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SOME APPLICATIONS OF NEVANLINNA THEORY TO ENTIRE FUNCTIONS THAT SHARE A SMALL FUNCTION WITH TWO DIFFERENCE OPERATORS

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Abstract. In this work, we are implementing some applications of Nevanlinna theory to entire functions that share a small function with two difference operators and we also generalize one of the results in the paper [3].

1. Introduction

Nevanlinna theory is considered one of the most important theories in complex analysis, especially in the study of entire functions. We say that an entire function $a(z)$ is a small function of $f(z)$ if $T(r,a) = S(r,f)$, where $S(r,f) = o(T(r,f))$, as $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. We use $S(f)$ to denote the family of all small functions with respect to $f(z)$. For an entire function $f(z)$ we define its shift by

$$f_c(z) = f(z+c) \,,$$

and its difference operators by

$$L_c^n f(z) = \alpha_n f(z+nc) + \dots + \alpha_1 f(z+c) + \alpha_0 f(z) \,, \quad n \in \mathbb{N},\ n \geq 1$$

where $\alpha_n\,(\neq 0), \dots, \alpha_1, \alpha_0$ are complex numbers. In particular for the case

$$\alpha_j = \binom{n}{j} (-1)^{n-j} \,, \quad j \in \mathbb{N}, \quad 0 \leq j \leq n$$

we define its difference operators by

$$\Delta_c f(z) = f(z+c) - f(z) \,,\ L_c^n f(z) = \Delta_c^n f(z) = \Delta_c^{n-1}(\Delta_c f(z)) \,,\ n \in \mathbb{N},\ n \geq 2 \,.$$

We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities), provided that $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros counting multiplicities. The uniqueness of meromorphic functions sharing values with their difference operators has been studied in many papers see e.g. [1, 2, 3, 4, 8, 11], and sharing values with their shifts has been investigated by many authors see e.g. [3, 6, 7, 9].

In 2015, A. El Farissi, Z. Latreuch and A. Asiri [3] proved:

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Theorem A. *Let $f(z)$ be a transcendental entire function of finite order such that $f(z) \not\equiv f(z+c)$. Then $f(z)$, $f(z+c)$ and $\Delta_c f(z)$ can not share any finite value $a \neq 0$ CM. Furthermore; if $a = 0$, $f(z)$ must be of the following form $f(z) = h(z)e^{\frac{\alpha}{c}z}$, where $\alpha \neq 0$ and $h(z)$ is a periodic entire function of period c.*

It is interesting now to see what is happening when $f(z)$, $f(z+c)$ and $L_c^n f(z)$ $(n \geq 1)$ share $a(z)$ CM. The main result of this paper is to prove that the conclusion of Theorem A, remains valid when we replace $\Delta_c f(z)$ by $L_c^n f(z)$, and we obtain the following results.

Theorem 1.1. *Let $f(z)$ be an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $f(z+c)$ and $L_c^n f(z)$ $(n \geq 1)$ share $a(z)$ CM, then*

$$f(z) = h(z)e^{\frac{\beta}{c}z} + a(z) \quad \textit{and} \quad a(z) = 0 \quad \textit{or} \quad \sum_{i=0}^{n} \alpha_i - 1 = 0\,,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c.

In the following examples, we take several cases to illustrate Theorem 1.1:

Example 1.1. In this example we illustrate the case $\sum_{i=0}^{n} \alpha_i - 1 = 0$ and $a(z) \neq 0$. The entire function $f(z) = \cos(z)\, e^{\frac{1}{2\pi}z} + e$ satisfies

$$f(z + 2\pi n) = e^n \cos(z)\, e^{\frac{1}{2\pi}z} + e, \quad n \in \mathbb{N}\,.$$

We put $\alpha_i = -1$, $1 \leq i \leq n$, $\alpha_0 = n+1$ and $a(z) = e$, then

$$L_{2\pi}^n f(z) = -f(z+2\pi n) - \cdots - f(z+2\pi) + (n+1) f(z)\,, \quad n \in \mathbb{N}^*\,.$$

