ARCHIVUM MATHEMATICUM (BRNO)
Tomus 60 (2024), 145–152

AN IMPROVED OSCILLATION THEOREM FOR NONLINEAR DIFFERENTIAL EQUATIONS OF ADVANCED TYPE

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Abstract. This paper deals with the oscillatory solutions of the first order nonlinear advanced differential equation. The aim of the present paper is to obtain an oscillation condition for this equation. This result is new and improves and correlates many of the well-known oscillation criteria that were in the literature. Finally, an example is given to illustrate the main result.

1. Introduction

We consider the first order nonlinear differential equation with advanced argument

$$x'(t) - p(t)f(x(\tau(t))) = 0\,, \quad t \geq t_0\,, \tag{1.1}$$

where the functions $p(t)$, $\tau(t) \in C([t_0,\infty),\mathbb{R}_+)$ $(\mathbb{R}_+ = [0,\infty))$ and $\tau(t)$ is not necessarily monotone such that

$$\tau(t) \geq t \quad \text{for} \quad t \geq t_0\,, \quad \lim_{t\to\infty} \tau(t) = \infty \tag{1.2}$$

and

$$f \in C(\mathbb{R},\ \mathbb{R}) \quad \text{and} \quad xf(x) > 0 \quad \text{for} \quad x \neq 0\,. \tag{1.3}$$

By a solution of (1.1), we mean a continuously differentiable function defined on $[t_0,\infty)$ such that (1.1) is satisfied for $t \geq t_0$. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros. Otherwise, it is called nonoscillatory. Equation (1.1) is called oscillatory if all its solutions are oscillatory.

When $f(x) = x$, we have the following equation which is the linear form of (1.1)

$$x'(t) - p(t)x(\tau(t)) = 0\,, \quad t \geq t_0\,. \tag{1.4}$$

Many scientists investigated the question of obtaining sufficient oscillation criteria for the solutions of (1.4). Ladas and Stavroulakis [5], Li and Zhu [7], Koplatadze and Chanturija [2], Kusano [4], Kulenovic and Grammatikopoulos [3] studied equation (1.4) with constant argument and obtained some oscillation results.

In 1987, Ladde et al. [6] obtained the following criteria for (1.4).

2020 *Mathematics Subject Classification*: primary 34C10; secondary 34K06, 34K11.
Key words and phrases: advanced equations, nonmonotone argument, oscillatory solution.
Received November 16, 2023, revised February 2024. Editor R. Šimon Hilscher.
DOI: 10.5817/AM2024-3-145

If $\tau(t)$ is nondecreasing and

$$\liminf_{t\to\infty}\int_t^{\tau(t)} p(s)\,ds > \frac{1}{e} \tag{1.5}$$

or

$$\limsup_{t\to\infty}\int_t^{\tau(t)} p(s)\,ds > 1 \tag{1.6}$$

then, all solutions of (1.4) are oscillatory.

Now, we define the following function

$$h(t) := \inf_{s\geq t}\{\tau(s)\}\ , \quad t\geq 0\,. \tag{1.7}$$

Apparently, $h(t)$ is nondecreasing and $h(t)\leq\tau(t)$ for all $t\geq 0$.

Also, suppose that f in (1.1) satisfies the following condition

$$0 < \widetilde{N} := \limsup_{|x|\to\infty}\frac{x}{f(x)} < \infty\,. \tag{1.8}$$

In 2019, Öcalan et al. [8] studied equation (1.1) with nonmonotone advanced argument and established the theorem given below.

Theorem 1.1. *Assume that* (1.2) *and* (1.3) *hold. If* $\tau(t)$ *is not necessarily monotone and*

$$\liminf_{t\to\infty}\int_t^{\tau(t)} p(s)\,ds > \frac{\widetilde{N}}{e}\,, \quad 0\leq\widetilde{N}<\infty \tag{1.9}$$

or

$$\limsup_{t\to\infty}\int_t^{h(t)} p(s)ds > \widetilde{N}\,, \quad 0<\widetilde{N}<\infty \tag{1.10}$$

where $h(t)$ *and* $\widetilde{N}$ *are defined by* (1.7) *and* (1.8)*, respectively, then all solutions of* (1.1) *are oscillatory.*

There are many papers about linear advanced differential equations, but there are a few articles about nonlinear differential equations with advanced argument. As far as we know, there are only two criteria for the oscillatory solutions of (1.1) with nonmonotone argument in the literature. In view of this, an interesting question that arises in the case that $\tau(t)$ is not necessarily monotone and (1.9) and (1.10) do not hold, is whether we can obtain a new oscillation criterion for (1.1). In this article, we will answer this question in a positive way. So, our purpose is to essentialy improve the conditions given above and to present new sufficient conditions for the oscillation of all solutions of (1.1) by using the ratio $\frac{x(h(t))}{x(t)}$.

