |
| (± 77/6,5897/36, 2, 0, 1) | 6 | 4 | 1 | 5897 | 77 |
| (± 7/2,47/4, 2, 1, 0) | 6 | 1 | 0 | 47 | 7 |
| (± 11/2,25/4, 2, 1, 1) | 6 | 1 | 1 | 25 | 11 |
| (± 29/4,695/16, 2, 1, 1) | 10 | 1 | 1 | 695 | 29 |
| (± 5/2,7/4, 2, 2, 0) | 6 | 2 | 0 | 7 | 5 |
| (± 89/6,2737/36, 3, 0, 2) | 7 | 4 | 2 | 2737 | 89 |
| (± 13/2,23/4, 3, 1, 1) | 7 | 1 | 1 | 23 | 13 |
| (± 157/8,24503/64, 3, 1, 1) | 15 | 1 | 1 | 24503 | 157 |
| (± 17/2,287/4, 3, 2, 0) | 7 | 2 | 0 | 287 | 17 |
| (± 19/2,215/4, 3, 2, 1) | 7 | 2 | 1 | 215 | 19 |
| (± 611/12,373175/144, 3, 2, 1) | 11 | 6 | 1 | 373175 | 611 |
| (± 145/2,21023/4, 3, 2, 1) | 7 | 2 | 1 | 21023 | 145 |

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