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ON MANIFOLDS HOMOTOPY EQUIVALENT TO THE TOTAL SPACES OF S^7-BUNDLES OVER S^8

Ajay Raj and Tibor Macko

Abstract. We calculate the structure sets in the sense of surgery theory of total spaces of bundles over eight-dimensional sphere with fibre a seven-dimensional sphere, in which manifolds homotopy equivalent to the total spaces are organized, and we investigate the question, which of the elements in these structure sets can be realized as such bundles.

1. Introduction

For a closed topological manifold X, the topological structure set, denoted by $\mathcal{S}(X)$, is the set of equivalence classes of homotopy equivalences $f\colon Y \to X$, where Y is a closed topological manifold, modulo the equivalence relation given by the homeomorphism of the source manifolds which is compatible with given homotopy equivalences (see Section 2). For example, the Generalized Poincare Conjecture tells us that $\mathcal{S}(S^n)$ consists of a single element represented by the identity map for all n. We are interested in the structure set $\mathcal{S}(E)$, where E is the total space of an S^7-bundle over S^8.

In [4] Crowley and Escher classified the total spaces of S^3-bundles over S^4. As a by-product of their Lemma 5.3 and the exact sequence (7), it can be stated that each element in the structure set $\mathcal{S}(E)$, where E is any S^3-bundle over S^4, can be realized with the source manifold the total space of some S^3-bundle over S^4, see Remark 3.7 for more details.

We consider fibre bundles over S^8 with total space M, fibre S^7, and the structure group $SO(8)$. Equivalence classes of such bundles are in one-to-one correspondence with elements in $\pi_7(SO(8)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Consider the generators ρ and σ of $\pi_7(SO(8))$ defined as follows (see [17]):

$$\rho(x)y = xyx^{-1} \quad \text{and} \quad \sigma(x)y = xy\,, \tag{1}$$

where x and y are vectors in S^7, which is identified with octonions of norm 1. For every pair (m,n) of integers we get an element $m\rho + n\sigma \in \pi_7(SO(8))$. This gives

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us a vector bundle $\gamma_{m,n}$ and corresponding sphere bundle

$$p_{m,n}\colon M_{m,n} := S(\gamma_{m,n}) \to S^8. \tag{2}$$

Tamura [15], [16] used methods similar to Milnor [10] to study S^7-bundles over S^8, and he constructed explicit homeomorphisms between $M_{m,n}$ and $M_{m',n'}$ for some specific values of (m,n) and (m',n') using foliations. Some results on smooth structures of S^7-bundles over S^8 were obtained by Shimada [14] and later complete diffeomorphism classification was achieved by Grey [6], who also gave some partial results on the homeomorphism classification.

Note that a change in orientation of the fibre yields $M_{m,n} \cong M_{m+n,-n}$ and a change in orientation of the base yields $M_{m,n} \cong M_{-m,-n}$. Hence we may assume $n \geq 0$.

Our main result is the following theorem and its corollary below.

Theorem 1.1. *Let N be a closed 15-dimensional manifold. If $(n,28) = 1$ then N is homotopy equivalent to $M_{m,n}$ from (2) if and only if N is homeomorphic to $M_{m',n}$ where $m' = m + 120j$ for some $j \in \mathbb{Z}$.*

Here $(n,28)$ denotes the greatest common divisor of n and 28. As a consequence of this, and of the calculation of $\mathcal{S}(M_{m,n})$ in display (3), we obtain:

Corollary 1.2. *If $(n,28) = 1$, then all the elements of $\mathcal{S}(M_{m,n}) \cong \mathbb{Z}_n$ can be realized with the source manifold an S^7-bundle over S^8.*

What happens when $(n,28) \neq 1$, including the case $n = 0$, is described in detail in Remark 3.4. The methods of the proofs are an adaptation of those used in [4] to the dimensions in which we work, combined with some additional necessary considerations, such as those in our Proposition 3.2 below.

Parts of this work will be used in the PhD thesis of Ajay Raj.