The paper is arranged as below. Firstly, we give some information about the advanced differential equations. Next, we establish a new condition involving lim sup and lim inf for all the oscillatory solutions of (1.1). We present an example to confirm the applicability of the main results.

2. Main results

In this section, we establish a new oscillation result for the solutions of (1.1), under the assumption that the advanced argument $\tau(t)$ is not necessarily monotone.

The following lemmas are useful to prove the main theorem.

Lemma 2.1 ([9, Lemma 2.2]). *Assume that*

$$\liminf_{t\to\infty} \int_t^{\tau(t)} p(s)\,ds > 0\,.$$

Then, we get

$$\liminf_{t\to\infty} \int_t^{\tau(t)} p(s)\,ds = \liminf_{t\to\infty} \int_t^{h(t)} p(s)\,ds\,, \tag{2.1}$$

where $h(t)$ is given by (1.7)*.*

Lemma 2.2. *Assume that $x(t)$ is an eventually positive solution of* (1.1)*. If*

$$\limsup_{t\to\infty} \int_t^{h(t)} p(s)\,ds > 0\,, \tag{2.2}$$

where $h(t)$ is given by (1.7)*, then $\lim\limits_{t\to\infty} x(t) = \infty$.*

Proof. If we take $m = 1$ in [1, Lemma 2], we obtained the above result. So, the proof of the lemma is omitted here. ☐

Lemma 2.3. *Assume that $x(t)$ is an eventually positive solution of* (1.1) *and*

$$\alpha := \liminf_{t\to\infty} \int_t^{\tau(t)} p(s)\,ds > 0\,. \tag{2.3}$$

Then, we have

$$\limsup_{t\to\infty} \frac{x(h(t))}{x(t)} \le \Big(\frac{2\widetilde{N}}{\alpha}\Big)^2, \tag{2.4}$$

where $h(t)$ and $\widetilde{N}$ are given by (1.7) *and* (1.8)*, respectively.*

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then, there exists $t_1 > t_0$ such that $x(t)$, $x(\tau(t))$, $x(h(t)) > 0$ for all $t \ge t_1$. Thus, from (1.1), we have

$$x'(t) = p(t) f(x(\tau(t))) \ge 0$$

for all $t \ge t_1$, which means that $x(t)$ is an eventually nondecreasing. Also, with the help of Lemma 2.1, (2.3) implies (2.2), then from Lemma 2.2, we know that

$\lim\limits_{t\to\infty} x(t) = \infty$. Then, from (1.8), we can choose $t_2 > t_1$ and there exists ε_1 such that

$$f\big(x(\tau(t))\big) > \frac{1}{\widetilde{N}+\varepsilon_1} x(\tau(t)) \quad \text{for} \quad t \geq t_2. \tag{2.5}$$

Using the fact that $x(t)$ is nondecreasing, $h(t) \leq \tau(t)$ and (2.5), from (1.1), we have

$$x'(t) - \frac{1}{\widetilde{N}+\varepsilon_1} p(t) x(h(t)) > 0\,. \tag{2.6}$$

Moreover, from (2.3) and Lemma 2.1, we have

$$\int\limits_{t}^{h(t)} p(s)\,ds \geq \alpha - \varepsilon_2\,, \quad \varepsilon_2 \in (0,\alpha)\,, \tag{2.7}$$

then, there exists $t^* < t$ such that

$$\int\limits_{t^*}^{t} p(s)\,ds \geq \frac{\alpha-\varepsilon_2}{2} \quad \text{and} \quad \int\limits_{t}^{h(t^*)} p(s)\,ds \geq \frac{\alpha-\varepsilon_2}{2}\,. \tag{2.8}$$