We can get

$$\frac{f(z+2\pi) - a(z)}{f(z) - a(z)} = \frac{e \cos(z)\, e^{\frac{1}{2\pi}z}}{\cos(z)\, e^{\frac{1}{2\pi}z}} = e\,,$$

and

$$\frac{L_{2\pi}^n f(z) - a(z)}{f(z) - a(z)} = \frac{-\cos(z)\, e^{\frac{1}{2\pi}z}(e^n + \cdots + e) - ne}{\cos(z)\, e^{\frac{1}{2\pi}z}} + \frac{(n+1)\cos(z)\, e^{\frac{1}{2\pi}z} + (n+1)e - e}{\cos(z)\, e^{\frac{1}{2\pi}z}} = \frac{e^{n+1} - e}{1-e} + n + 1\,,$$

and hence $f(z)$, $f(z+2\pi)$ and $L_{2\pi}^n f(z)$ share $a(z)$ CM.

Example 1.2. In this example we illustrate the case $a(z) = 0$ and $\sum_{i=0}^{n} \alpha_i - 1 \neq 0$. The entire function $f(z) = e^z$ satisfies

$$f(z+n) = e^n e^z, \quad n \in \mathbb{N}\,.$$

We put $\alpha_i = 1$, $0 \leq i \leq n$, then

$$L_1^n f(z) = f(z+n) + \cdots + f(z+1) + f(z)\,,\ n \in \mathbb{N}^*.$$

We can get

$$\frac{f(z+1)}{f(z)} = \frac{ee^z}{e^z} = e\,,$$

and

$$\frac{L_1^n f(z)}{f(z)} = \frac{e^z\left(e^n + \cdots + 1\right)}{e^z} = \frac{1-e^{n+1}}{1-e}\,,$$

and hence $f(z)$, $f(z+1)$ and $L_1^n f(z)$ share 0 CM.

Example 1.3. In this example we illustrate the case $\sum\limits_{i=0}^{n}\alpha_i - 1 = 0$ and $a(z) = 0$. The entire function $f(z) = \sin(z)\, e^z$ satisfies

$$f(z+2\pi n) = e^{2\pi n}\sin(z)\, e^z\,, \quad n \in \mathbb{N}\,.$$

We put $\alpha_i = 1$, $1 \leq i \leq n$, $\alpha_0 = -(n-1)$ and $a(z) = 0$, then

$$L_{2\pi}^n f(z) = f(z+2\pi n) + \cdots + f(z+2\pi) - (n-1)\, f(z)\,, \quad n \in \mathbb{N}^*\,.$$

We can get

$$\frac{f(z+2\pi)}{f(z)} = \frac{e^{2\pi}\sin(z)\, e^z}{\sin(z)\, e^z} = e^{2\pi}\,,$$

and

$$\frac{L_{2\pi}^n f(z)}{f(z)} = \frac{\sin(z)\, e^z\left(e^{2\pi n} + \cdots + e^{2\pi} - (n-1)\right)}{\sin(z)\, e^z} = \frac{e^{2\pi(n+1)} - e^{2\pi}}{e^{2\pi}-1} - n + 1\,,$$

and hence $f(z)$, $f(z+2\pi)$ and $L_{2\pi}^n f(z)$ share 0 CM.

In this corollary, we have replaced $f(z+c)$ with $L_c^n f(z+c)$ in Theorem 1.1 and we obtained the same result.

Corollary 1.1. *Let $f(z)$ be an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $L_c^n f(z)$ and $L_c^n f(z+c)$ $(n \geq 1)$ share $a(z)$ CM, then*

$$f(z) = h(z)e^{\frac{\beta}{c}z} + a(z) \quad \textit{and} \quad a(z) = 0 \quad \textit{or} \quad \sum_{i=0}^{n}\alpha_i - 1 = 0\,,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c.

