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LIE ALGEBRA STRUCTURE IN THE MODEL OF 3-LINK SNAKE ROBOT

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Abstract. In this paper, we study a 5 dimensional configuration space of a 3-link snake robot model moving in a plane. We will derive two vector fields generating a distribution which represents a space of the robot's allowable movement directions. An arbitrary choice of such generators generates the entire tangent space of the configuration space, i.e. the distribution is bracket-generating, but our choice additionally generates a finite dimensional Lie algebra over real numbers. This allows us to extend our model to a model with local Lie group structure, which may have interesting consequences for our original model.

1. Introduction

The model of 3-link snake robot has been studied by mathematicians for decades as one of the $(2,3,5)$-distributions. Élie Cartan classified all $(2,3,5)$-distributions in his famous paper [2] and found that the maximal possible group of symmetries is the 14-dimensional G_2. Although the snake robot model probably has only a 3-dimensional symmetry group, there is still the possibility that some slight modification of the model has a larger symmetry group. Regardless, the model is studied in control theory as one of the simplest snake-like model, e.g. [3].

This paper is intended for a wider audience. We first show in detail how the model and many similar models can be described, then we focus on a specific structure that emerges and consider the consequences of the existence of this structure.

The 3-link snake robot moves on a plane and is composed of three segments of fixed length with two movable joints. In the center of each segment, it has a wheel which force the movement direction of each center to be collinear with the segment direction. Due to that, it has two degrees of freedom. In control theory, it is normally studied how to control it by the two angles between jointed segments [3].

In the next section, we describe the configuration space as a smooth manifold $\mathcal{M} = S^1 \times S^1 \times S^1 \times \mathbb{R}^2$ that can be obtained as a submanifold of $\left(\mathbb{R}^2\right)^4$. Its

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2-rank distribution $\mathcal{D}$ is a subbundle of the tangent bundle $T\mathcal{M}$ that consists of vectors corresponding to motion directions that satisfy the constraints caused by the wheels and requirement of a fixed length of the segments in each state $q \in \mathcal{M}$ of the model. We will find two vector fields which locally span the distribution and correspond to the motions caused by a forward shift and a rotation of the central segment. Since the distribution $\mathcal{D}$ is not involutive it is reasonable to define for each $q \in \mathcal{M}$ recurrently

$$\mathcal{D}^1_q := \mathcal{D}_q\,, \quad \mathcal{D}^{k+1}_q := \mathcal{D}^k_q + [\mathcal{D}, \mathcal{D}^k]_q\,. \tag{1}$$

It will be shown that distribution $\mathcal{D}$ is *bracket-generating*, which means that for all $q \in \mathcal{M}$ there is some $r = r(q) \in \mathbb{N}$ such that $\mathcal{D}^r_q = T_q\mathcal{M}$ and $\mathcal{D}^l_q = \mathcal{D}^r_q$ for $l \geq r$. We consider only a neighborhood of a regular point, so $r = r(q)$ is constant. We call the sequence of numbers $(\dim \mathcal{D}^1_q, \dim \mathcal{D}^2_q, \dots \dim D^r_q)$ a *growth vector* of $\mathcal{D}$. Let us also define *growth vector for the generators* $X, Y \in \mathcal{D}$ as the increasing sequence of numbers of linearly independent vector fields over real numbers gained by each Lie bracket iteration of the generators. The difference between these two definitions is that in the first one we consider subbundles of $T\mathcal{M}$ in some neighborhood of $q \in \mathcal{M}$ and in the second we consider real subspaces of the space of all vector fields $\mathfrak{X}(\mathcal{M})$. The growth vector of a distribution is always a lower bound for the growth vector for some its generators and the latter is usually an infinite sequence.

By Chow-Rashevskii theorem [4, p. 24], for bracket-generating distribution $\mathcal{D}$, each point is achievable, i.e. there is a path tangent to $\mathcal{D}$ between each two points of $\mathcal{M}$. It is crucial from the perspective of control theory, where they call such model controllable.