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2. Preliminaries

Let $n > 0$ and α_8 be the standard generator of $H^8(S^8)$. The Euler class of the bundle $\gamma_{m,n}$ is $e(\gamma_{m,n}) = n\alpha_8$ (see [6], Lemma 2.2.1). A simple calculation using Gysin sequence tells us that

$$\begin{aligned} H^0(M_{m,n}) &\cong H^{15}(M_{m,n}) \cong \mathbb{Z}\,; \\ H^8(M_{m,n}) &\cong \mathbb{Z}_n\,; \\ H^i(M_{m,n}) &\cong 0 \quad \text{for} \;\; i \neq 0, 8, 15\,. \end{aligned}$$

Let us also adopt the notation $W_{m,n}$ for the total space of the disk bundle $D^8 \hookrightarrow W_{m,n} \xrightarrow{\overline{p}_{m,n}} S^8$ associated to the bundle $\gamma_{m,n}$, so that $\partial(W_{m,n}) = M_{m,n}$. Obviously, both $M_{m,n}$ and $W_{m,n}$ are simply connected.

Given a vector bundle ξ over a compact manifold B, we denote by $D(\xi)$ the total space of the associated disk bundle, by $-\xi$ the stable inverse of ξ, by τ_B the stable tangent bundle of B and by $\nu_B = -\tau_B$ its stable normal bundle. As in [4, Fact 3.1] we use the following observation:

Fact 2.1. *We have bundle isomorphisms*

$$\nu_{S(\xi)} \cong p_\xi^*(\nu_B \oplus -\xi) \quad \textit{and} \quad \tau_{S(\xi)} \cong p_\xi^*(\tau_B \oplus \xi)\,.$$

Analogous statement holds for the associated disk bundle $D(\xi)$ instead of $S(\xi)$.

For the second Pontryagin class of the vector bundle $\gamma_{m,n}$ we have that $p_2(\gamma_{m,n}) = 6(2m+n)\cdot\alpha_8 \in H^8(S^8)$, see [15, Theorem 4.4], and for the second Pontryagin class of $M_{m,n}$ we have that $p_2(M_{m,n}) = 12m \cdot p_{m,n}^*(\alpha_8) \in H^8(M_{m,n})$, see [15, Theorem 6.2], and [16, Theorem 2.2].

Next we summarize some background from surgery theory in the topological category and in the simply connected situation that we use. For a compact manifold X with boundary ∂X, the structure set $\mathcal{S}(X)$ consists of the equivalence classes of pairs (Y, f) where Y is a compact manifold with boundary ∂Y and $f\colon (Y, \partial Y) \to (X, \partial X)$ is a homotopy equivalence of pairs modulo the following equivalence relation. Two pairs (Y_1, f_1), (Y_2, f_2) are called equivalent if there exists a homeomorphism $g : (Y_1, \partial Y_1) \to (Y_2, \partial Y_2)$ such that $f_2 \circ g$ is homotopic to f_1.

Given an oriented manifold X with boundary ∂X, a degree one normal to $(f, \bar{f})\colon (Y, \partial Y) \to (X, \partial X)$ is a map of manifolds with boundary which is of degree one and it is covered by a bundle map

$$\begin{array}{ccc} (\nu_Y, \nu_{\partial Y}) & \xrightarrow{\bar{f}} & (\zeta, \zeta|_{\partial X}) \\ \downarrow & & \downarrow \\ (Y, \partial Y) & \xrightarrow{f} & (X, \partial X) \end{array}$$

where ζ is some topological microbundle and ν_Y is the stable topological normal microbundle [7]. The set $\mathcal{N}(X)$ of normal invariants of X is the set of equivalence classes of degree one normal maps with target X modulo the normal bordism relation, which means bordism in the source equipped with suitable bundle data.

For an oriented $(X, \partial X)$ we can map elements $[f\colon (Y, \partial Y) \to (X, \partial X)]$ of the structure set $\mathcal{S}(X)$ to the set of normal invariants $\mathcal{N}(X)$, by equipping $(Y, \partial Y)$ with an orientation so that the homotopy equivalence $f\colon (Y, \partial Y) \to (X, \partial X)$ is of degree one and using any homotopy inverse $f^{-1}\colon (X, \partial X) \to (Y, \partial Y)$ to obtain the required bundle data by defining $\zeta := (f^{-1})^*(\nu_Y)$.