Example 1.4. The entire function $f(z) = e^{\frac{1}{b}z}$, where $b \neq 0$ satisfies

$$f(z+nb) = e^n e^{\frac{1}{b}z}\,, \quad n \in \mathbb{N}\,.$$

We put $\alpha_i = 1$, $0 \le i \le n$ and $a(z) = 0$, then

$$L_b^n f(z) = f(z+nb) + \cdots + f(z+b) + f(z), \quad n \in \mathbb{N}^*$$

and

$$L_b^n f(z+b) = f(z+(n+1)b) + \cdots + f(z+2b) + f(z+b), \quad n \in \mathbb{N}^*,$$

we can get

$$\frac{L_b^n f(z) - a(z)}{f(z) - a(z)} = \frac{f(z)(e^n + \cdots + e + 1)}{f(z)} = \frac{1 - e^{n+1}}{1 - e},$$

and

$$\begin{aligned} \frac{L_b^n f(z+b) - a(z)}{f(z) - a(z)} &= \frac{f(z)\left(e^{n+1} + \cdots + e\right)}{f(z)} \\ &= \frac{e - e^{n+2}}{1 - e}, \end{aligned}$$

and hence $f(z)$, $L_b^n f(z)$ and $L_b^n f(z+b)$ share 0 CM.

It is natural to ask what happens if $L_c^n f(z)$ is replaced by $\Delta_c^n f(z)$ in Theorem 1.1. Corresponding to this question, we obtain the following result.

Theorem 1.2. *Let $f(z)$ be an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ $(n \ge 1)$ share $a(z)$ CM, then*

$$f(z) = h(z) e^{\frac{\beta}{c} z}, \text{ and } a(z) = 0,$$

where $\beta \ne 0$ and $h(z)$ is a periodic entire function of period c.

Example 1.5. The entire function $f(z) = \sin(z)\, e^{\frac{1}{2\pi} z}$ satisfies

$$f(z + 2\pi n) = e^n f(z), \quad n \in \mathbb{N}.$$

We can get

$$\frac{f(z+2\pi)}{f(z)} = \frac{e f(z)}{f(z)} = e,$$

and

$$\frac{\Delta_{2\pi}^n f(z)}{f(z)} = \frac{f(z) \sum\limits_{i=0}^{n} \binom{n}{i} (-1)^{n-i} e^i}{f(z)} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} e^i,$$

and hence $f(z)$, $f(z+2\pi)$ and $\Delta_{2\pi}^n f(z)$ share 0 CM.

In the following corollary we have replaced $f(z+c)$ with $\Delta_c^n f(z+c)$ in Theorem 1.2, and we obtained the same result.

Corollary 1.2. *Let $f(z)$ be an entire function of finite order such that $f(z) \not\equiv f_c(z)$, and let $a(z) \in S(f)$ be a periodic entire function with period c. If $f(z)$, $\Delta_c^n f(z)$ and $\Delta_c^n f(z+c)$ $(n \geq 1)$ share $a(z)$ CM, then*

$$f(z) = h(z) e^{\frac{\beta}{c} z}, \quad \textit{and} \quad a(z) = 0,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c.

Example 1.6. The entire function $f(z) = \sin(z)\, e^{\frac{1}{2\pi} z}$ satisfies

$$f(z + 2\pi n) = e^n f(z), \quad n \in \mathbb{N}.$$

We can get

$$\frac{\Delta_{2\pi}^n f(z)}{f(z)} = \frac{f(z) \sum\limits_{i=0}^{n} \binom{n}{i} (-1)^{n-i} e^i}{f(z)} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} e^i,$$

and

$$\frac{\Delta_{2\pi}^n f(z+2\pi)}{f(z)} = \frac{f(z) \sum\limits_{i=0}^{n} C_n^i (-1)^{n-i} e^{i+1}}{f(z)} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} e^{i+1},$$

and hence $f(z)$, $f(z+2\pi)$ and $\Delta_{2\pi}^n f(z)$ share 0 CM.

2. Lemmas

For the proof of our results, we need the following lemmas.

Lemma 2.1 ([10])**.** *Let $c \in \mathbb{C}$, $n \in \mathbb{N}$, and let $f(z)$ be a meromorphic function of finite order. Then for any small periodic function $a(z)$ with period c, with respect to $f(z)$,*

$$m\left(r, \frac{\Delta_c^n f}{f - a}\right) = S(r, f)$$

where the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

Lemma 2.2 ([5])**.** *Suppose that $f_1(z)$, $f_2(z), \ldots, f_n(z)$ $(n \geq 2)$ are some meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ $(n \geq 1)$ entire functions satisfying the following conditions:*

(i) $\sum\limits_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$;

(ii) *$g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;*

(iii) *for $1 \leq j \leq n$, $1 \leq j < k \leq n$. $T(r, f_j) = o\left\{T(r, e^{g_j(z) - g_k(z)})\right\}$ $(r \to \infty$, $r \notin E)$.*

Then $f_j(z) \equiv 0$, $(j = 1, 2, \ldots, n)$.