Then, integrating (2.6) from t^* and t and using the fact that $x(t)$ and $h(t)$ are nondecreasing and (2.8), we have

$$x(t) - x(t^*) - \frac{1}{\widetilde{N}+\varepsilon_1} \int\limits_{t^*}^{t} p(s) x(h(s))\,ds > 0$$

so,

$$x(t) - x(t^*) - \frac{1}{\widetilde{N}+\varepsilon_1} x\big(h(t^*)\big) \int\limits_{t^*}^{t} p(s)\,ds > 0$$

and

$$x(t) > \frac{1}{\widetilde{N}+\varepsilon_1} x\big(h(t^*)\big) \frac{\alpha-\varepsilon_2}{2}\,. \tag{2.9}$$

By using the same facts as above, integrating (2.6) from t to $h(t^*)$, we have

$$x\big(h(t^*)\big) - x(t) - \frac{1}{\widetilde{N}+\varepsilon_1} \int\limits_{t}^{h(t^*)} p(s) x\big(h(s)\big)\,ds > 0$$

so,

$$x\big(h(t^*)\big) - x(t) - \frac{1}{\widetilde{N}+\varepsilon_1} x\big(h(t)\big) \int\limits_{t}^{h(t^*)} p(s)\,ds > 0$$

and

$$x\big(h(t^*)\big) > \frac{1}{\widetilde{N}+\varepsilon_1} x\big(h(t)\big) \frac{\alpha-\varepsilon_2}{2}\,. \tag{2.10}$$

Finally, combining (2.9) and (2.10), we get

$$x(t) > \Big(\frac{1}{\widetilde{N}+\varepsilon_1}\Big)\Big(\frac{1}{\widetilde{N}+\varepsilon_1}\Big)x\big(h(t)\big)\Big(\frac{\alpha-\varepsilon_2}{2}\Big)\Big(\frac{\alpha-\varepsilon_2}{2}\Big)$$

so,

$$\frac{x(h(t))}{x(t)} < \Big(\frac{2(\widetilde{N}+\varepsilon_1)}{\alpha-\varepsilon_2}\Big)^2 .$$

Hence, we have

$$\limsup_{t\to\infty} \frac{x(h(t))}{x(t)} \le \Big(\frac{2(\widetilde{N}+\varepsilon_1)}{\alpha-\varepsilon_2}\Big)^2 .$$

Because of ε_1 and ε_2 are arbitrary, by letting $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$, we obtain (2.4), and this completes the proof. □

Theorem 2.4. *Assume that* (1.2) *and* (1.3) *hold. If* $\tau(t)$ *is not necessarily monotone,* $0 < \alpha \le \frac{\widetilde{N}}{e}$ *and*

$$\limsup_{t\to\infty} \int_t^{h(t)} \frac{p(s)}{\widetilde{N}} ds > 1 - \Big(\frac{\alpha}{2\widetilde{N}}\Big)^2, \tag{2.11}$$

where $h(t)$ *and* $\widetilde{N}$ *are given by* (1.7) *and* (1.8)*, respectively, then all solutions of* (1.1) *are oscillatory.*

Proof. Assume, for the sake of contradiction, that there is an eventually positive solution $x(t)$ of (1.1). If $x(t)$ is an eventually negative solution of (1.1), the proof can be done in a similar way. Then, there exists $t_1 > t_0$ such that $x(t)$, $x\left(\tau(t)\right)$, $x\left(h(t)\right) > 0$ for all $t \ge t_1$. So, from (1.1), we get

$$x'(t) = p(t) f\big(x\left(\tau(t)\right)\big) \ge 0$$

for all $t \ge t_1$, which implies that $x(t)$ is nondecreasing function. Lemma 2.2 and the condition (2.11) imply that $\lim\limits_{t\to\infty} x(t) = \infty$. Then from (1.8), we can choose $t_2 > t_1$ and there exists ε_1 such that

$$f\big(x\left(\tau(t)\right)\big) > \frac{1}{\widetilde{N}+\varepsilon_1} x\left(\tau(t)\right) \quad \text{for} \quad t \ge t_2. \tag{2.12}$$