Let X be a simply connected closed oriented manifold of dimension $k \geq 5$. The structure set $\mathcal{S}(X)$ fits into the surgery exact sequence, which in this case is a short exact sequence of abelian groups of the shape:

$$0 \to \mathcal{S}(X) \xrightarrow{\eta_k} \mathcal{N}(X) \xrightarrow{\sigma_k} L_k(\mathbb{Z}) \to 0\,,$$

see [11, Proposition 20.3], where

$$L_k(\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if} \quad k \equiv 0 \pmod 4\,; \\ 0, & \text{if} \quad k \equiv 1 \pmod 4\,; \\ \mathbb{Z}_2, & \text{if} \quad k \equiv 2 \pmod 4\,; \\ 0, & \text{if} \quad k \equiv 3 \pmod 4\,. \end{cases}$$

For X a simply connected compact oriented manifold of dimension $k \geq 5$ with non-empty simply connected boundary ∂X the corresponding surgery exact sequence has trivial L-groups, so that we have an isomorphism of abelian groups

$$\eta_k \colon \mathcal{S}(X) \xrightarrow{\cong} \mathcal{N}(X)\,,$$

see [18, §4, §10] or [3, Theorem 1.65] and [11, Remark 19.10].

There exists the classifying space of stable topological microbundles BTOP, such that the stable isomorphism classes of topological microbundles over a manifold X are in bijection with the set $[X, \mathrm{BTOP}]$ of homotopy classes of maps from X to BTOP. Any topological microbundle has an associated spherical fibration, which is reflected by the existence of the canonical map $\mathrm{BTOP} \to \mathrm{BG}$, with the target the classifying space BG of the stable spherical fibrations.

Further information from surgery theory that is needed is the following (see [13, Chapter 13] and [18, §10]):

Fact 2.2.

(1) *Denoting by* $\mathrm{G}\,/\,\mathrm{TOP}$ *the homotopy fibre of the canonical map* $j\colon \mathrm{BTOP} \to \mathrm{BG}$ *we have a bijection*

$$\mathcal{N}(X) \cong [X, \mathrm{G}\,/\,\mathrm{TOP}]$$

(2) *For* $k > 5$ *and denoting by* $i\colon \partial X \to X$ *the inclusion of the boundary we have the commutative diagram*

$$\begin{array}{ccc} \mathcal{S}(X) & \xrightarrow{\eta} & \mathcal{N}(X) \\ \downarrow i^* & & \downarrow i^* \\ \mathcal{S}(\partial X) & \xrightarrow{\eta} & \mathcal{N}(\partial X)\,, \end{array}$$

where both maps i^* *are given by restriction.*

We apply topological surgery theory to the 15-dimensional manifold with boundary $(W_{m,n}, M_{m,n})$. By the above facts the structure set and normal invariant set coincide for both $W_{m,n}$ and $M_{m,n}$. Via primary (and the only) obstruction to null homotopy, we can identify $[W_{m,n}, G/TOP] \cong H^8(W_{m,n})$ and $[M_{m,n}, G/TOP] \cong H^8(M_{m,n})$ (see [2, Theorem 13.11 in Chapter VII] and note that for $k > 0$ we have $\pi_k(G/TOP) \cong L_k(\mathbb{Z})$). This gives us in the case $n > 0$ the following commutative diagram

$$\begin{array}{ccccccc} \mathcal{S}(W_{m,n}) & \overset{\eta}{\xrightarrow{\cong}} & \mathcal{N}(W_{m,n}) & \xrightarrow{\cong} & H^8(W_{m,n}) & \xrightarrow{\cong} & \mathbb{Z} \\ \downarrow i^* & & \downarrow i^* & & \downarrow i^* & & \downarrow \\ \mathcal{S}(M_{m,n}) & \overset{\eta}{\xrightarrow{\cong}} & \mathcal{N}(M_{m,n}) & \xrightarrow{\cong} & H^8(M_{m,n}) & \xrightarrow{\cong} & \mathbb{Z}_n, \end{array} \tag{3}$$

where the right vertical map is simply just reduction modulo n. When $n = 0$ the bottom right corner is $\mathbb{Z}$ and the right vertical map is the identity.

We further need some background on spherical fibrations and their relation to vector bundles, following [5]. Note that the fibre homotopy equivalence classes of spherical fibration are in bijection with homotopy equivalence classes of their total spaces. Given $k \geq 1$ denote by $SH(k)$, the topological monoid of orientation preserving self homotopy equivalences of S^{k-1}. Via clutching construction we have that the fibre homotopy equivalence classes of S^{k-1}-spherical fibrations over S^l for $l \geq 3$ are in bijection with $\pi_{l-1}(SH(k))$, an analogue of the fact that oriented vector bundles over S^l are in bijection with $\pi_{l-1}(SO(k))$. We have natural inclusions $i_k : SO(k) \to SH(k)$ obtained by restricting the transformations from $SO(k)$ to S^{k-1}. For any two bundles ξ_1 and ξ_2 the corresponding sphere fibrations $S(\xi_1)$ and $S(\xi_2)$ are fibre homotopy equivalent if and only if we have $(i_k)_*(\xi_1) = (i_k)_*(\xi_2) \in \pi_{l-1}(SH(k))$ (see [5], Theorem 6.2 and Corollary 7.4).