Lemma 2.3 ([5]). *Let $f(z)$ be a non-constant meromorphic function in the complex plane and*

$$R(f)=\frac{P(f)}{Q(f)}\,,$$

where $P(f)=\sum\limits_{k=0}^{p}a_kf^k$ and $Q(f)=\sum\limits_{j=0}^{q}b_jf^j$ are two mutually prime polynomials in $f(z)$. If the coefficients a_k, b_j are small functions of $f(z)$ and $a_p(z)\not\equiv 0$, $b_q(z)\not\equiv 0$, then

$$T\big(r,R(f)\big)=\max\{p,q\}\,T(r,f)+S\left(r,f\right)\,.$$

3. Proofs of the Theorems and Corollaries

Proof of Theorem 1.1. Suppose that $f(z)$, $f\left(z+c\right)$ and $L_c^nf\left(z\right)$ share $a\left(z\right)$ CM. Then

$$\frac{f\left(z+c\right)-a\left(z\right)}{f\left(z\right)-a\left(z\right)}=e^{p(z)}\,, \tag{3.1}$$

and

$$\frac{L_c^nf\left(z\right)-a\left(z\right)}{f\left(z\right)-a\left(z\right)}=e^{q(z)}, \tag{3.2}$$

where p and q are polynomials. From (3.1) it's easy to prove the following

$$f\left(z+nc\right)-a\left(z\right)=\left[f\left(z\right)-a\left(z\right)\right]e^{\sum\limits_{i=0}^{n-1}p(z+ic)}\,, \tag{3.3}$$

by using equations (3.2), and (3.3), we obtain

$$\frac{\alpha_n\left[f\left(z\right)-a\left(z\right)\right]e^{\sum\limits_{i=0}^{n-1}p(z+ic)}+a\left(z\right)\alpha_n+\cdots+\alpha_0\left[f\left(z\right)-a\left(z\right)\right]+a\left(z\right)\alpha_0-a\left(z\right)}{f\left(z\right)-a\left(z\right)}$$
$$=e^{q(z)}\,,$$

then

$$\alpha_ne^{\sum\limits_{i=0}^{n-1}p(z+ic)}+\cdots+\alpha_1e^{p(z)}+\alpha_0+\frac{a\left(z\right)\big(\sum\limits_{i=0}^{n}\alpha_i-1\big)}{f\left(z\right)-a\left(z\right)}=e^{q(z)}\,. \tag{3.4}$$

From (3.1) and (3.2), we get

$$\begin{aligned}\frac{L_c^nf\left(z+c\right)-L_c^nf\left(z\right)}{f\left(z\right)-a\left(z\right)}&=\frac{\alpha_n\Delta_cf\left(z+nc\right)+\cdots+\alpha_0\Delta_cf\left(z\right)}{f\left(z\right)-a\left(z\right)}\\&=e^{p(z)+q_c(z)}-e^{q(z)}\,.\end{aligned} \tag{3.5}$$

Set

$$\varphi\left(z\right)=e^{p(z)+q_c(z)}-e^{q(z)}\,.$$

We show that $\varphi(z) \not\equiv 0$. If $\varphi(z) \equiv 0$, then

$$e^{p(z)} = e^{q(z)-q_c(z)}\,, \tag{3.6}$$

thus, by equations (3.1) and (3.6), we have

$$\frac{f(z+c)-a(z)}{f(z)-a(z)} = e^{q(z)-q_c(z)}\,. \tag{3.7}$$

If $e^{p(z)}$ is a constant or $\deg q(z) = 1$, then by equations (3.1) and (3.7) respectively, we get

$$\frac{f(z+c)-a(z)}{f(z)-a(z)} = e^{\beta}\,,$$

where $\beta \neq 0$, we leave this situation to come back to it in the end.
If $q(z)$ is a constant, then by (3.7) we get the following contradiction

$$f(z+c) = f(z)\,.$$

By equation (3.7) it is easy to see

$$\frac{f(z+nc)-a(z)}{f(z)-a(z)} = e^{q(z)-q_{nc}(z)}\,. \tag{3.8}$$

By using equations (3.3), (3.8) and (3.4), we have

$$-\frac{a(z)\left(\sum\limits_{i=0}^{n}\alpha_i - 1\right)}{f(z)-a(z)} = \alpha_n e^{q(z)-q_{nc}(z)} + \cdots + \alpha_1 e^{q(z)-q_c(z)} + \alpha_0 - e^{q(z)}\,. \tag{3.9}$$