In this paper, we also consider the symmetries of the model and its extension. Symmetries of a model are automorphisms of the configuration space $F\colon \mathcal{M} \to \mathcal{M}$ that map the distribution $\mathcal{D}$ to itself, i.e. $\mathrm{d}F(\mathcal{D}_q) = \mathcal{D}_{F(q)}$. Symmetries map paths in $\mathcal{M}$ tangent to $\mathcal{D}$ to paths again tangent to $\mathcal{D}$. The Lie algebra of this automorphism group corresponds to *infinitesimal symmetries*, they are vector fields $Y \in \mathfrak{X}(\mathcal{M})$ such that

$$[Y, \mathcal{D}] \subseteq \mathcal{D}\,. \tag{2}$$

Let us consider a Lie group G, its Lie algebra $\mathfrak{g}$ and the left-invariant vector fields X_i corresponding to some vectors A_i generating $\mathfrak{g}$ by Lie brackets, i.e. $\mathfrak{g}$ is the smallest Lie algebra containing them. If we consider distribution generated by these vector fields, then its growth vector is the same as the growth vector for the generators A_i. Since right-invariant vector fields are precisely those that commute with all left-invariant vector fields, the distribution has at least right-invariant

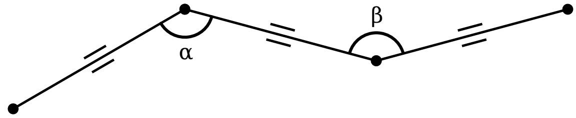

FIG. 1: The snake robot model, the angles α, β are the steering parameters.

vector fields as infinitesimal symmetries, because the left hand side of (2) is even equal to zero. Conversely, if we find vector fields that locally span a distribution such that they have the same growth vector as the distribution itself, we can declare them as left-invariant vector fields and locally define a group structure by the Baker-Campbell-Hausdorff formula.

Later, we will use this observation for the extended model $\mathcal{M}'$ with a Lie group structure.

2. Allowable movements

Our snake robot model is moving on the plane, so we can describe its state by four points $P_i \in \mathbb{R}^2$.

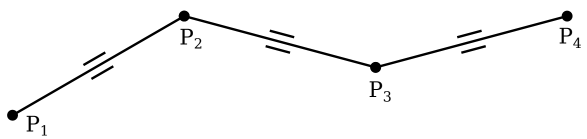

FIG. 2: Description of a state of the model by points $P_i \in \mathbb{R}^2$ with a fixed length of P_iP_{i+1}.

The requirement for velocity vectors of the segment centers corresponds to the condition that there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that the following holds.

$$\begin{aligned} \tfrac{1}{2}(\dot{P}_1 + \dot{P}_2) &= \alpha(P_2 - P_1) \\ \tfrac{1}{2}(\dot{P}_2 + \dot{P}_3) &= \beta(P_3 - P_2) \\ \tfrac{1}{2}(\dot{P}_3 + \dot{P}_4) &= \gamma(P_4 - P_3) \end{aligned} \tag{3}$$

The requirement for fixed length corresponds to the following.

$$\|P_2 - P_1\| = const.,\ \|P_3 - P_2\| = const.,\ \|P_4 - P_3\| = const. \tag{4}$$

Putting $Z_i = P_{i+1} - P_i$, $S = \frac{1}{2}(P_2 + P_3)$ we can rewrite the system of equations (3) as follows.

$$\begin{aligned} 2\dot{S} - \dot{Z}_2 - \dot{Z}_1 &= 2\alpha Z_1 \\ \dot{S} &= \beta Z_2 \\ 2\dot{S} + \dot{Z}_2 + \dot{Z}_3 &= 2\gamma Z_3\,. \end{aligned} \tag{5}$$

Although all points lie in the plane, we will consider a unit vector $\mathbf{e_3}$ perpendicular to the plane to describe rotations using the cross product. Moreover, we will denote the dot product of X and Y as $X \cdot Y$.

Using the condition (4) we can put $\dot{Z}_2 = \delta Z_2 \times \mathbf{e_3}$ for some parameter $\delta \in \mathbb{R}$ and we can express α and γ in β and δ to get the following.

$$\begin{aligned} \dot{Z}_1 &= 2\beta\, Z_2 - \tfrac{1}{\|Z_1\|^2}(2\beta\, Z_1 \cdot Z_2 - \delta\, Z_1 \cdot (Z_2 \times \mathbf{e_3}))Z_1 - \delta\, Z_2 \times \mathbf{e_3} \\ \dot{Z}_2 &= \delta\, Z_2 \times \mathbf{e_3} \\ \dot{Z}_3 &= -2\beta\, Z_2 + \tfrac{1}{\|Z_3\|^2}(2\beta\, Z_3 \cdot Z_2 + \delta\, Z_3 \cdot (Z_2 \times \mathbf{e_3}))Z_3 - \delta\, Z_2 \times \mathbf{e_3} \\ \dot{S} &= \beta\, Z_2\,. \end{aligned} \tag{6}$$

All the allowable movements of the snake model are paths in $\left(\mathbb{R}^2\right)^4$ with components of the tangent vector $\frac{\mathrm{d}}{\mathrm{d}t}\left(Z_1(t), Z_2(t), Z_3(t), S(t)\right)$ in each point equal to (6) for some $\beta, \delta \in \mathbb{R}$.