3. Proofs of main results

The Propositions 3.1 and 3.3 below are adaptations of Lemmas 5.2 and 5.3 respectively from [4] to the case of S^7-bundles over S^8 in the topological category. In Proposition 3.3 a calculation using Kervaire–Milnor braid is needed, which is more complicated than its analogue in Lemma 5.3 of [4], so we separate it to Proposition 3.2.

Proposition 3.1. *Let $n \geq 0$ and let m, $j \in \mathbb{Z}$ be arbitrary. Then there exists a fibre homotopy equivalence $f_j \colon M_{m+120j,n} \to M_{m,n}$.*

Proof. First we show that there is an isomorphism $\pi_7(SH(8)) \cong \mathbb{Z}_{120} \oplus \mathbb{Z}$ such that

$$(i_8)_*(\gamma_{m,n}) = (m \bmod 120, n).$$

Let $SF(8)$ be the subspace of $SH(8)$ consisting of orientation preserving self homotopy equivalence of S^7 which fix a base point u. Observe that $SF(8) = \Omega^7_1(S^7)$, the component of the identity in $\Omega^7(S^7)$. Note that $\Omega^7_1(S^7) \simeq \Omega^7_0(S^7)$, the component of the constant map (see [19, (2.6)]). This together with the usual adjoint correspondence gives us an isomorphism $I_{7,7} \colon \pi_7(SF(8)) \cong \pi_{14}(S^7)$ (see [9, Pages 46-47]). The space SF(8) is the fibre of the fibration

$$\begin{aligned} \pi \colon SH(8) &\to S^7 \\ f &\mapsto f(u). \end{aligned}$$

The restriction $\pi|_{SO(8)} \colon SO(8) \to S^7$ is the usual fibration with fibre $SO(7) \subset SF(8)$. This fibration has a section $s \colon S^7 \to SO(8)$ given by $s(x) = J_x$ where $J_x(y) = x \cdot y$ is octonion multiplication. The long exact homotopy sequences of the above two fibrations give the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \mathbb{Z} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \pi_7(SO(7)) \arrow[r, "i_*"] \arrow[d, "(i_7)_*"] & \pi_7(SO(8)) \arrow[r, "\pi|_{SO(8)}"] \arrow[d, "(i_8)_*"] & \pi_7(S^7) \arrow[r] \arrow[d, "Id"] & 0 \\
0 \arrow[r] & \pi_7(SF(8)) \arrow[r] \arrow[d, "\cong"] & \pi_7(SH(8)) \arrow[r, "\pi_*"] \arrow[d, "\cong"] & \pi_7(S^7) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathbb{Z}_{120} \arrow[r] & \mathbb{Z}_{120}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The proof follows by observing first that if ω_7 and ρ denote suitable generators of $\pi_7(S^7)$ and $\pi_7(SO(7))$ respectively, then for any $m, n \in \mathbb{Z}$

$$s_*(n\omega_7) = \gamma_{0,n} \text{ and } i_*(m\rho) = \gamma_{m,0}.$$

Secondly, if $i_7\colon SO(7) \rightarrowtail SF(8)$ is the inclusion of one fibre into the other, then the composition $I_{7,7} \circ (i_7)_*\colon \pi_7(SO(7)) \to \pi_{14}(S^7)$ is the usual J-homomorphism (see [19, Section 5], [20, Section 9], [9, Page 47]). Similar to as done in [8], we can also prove that the unstable J-homomorphism $J_{7,7}\colon \pi_7(SO(7)) \to \pi_{14}(S^7)$ is surjective using the stabilized commutative diagram below:

$$\begin{tikzcd}
\pi_7(SO(7)) \arrow[r, "J_*"] \arrow[d, "\times 2"'] & \pi_7(SF(8)) \arrow[d, "\times 2"] \\
\pi_7(SO) \arrow[r, "J^s_*"'] & \pi_7^S
\end{tikzcd}$$