If $a(z)\left(\sum\limits_{i=0}^{n}\alpha_i - 1\right) = 0$, then

$$e^{q(z)} = \alpha_n e^{q(z)-q_{nc}(z)} + \cdots + \alpha_1 e^{q(z)-q_c(z)} + \alpha_0\,,$$

we get the following contradiction

$$T(r, e^q) = S(r, e^q)\,.$$

If $a(z)\left(\sum\limits_{i=0}^{n}\alpha_i - 1\right) \neq 0$, then by using equations (3.7) and (3.9), we have

$$-\frac{a(z)\left(\sum\limits_{i=0}^{n}\alpha_i - 1\right)}{f(z)-a(z)} = \alpha_n e^{q(z)-q_{(n+1)c}(z)} + \cdots + \alpha_0 e^{q(z)-q_c(z)} - e^{q(z)}\,, \tag{3.10}$$

thus, by equations (3.9) and (3.10), we have

$$\alpha_n e^{q(z)-q_{(n+1)c}(z)} + (\alpha_{n-1}-\alpha_n)\, e^{q(z)-q_{nc}(z)} + \cdots + (\alpha_0-\alpha_1)\, e^{q(z)-q_c(z)} - \alpha_0 = 0\,,$$

as we know from the above that $\deg q(z) \geq 2$, then by Lemma 2.2 we get the following contradiction

$$\alpha_n = \alpha_{j-1} - \alpha_j = \alpha_0 = 0,\ 0 \leq j \leq n\,,$$

thus, we deduce $\varphi(z) \not\equiv 0$.

Since $\varphi(z) \not\equiv 0$, by Lemma 2.1 and equation (3.5), we deduce that

$$T(r,\varphi) = m(r,\varphi)$$

$$\leq m\Big(r, \frac{\Delta_c f(z+nc)}{f-a(z)}\Big) + \cdots + m\Big(r, \frac{\Delta_c f(z)}{f-a(z)}\Big) + S(r,f) = S(r,f)\ . \tag{3.11}$$

Note that $\dfrac{e^{p(z)+q_c(z)}}{\varphi(z)} - \frac{e^{q(z)}}{\varphi(z)} = 1$. By using the second main theorem and equation (3.11), we have

$$T\Big(r, \frac{e^q}{\varphi}\Big) \leq \overline{N}\Big(r, \frac{e^q}{\varphi}\Big) + \overline{N}\Big(r, \frac{\varphi}{e^q}\Big) + \overline{N}\Big(r, \frac{1}{\frac{e^q}{\varphi}+1}\Big) + S\Big(r, \frac{e^q}{\varphi}\Big)$$

$$= \overline{N}\Big(r, \frac{e^q}{\varphi}\Big) + \overline{N}\Big(r, \frac{\varphi}{e^q}\Big) + \overline{N}\Big(r, \frac{\varphi}{e^{p+q_c}}\Big) + S\Big(r, \frac{e^q}{\varphi}\Big)$$

$$= S(r,f) + S\Big(r, \frac{e^q}{\varphi}\Big)\,. \tag{3.12}$$

Thus, by equations (3.11) and (3.12), we have

$$T(r,e^q) = S(r,f)\,. \tag{3.13}$$

Similarly, we get

$$T(r,e^p) = S(r,f)\,. \tag{3.14}$$

By using the first main theorem, we have

$$T\left(r, \frac{a(z)(\sum\limits_{i=0}^{n}\alpha_i - 1)}{f-a(z)}\right) = T(r,f) + S(r,f)\,. \tag{3.15}$$

From equations (3.4) and (3.15), we deduce that

$$T(r,f) \leq T(r, e^{\sum\limits_{i=0}^{n-1} p_{ic}}) + \cdots + T(r,e^p) + T(r,e^q) + S(r,f)\,. \tag{3.16}$$