By setting $\beta = 0, \delta = 1$ and $\beta = 1, \delta = 0$ we obtain the local basis of the distribution $\mathcal{D}$ described by (6). Considering a dot product with ∂_{Z_i} just as formal product, we write the local basis

$$\begin{aligned} X_1 &= \Big(\tfrac{1}{\|Z_1\|^2} Z_1 \cdot (Z_2 \times \mathbf{e_3}) Z_1 - Z_2 \times \mathbf{e_3}\Big) \cdot \partial_{Z_1} + (Z_2 \times \mathbf{e_3}) \cdot \partial_{Z_2} \\ &\quad + \Big(\tfrac{1}{\|Z_3\|^2} Z_3 \cdot (Z_2 \times \mathbf{e_3}) Z_3 - Z_2 \times \mathbf{e_3}\Big) \cdot \partial_{Z_3} \\ X_2 &= \Big(2Z_2 - \tfrac{2}{\|Z_1\|^2}(Z_1 \cdot Z_2) Z_1\Big) \cdot \partial_{Z_1} + \Big(-2Z_2 + \tfrac{2}{\|Z_3\|^2}(Z_3 \cdot Z_2) Z_3\Big) \cdot \partial_{Z_3} \\ &\quad + Z_2 \cdot \partial_S\,, \end{aligned} \tag{7}$$

where $Z_i = (z_{i1}, z_{i2}, 0)$, $\partial_S = (\partial_x, \partial_y, 0)$, $\partial_{Z_i} = (\partial_{z_{i1}}, \partial_{z_{i2}}, 0)$ and $\mathbf{e_3} = (0, 0, 1)$ where

$$(z_{11}, z_{12}, z_{21}, \dots, z_{32}, x, y)$$

are coordinates on $\left(\mathbb{R}^2\right)^4$.

Now, we choose the submanifold corresponding to the length (4) of all three segments equal one ($\|Z_1\| = \|Z_2\| = \|Z_3\| = 1$) and choose coordinates (v_1, v_2, v_3, x, y) such that

$$\begin{aligned} z_{21} &= \cos v_2, & z_{22} &= \sin v_2 \\ z_{11} &= \cos(v_2 + v_1), & z_{12} &= \sin(v_2 + v_1) \\ z_{31} &= \cos(v_3 - v_1), & z_{22} &= \sin(v_3 - v_1) \end{aligned} \tag{8}$$

as we can see on the Figure 3.

In these new coordinates $(v_1, v_2, v_3, x, y) \in \mathcal{M} = S^1 \times S^1 \times S^1 \times \mathbb{R}^2$, the vector fields (7) are of the form

$$\begin{aligned} X_1 &= (1 + \cos(v_1))\, \partial_{v_1} - \partial_{v_2} + (1 + \cos(v_3))\, \partial_{v_3}\,, \\ X_2 &= -2\sin(v_1)\, \partial_{v_1} + 2\sin(v_3)\, \partial_{v_3} + \cos(v_2)\, \partial_x + \sin(v_2)\, \partial_y\,. \end{aligned} \tag{9}$$

The mechanical interpretation of the two vector fields is the following: X_1 represents rotation and X_2 forward shift of the central segment together with movements forced by the constraints, of course.

By a suitable change of coordinates, one can see that X_1, X_2 generates the same subspaces of $T_x\mathcal{M}$ in each regular $x \in \mathcal{M}$ as the classical well-known vector fields that describe motion by changing angles in the individual joints.

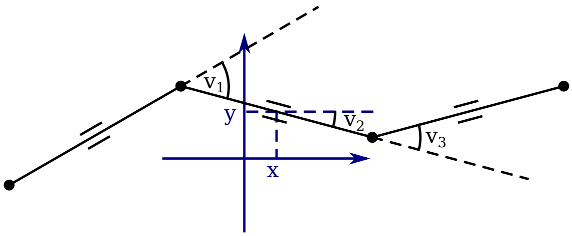

FIG. 3: The coordinates (v_1, v_2, v_3, x, y).