It is well known that $\pi_7(SO) \cong \mathbb{Z}$, $\pi_7(SO(7)) \cong \mathbb{Z}$, $\pi_7(SF(8)) \cong \pi_{14}(S^7) \cong \mathbb{Z}_{120}$ and $\pi_7^S \cong \mathbb{Z}_{240}$, and that the vertical maps are given by multiplication by 2, see [15, page 252 and (2.2)]. The stable J^s_* in the above diagram is surjective (see [1], Theorem 1.6) hence J_* is too. $\square$

Note that the proposition above re-establishes the result [15, Theorem 2.3(i)]

Recall the Kervaire-Milnor braid, for example from [13, Remark 13.25], which links the homotopy groups of various classifying spaces, in particular those of G / TOP, BO and BTOP which are of interest to us. We use known results about it to show the following proposition.

Proposition 3.2. *The composition*

$$\pi_7(SO(8)) \cong \pi_8(BSO(8)) \xrightarrow{i_*} \pi_8(BSO) \xrightarrow{\mu_*} \pi_8(BTOP) \xrightarrow{\pi} \mathbb{Z}\oplus\mathbb{Z}_4\,,$$

where i is the inclusion, μ the canonical map, and $\pi\colon \pi_8(BTOP) \cong \mathbb{Z}\oplus\mathbb{Z}_4$, sends the difference of equivalence classes of vector bundles $[\gamma_{m+120j,n}] - [\gamma_{m,n}]$ to $((7\times 240)\cdot j, 0)$.

Proof. We have the following commutative square

$$\begin{array}{ccc} \pi_8(BSO(8)) & \xrightarrow{i_*} & \pi_8(BSO) \\ \cong\downarrow & & \cong\downarrow \\ \mathbb{Z}\oplus\mathbb{Z} & \longrightarrow & \mathbb{Z}\,, \end{array}$$

where the bottom horizontal map sends $(1,0)$ to 2 and $(0,1)$ to 1 hence it is given by $(m,n) \mapsto 2m+n$ [15, Equations 2.1 and 2.2].

From the portion of the Kervaire–Milnor braid depicted in the diagram on page 311 in Remark 13.25 of [13], we observe that the map $\mu_*\colon \pi_8(BSO) \to \pi_8(BTOP)$ is $\mu_*\colon \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}_4$ which is given by $x \mapsto (7x, 2x)$.

Composition of the two maps yields the desired formula. □

In the following proposition we shall abuse the notation a little bit and denote the corresponding image of $\gamma_{m,n}$ in $\pi_8(BTOP)$ under the above composition by same notation.

Proposition 3.3. *The fibre homotopy equivalences $f_j\colon M_{m+120j,n} \to M_{m,n}$ have normal invariant $\eta(f_j) = 28j \in \mathcal{N}(M_{m,n}) \cong \mathbb{Z}_n$.*

Proof. Given a map $f\colon S(\gamma) \to S(\chi)$ of sphere bundles, the cone of f can be defined as the map of disk bundles $F\colon D(\gamma) \to D(\chi)$ which for $0 \le r \le 1$ and $r\cdot v \in D(\gamma)$ takes the value $F(r\cdot v) = r\cdot f(v)$. It is easy to check that $(F_j, f_j)\colon (W_{m+120j,n}, M_{m+120j,n}) \to (W_{m,n}, M_{m,n})$ is a fibre homotopy equivalence of pairs. Fact 2.2 (2) implies that $\eta(f_j) = i^*\eta(F_j)$. Moreover, Diagram (3) shows that it is enough to prove that $\eta(F_j) \in \mathcal{N}(W_{m,n})$ takes on the value $28j \in \mathbb{Z}$.

Note that the inclusion $i_{m,n}$ of the zero section S^8 into $W_{m,n}$ is homotopy equivalence. Consider the canonical map $j\colon G/TOP \to BTOP$ from Fact 2.2. We obtain the following commutative diagram

$$\begin{array}{ccccc} \mathcal{N}(W_{m,n}) & \overset{\cong}{\to} & [W_{m,n}, G/TOP] & \overset{j_*}{\to} & [W_{m,n}, BTOP] \\ \downarrow & & \downarrow i^*_{m,n} & & \downarrow i^*_{m,n} \\ \mathcal{N}(S^8) & \xrightarrow{\cong} & \left[S^8, G/TOP\right] & \xrightarrow{j_*} & \left[S^8, BTOP\right] \end{array}$$