If $a(z)\big(\sum_{i=0}^{n}\alpha_i - 1\big) \neq 0$, by equations (3.13), (3.14) and (3.16), we deduce the contradiction

$$T(r,f) \leq S(r,f)$$

and from this, we deduce that either $a(z) = 0$ or $\sum_{i=0}^{n}\alpha_i - 1 = 0$, and by equation (3.4), we have

$$\alpha_n e^{\sum\limits_{i=0}^{n-1} p(z+ic)} + \cdots + \alpha_1 e^{p(z)} + \alpha_0 = e^{q(z)}\,. \tag{3.17}$$

Next, we prove that $p(z)$ and $q(z)$ are constants. We need to treat the following cases:

First of all, we set

$$p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 = a_n z^n + \alpha(z)$$

and

$$q(z) = b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0 = b_m z^m + \gamma(z) ,$$

where $a_n \neq 0$, $a_{n-1}, \ldots, a_0$, $b_m \neq 0, b_{m-1}, \ldots, b_0$ are constants, α and γ are polynomials where $\deg \alpha \leq n-1$, and $\deg \gamma \leq m-1$.
On the other hand we have

$$\begin{aligned}\sum_{i=0}^{j-1} p(z+ic) &= p(z) + p(z+c) + \cdots + p(z+(j-1)c) \\ &= j a_n z^n + \lambda_j(z) ,\end{aligned}$$

where λ_j are polynomials with degree at most $n-1$ for $j = 1, 2, \ldots, n$. By equation (3.17), we have

$$\alpha_n \left(e^{a_n z^n}\right)^n e^{\lambda_n(z)} + \cdots + \alpha_1 e^{a_n z^n} e^{\lambda_1(z)} + \alpha_0 = e^{b_m z^m} e^{\gamma(z)} . \tag{3.18}$$

Define functions $H(z) = e^{a_n z^n}$, and $G(z) = e^{b_m z^m}$. Then, equation (3.18) becomes

$$\alpha_n [H(z)]^n e^{\lambda_n(z)} + \cdots + \alpha_1 H(z) e^{\lambda_1(z)} + \alpha_0 = G(z) e^{\gamma(z)} . \tag{3.19}$$

(i) If $m \neq n$, then we have two subcases:

Case (A): If $m < n$, then by using equation (3.19) and applying Lemma 2.3, we see that

$$nT(r, H) = S(r, H) ,$$

which is impossible.

Case (B): If $n < m$, then, by using equation (3.19) and applying Lemma 2.3, we see that

$$T(r, G) = S(r, G) ,$$

which is impossible.

(ii) If $n = m \neq 0$, then we have two subcases:

Case (A): If $b_m = j a_n$, $1 \leq j \leq n$, then by using equation (3.18), we have

$$\alpha_n [H(z)]^n e^{\lambda_n(z)} + \cdots + \alpha_j [H(z)]^j \left(e^{\lambda_j(z)} - e^{\gamma(z)}\right) + \cdots + H(z) e^{\lambda_1(z)} + \alpha_0 = 0 ,$$

then by Lemma 2.3 we deduce the contradiction

$$nT(r, H) = S(r, H) .$$

Case (B): If $b_m \neq j a_n$, $1 \leq j \leq n$, then by using equation (3.18), we have

$$\alpha_n e^{n a_n z^n + \lambda_n(z)} + \cdots + \alpha_1 e^{a_n z^n + \lambda_1(z)} + \alpha_0 = e^{b_m z^n + \gamma(z)} ,$$

then by Lemma 2.2 we deduce the contradiction

$$1 = \alpha_n = \alpha_j = \alpha_0 = 0 , \quad 0 \leq j \leq n .$$

Finally, we conclude that $p(z)$ and $q(z)$ are constants, suppose that $e^{p(z)} = e^{\beta}$ (the same situation we left earlier) where $\beta \neq 0$, from equation (3.1), we have

$$f(z+c) - a(z) = e^{\beta} [f(z) - a(z)] . \tag{3.20}$$

If $f(z)$ and $g(z)$ are two solutions of the equation (3.20), then $h(z) = \frac{f(z)-a(z)}{g(z)-a(z)}$ is a periodic function of period c. Obviously $g(z) = e^{\frac{\beta}{c}z} + a(z)$ is solution of (3.20). Hence the entire solution of (3.20) must be of the form $f(z) = h(z)\, e^{\frac{\beta}{c}z} + a(z)$, where $h(z)$ is a periodic entire function of period c. □