Using the fact that inequality (2.12), $x(t)$ is nondecreasing and $h(t) \le \tau(t)$, from (1.1), we obtain

$$x'(t) - \frac{1}{\widetilde{N}+\varepsilon_1} p(t) x(h(t)) > 0 . \tag{2.13}$$

Integrating (2.13) from t to $h(t)$, we have

$$x\big(h(t)\big) - x(t) - \frac{1}{\widetilde{N}+\varepsilon_1} \int_t^{h(t)} p(s) x\big(h(s)\big)\, ds > 0$$

so,

$$x\big(h(t)\big) - x(t) - \frac{1}{\widetilde{N}+\varepsilon_1} x\big(h(t)\big) \int_t^{h(t)} p(s)\,ds > 0$$

and

$$x\big(h(t)\big)\Big[1 - \frac{1}{\widetilde{N}+\varepsilon_1} \int_t^{h(t)} p(s)\,ds\Big] - x(t) > 0\,.$$

Hence, we get

$$\frac{1}{\widetilde{N}+\varepsilon_1} \int_t^{h(t)} p(s)\,ds < 1 - \frac{x(t)}{x(h(t))}\,.$$

Then, by Lemma 2.3, we obtain

$$\limsup_{t\to\infty} \int_t^{h(t)} \frac{p(s)}{\widetilde{N}}\,ds \le 1 - \Big(\frac{\alpha}{2\widetilde{N}}\Big)^2,$$

which contradicts to (2.11), so this completes the proof. □

Example 2.5. We consider the following first order nonlinear advanced differential equation

$$x'(t) - 0.213x\,(\tau(t))\ln(e^{-|x(\tau(t))|} + 3.03739) = 0\,, \quad t \ge 0\,, \tag{2.14}$$

where

$$\tau(t) = \begin{cases} 5k+3, & t \in [5k,\ 5k+1] \\ 4t-15k-1, & t \in [5k+1,\ 5k+2] \\ -3t+20k+13, & t \in [5k+2,\ 5k+3] \\ 5t-20k-11, & t \in [5k+3,\ 5k+4] \\ -t+10k+13, & t \in [5k+4,\ 5k+5] \end{cases}, \quad k \in \mathbb{N}_0$$

and

$$h(t) := \inf_{s\ge t}\{\tau(s)\} = \begin{cases} 5k+3, & t \in [5k,\ 5k+1] \\ 4t-15k-1, & t \in [5k+1,\ 5k+1.25] \\ 5k+4, & t \in [5k+1.25,\ 5k+3] \\ 5t-20k-11, & t \in [5k+3,\ 5k+3.8] \\ 5k+8, & t \in [5k+3.8,\ 5k+5] \end{cases}, \quad k \in \mathbb{N}_0\,,$$

then, we have

$$\widetilde{N} = \limsup_{|x|\to\infty} \frac{x\,(\tau(t))}{x\,(\tau(t))\ln(e^{-|x(\tau(t))|} + 3.03739)} = \frac{1}{\ln(3.03739)} \cong 0.9\,.$$

On the other hand, as we can see from the following process, the previous results, which are $\liminf\limits_{t\to\infty} \int_t^{\tau(t)} p(s)\,ds > \frac{\widetilde{N}}{e}$ and $\limsup\limits_{t\to\infty} \int_t^{h(t)} p(s)\,ds > \widetilde{N}$ for the first order nonlinear advanced differential equation, are not satisfied.

$$\alpha = \liminf_{t\to\infty} \int\limits_{t}^{\tau(t)} p(s)\,ds = \liminf_{t\to\infty} \int\limits_{5k+3}^{5k+4} (0.213)\,ds \cong 0.213 \ngtr \frac{\widetilde{N}}{e} \cong 0.33112$$

and

$$\limsup_{t\to\infty} \int\limits_{t}^{h(t)} p(s)ds = \limsup_{t\to\infty} \int\limits_{5k+3.8}^{5k+8} (0.213)\,ds \cong 0.8946 \ngtr \widetilde{N} = 0.9\,.$$

So, the earlier results which were given in the Theorem 1.1 ((1.9) and (1.10)) are not valid in this example.

However, by using the result given in the present paper, we obtain

$$\limsup_{t\to\infty} \int\limits_{t}^{h(t)} \frac{p(s)}{\widetilde{N}}\,ds \cong 0,994 > 1 - \left(\frac{\alpha}{2\widetilde{N}}\right)^2 = 0,986.$$

Then, all conditions of the main theorem are satisfied and all solutions of (2.14) are oscillatory.

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