Since $i_{m,n}$ is homotopy equivalence, vertical arrows induce bijection. From the Kervaire–Milnor braid, see page 311 in Remark 13.25 of [13], we observe that $j_*\colon \pi_8(G/TOP) \to \pi_8(BTOP)$ is the map $j_*\colon \mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}_4$, which is given by $x \mapsto (60x, 3x)$. For any compact space Y, the group $[Y, BTOP]$ may be regarded as formal differences of stable topological microbundles over Y. Hence for $Y = W_{m,n}$, which is a smooth manifold, we have that its stable normal vector bundle is also its stable normal microbundle, and we get

$$j_*(\eta(F_j)) = \nu(W_{m,n}) - F_j^{-1*}(\nu(W_{m+120j,n}))\,.$$

Fact 2.1 for disk bundles implies that

$$\nu(W_{m,n}) = p^*_{m,n}(\nu_{S^8} \oplus -\gamma_{m,n}) = p^*_{m,n}(-\gamma_{m,n})\,.$$

Since F_j commutes with $p_{m,n}$ and $p_{m+12j,n}$, we may choose F_j^{-1} to commute up to homotopy. We have now

$$\begin{aligned} i_{m,n}^*(j_*(\eta(F_j))) &= i_{m,n}^*(\nu(W_{m,n}) - F_j^{-1*}(\nu(W_{m+120j,n}))) \\ &= i_{m,n}^*(p_{m,n}^*(-\gamma_{m,n}) - F_j^{-1*}(p_{m+12j,n}^*(-\gamma_{m+120j,n}))) \\ &= i_{m,n}^*(p_{m,n}^*(-\gamma_{m,n} + \gamma_{m+120j,n}))) \\ &= \gamma_{m+120j,n} - \gamma_{m,n} \,. \end{aligned}$$

By Proposition 3.2 this difference is sent to the pair $(7 \times 240j, 0) \in \mathbb{Z} \oplus \mathbb{Z}_4$. This gives us $\eta(F_j) = \frac{7\times 240j}{60} = 28j \in \mathcal{N}(W_{m,n})$, hence $\eta(f_j) = 28j \in \mathcal{N}(M_{m,n})$. □

Proof of Theorem 1.1. Since $(n, 28) = 1$, multiplication by 28 is an isomorphism of $\mathbb{Z}_n$. Hence for any $l \in \mathbb{Z}_n$ we have $l = 28j$ for some $j \in \mathbb{Z}$. □

Remark 3.4.
(1) In the case where $(n, 28) \neq 1$ and $n \neq 0$ we obtain that the elements in the subgroup $28 \cdot \mathbb{Z}_n \subset \mathbb{Z}_n \cong \mathcal{S}(M_{m,n})$ can be realized as $f_j \colon M_{m+120j,n} \to M_{m,n}$ for suitable j.
(2) If $n = 0$ we have $\mathcal{N}(M_{m,0}) \cong \mathbb{Z}$. A manifold N will be homotopy equivalent to $M_{m,0}$ if and only if N is homeomorphic to $M_{m+120k,0}$ where $k = 28j$ for some $j \in \mathbb{Z}$. We cannot say anything about the other elements.
(3) $M_{m,0}$ is homeomorphic to $M_{m',0}$ if and only if $m = \pm m'$. It follows by the topological invariance of rational Pontryagin classes [12, Theorem 4.1] that are mentioned below Fact 2.1.

Remark 3.5. Given a smooth manifold X there exists also the smooth version of the structure set $\mathcal{S}^{\mathrm{DIFF}}(X)$, where all manifolds considered are smooth and we divide out the relation of diffeomorphism in the source. There is the obvious forgetful map $\mathcal{S}^{\mathrm{DIFF}}(X) \to \mathcal{S}(X)$ and it is an interesting question what are the properties of this map for a given X. Our results show that if $(n, 28) = 1$, then $\mathcal{S}^{\mathrm{DIFF}}(M_{m,n}) \to \mathcal{S}(M_{m,n})$ is surjective, since then all elements in $\mathcal{S}(M_{m,n})$ are represented by homotopy equivalences, whose source manifold is smooth.

Remark 3.6. The results in [4] were proved working with PL block bundles rather than topological microbundles and with PL surgery rather than topological surgery. However, it is well known that the difference between the classifying spaces BPL and BTOP is very small, namely the homotopy fibre TOP / PL of the canonical map BPL $\to$ BTOP has the homotopy type of the Eilenberg-Mac Lane space $K(\mathbb{Z}_2, 3)$ and hence, due to the dimensions that we work with, the proofs would be the same in the PL-category.