Proof of Corollary 1.1. Suppose that $f(z)$, $L_c^n f(z)$ and $L_c^n f(z+c)$ share $a(z)$ CM. Then

$$\frac{L_c^n f(z) - a(z)}{f(z) - a(z)} = e^{p(z)}, \tag{3.21}$$

and

$$\frac{L_c^n f(z+c) - a(z)}{f(z) - a(z)} = e^{q(z)}, \tag{3.22}$$

where p and q are polynomials. By using equation (3.21), we deduce that

$$\frac{L_c^n f(z+c) - a(z)}{f(z+c) - a(z)} = e^{p(z+c)}. \tag{3.23}$$

By equations (3.22) and (3.23), we get the following result

$$\frac{f(z+c) - a(z)}{f(z) - a(z)} = e^{q(z)-p(z+c)}, \tag{3.24}$$

and finally, using equations (3.21) and (3.24), we can deduce $f(z)$, $f(z+c)$ and $L_c^n f(z)$ $(n \geq 1)$ share $a(z)$ CM, then by Theorem 1.1 we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z} + a(z) \quad \text{and} \quad a(z) = 0 \quad \text{or} \quad \sum_{i=0}^{n} \alpha_i - 1 = 0,$$

Where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c. □

Proof of Theorem 1.2. Suppose that $f(z)$, $f(z+c)$ and $\Delta_c^n f(z)$ share $a(z)$ CM, we follow the same steps as in Proof of Theorem 1.1. Equation (3.4) becomes

$$\alpha_n e^{\sum_{i=0}^{n-1} p(z+ic)} + \cdots + \alpha_1 e^{p(z)} + \alpha_0 - \frac{a(z)}{f(z) - a(z)} = e^{q(z)},$$

because we know that, if

$$L_c^n f(z) = \Delta_c^n f(z),$$

then

$$\sum_{i=0}^{n} \alpha_i = 0.$$

We continue with the same steps without forgetting that

$$\sum_{i=0}^{n} \alpha_i = 0.$$

In the proof of the previous theorem, following equation (3.16), we concluded that

$$a\,(z) = 0 \quad \text{or} \quad \sum_{i=0}^{n} \alpha_i - 1 = 0\,,$$

but since

$$\sum_{i=0}^{n} \alpha_i = 0\,,$$

we conclude that

$$a\,(z) = 0\,.$$

From this, we continue with the same steps until the end of the previous proof. Finally, we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z}, \text{ and } a\,(z) = 0\,,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c. □

Proof of Corollary 1.2. Suppose that $f(z)$, $\Delta_c^n f\,(z)$ and $\Delta_c^n f\,(z+c)$ share $a\,(z)$ CM. Then

$$\frac{\Delta_c^n f\,(z) - a\,(z)}{f\,(z) - a\,(z)} = e^{p(z)}\,, \tag{3.25}$$

and

$$\frac{\Delta_c^n f\,(z+c) - a\,(z)}{f\,(z) - a\,(z)} = e^{q(z)}\,, \tag{3.26}$$

where p and q are polynomials. By using equation (3.25), we deduce that

$$\frac{\Delta_c^n f\,(z+c) - a\,(z)}{f\,(z+c) - a\,(z)} = e^{p(z+c)}\,, \tag{3.27}$$

By equations (3.26) and (3.27), we get the following result

$$\frac{f\,(z+c) - a\,(z)}{f\,(z) - a\,(z)} = e^{q(z)-p(z+c)}\,, \tag{3.28}$$

and finally, using equations (3.25) and (3.28), we can deduce $f\,(z)$, $f\,(z+c)$ and $\Delta_c^n f\,(z)$ $(n \geq 1)$ share $a\,(z)$ CM, then by Theorem 1.2 we conclude that

$$f(z) = h(z)e^{\frac{\beta}{c}z}, \quad \text{and} \quad a\,(z) = 0\,,$$

where $\beta \neq 0$ and $h(z)$ is a periodic entire function of period c. □

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