Remark 3.7. In [4] the authors provided various classification results of the total spaces of S^3-bundle over S^4. They did not explicitly asked the question whether elements in the structure set of such a bundle are represented with sources such bundles. However, it follows immediately from their Lemma 5.3 and display (7) that the answer is yes.

Remark 3.8. It would be interesting to further investigate the homeomorphism classification of $M_{m,n}$ and the question of the action of the group of self homotopy equivalences of $M_{m,n}$ on the structure set. We plan to study this in future.

References

[1] Adams, J.F., *On the groups $J(X)$, IV*, Topology **5** (1966), 21–71.

[2] Bredon, G.E., *Topology and Geometry*, Graduate Texts in Mathematics, vol. 139, Springer-Verlag Inc., New York, 1993.

[3] Chang, S., Weinberger, S., *A course on surgery theory*, Annals of Mathematics Studies, vol. 211, Princeton University Press, 2021.

[4] Crowley, D., Escher, C.M., *A classification of S^3-bundles over S^4*, Differential Geom. Appl. **18** (3) (2003), 363–380.

[5] Dold, A., Lashof, R., *Principal quasi-fibration and fibre homotopy equivalences of the bundles*, Illinois J. Math. **3** (1959), 285–305.

[6] Grey, M., *On the classification of total space of S^7-bundles over S^8*, Master's thesis, Humboldt University, Berlin, 2012.

[7] Kirby, R.C., Siebenmann, L.C., *Foundational essays on topological manifolds, smoothings, and triangulations*, Annals of Mathematics Studies, vol. 88, Princeton University Press, 1977.

[8] Kitchloo, N., Shankar, K., *On complex equivalent to S^3-bundles over S^4*, Int. Math. Res. Not. IMRN **2001** (8) (2001), 381–394.

[9] Madsen, I.B., Milgram, R.J., *The Classifying Spaces For Surgery and Cobordism of Manifolds*, Princeton University Press, New Jersey, 1979.

[10] Milnor, J., *On manifolds homeomorphic to the 7-sphere*, Ann. of Math. (2) **64** (2) (1956), 399–405.

[11] Ranicki, A., *Algebraic L-theory and topological manifolds*, Cambridge University Press, 1992.

[12] Ranicki, A., *On The Novikov Conjecture*, Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993), London Math. Soc. Lecture Note Ser., vol. 226, Cambridge Univ. Press, Cambridge, 1995, pp. 272–337.

[13] Ranicki, A., *Algebraic and Geometric Surgery*, Oxford Mathematical Monograph, Oxford Science Publication, 2002, ISBN 978-0198509240.

[14] Shimada, N., *Differentiable structures on the 15-sphere and Pontrjagin classes of certain manifolds*, Nagoya Math. J. **12** (1957), 59–69.

[15] Tamura, I., *On Pontrjagin classes and homotopy types of the manifolds*, J. Math. Soc. Japan **9** (2) (1957), 250–262.

[16] Tamura, I., *Homeomorphic classification of total spaces of sphere bundles over spheres*, J. Math. Soc. Japan **10** (1) (1958), 29–42.

[17] Toda, H., Saito, Y., Yokota, I., *Notes on the generator of $\pi_7(SO(n))$*, Mem. Coll. Sci. Univ. Kyoto **30** (1957), 227–230.

[18] Wall, C.T.C., *Surgery on Compact Manifolds*, Mathematical Surveys and Monographs, vol. 69, AMS, 1999.

[19] Whitehead, G.W., *On products in homotopy groups*, Ann. of Math. (2) **40** (3) (1946), 460–475.

[20] Whitehead, G.W., *A generalization of the Hopf invariant*, Ann. of Math. (2) **50** (1) (1950), 192–237.

Department of Algebra and Geometry,
FMFI, Comenius University,
Bratislava, SK-84248, Slovakia
E-mail: `ajay.raj@fmph.uniba.sk`

Department of Algebra and Geometry,
FMFI, Comenius University,
Bratislava, SK-84248, Slovakia,
and
Institute of Mathematics,
Slovak Academy of Sciences,
Štefánikova 49, Bratislava,
SK-81473, Slovakia
E-mail: tibor.macko@fmph.uniba.sk