3. The extended model

The distribution $\mathcal{D}$ generated in each point of $\mathcal{M}$ by (9) is bracket-generating, because by putting

$$X_3 := [X_1, X_2], \quad X_4 := [X_1, X_3], \quad X_5 := [X_2, X_3]$$

the tangent space $T\mathcal{M}$ is generated by them in each *regular* point, i.e. for all regular $q \in \mathcal{M}$

$$T_q\mathcal{M} = \langle X_{1q}, X_{2q}, X_{3q}, X_{4q}, X_{5q} \rangle$$

and the growth vector of $\mathcal{D}$ is $(2, 3, 5)$. Points that are not regular are those points with components $v_1 = -v_3$ (straight snake or snake on a circle) and the growth vector is $(2, 3, 4, 5)$ in them.

As one may expect, $[X_1, X_4]$ is linearly independent *over real numbers* on the vector fields $X_1, \ldots, X_5$. So, let us define

$$X_6 := [X_1, X_4] .$$

Now, we can easily verify that all $[X_1, X_i]$, $[X_2, X_i]$ for $i = 1, \ldots 6$ are linearly dependent *over real number* on the vector fields $X_1, \ldots, X_5, X_6$ and hence these vector fields form a Lie algebra with multiplication table 1. Thus the growth vector for generators X_1, X_2 is $(2, 3, 5, 6)$.

$[\cdot,\cdot]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	X_3	X_4	X_6	0	$-X_4$
X_2	$-X_3$	0	X_5	0	$4X_3 + 4X_6$	0
X_3	$-X_4$	$-X_5$	0	0	0	0
X_4	$-X_6$	0	0	0	0	0
X_5	0	$-4X_3 - 4X_6$	0	0	0	0
X_6	X_4	0	0	0	0	0

TAB. 1: Multiplication table of vector fields $X_1, \ldots X_6$

As we mentioned $X_1, \ldots, X_6$ form a Lie algebra. If they were also local span of a tangent space, they would generate a Lie group structure in a neighbourhood of a point. We would like to gain the most similar model to $\mathcal{M}$ with Lie group structure. To achieve that, we need to increase the dimension of $\mathcal{M}$ by one and slightly modify the generators without changing the multiplication table above. In order to make $X_1, \dots, X_6$ locally spanning $T\mathcal{M}$, it is enough to modify X_2 in a way that does not affect $X_3, \dots, X_6$, because X_6 can be uniquely written as

$$X_6 = A \cdot X_2 - X_3 - A \cdot X_4 - B \cdot X_5$$

for two functions $A, B \in C^\infty(\mathcal{M})$.

Let us define $\mathcal{M}' = \mathcal{M} \times \mathbb{R}$ with coordinates (v_1, v_2, v_3, x, y, t) and the distribution $\mathcal{D}'$ locally span by

$$\begin{aligned} X_1' &= (1 + \cos(v_1))\, \partial_{v_1} - \partial_{v_2} + (1 + \cos(v_3))\, \partial_{v_3}, \\ X_2' &= -2\sin(v_1)\, \partial_{v_1} + 2\sin(v_3)\, \partial_{v_3} + \cos(v_2)\, \partial_x + \sin(v_2)\, \partial_y + \partial_t. \end{aligned}$$

Now, the distribution $\mathcal{D}'$ is bracket-generating with the growth vector $(2, 3, 5, 6)$. Let us denote $X_3', \dots, X_6'$ similarly as X_is.

Because each point in a neighbourhood of a point $q \in \mathcal{M}$ can be uniquely described as $\mathrm{Fl}_a^{X'}(q)$ where $X' = c_1 X_1' + \dots + c_6 X_6'$ for $c_1, \dots, c_6 \in \mathbb{R}$, $a \in I \subseteq \mathbb{R}$, there is a multiplication defined on the neighbourhood by Baker-Campbell-Hausdorff formula and hence a Lie group structure is locally defined there, call it G. The vectors $X_1'(q), \dots, X_6'(q)$ form a basis of the Lie algebra $\mathfrak{g}$ of the Lie group G, vector fields $X_1', \dots, X_6'$ are left-invariant vector fields of G and so is the distribution $\mathcal{D}'$.

4. Infinitesimal symmetries

Right-invariant vector fields form a Lie algebra isomorphic to the Lie algebra formed by left-invariant ones. Moreover, right-invariant vector fields are exactly those that commute with all the left-invariant ones [5, p. 50]. Thanks to that, we can find six linearly independent vector fields that commute with the distribution $\mathcal{D}'$ and hence are infinitesimal symmetries by (2)

$$\partial_x,\ \partial_y,\ \partial_{v_2} - y\, \partial_x + x\, \partial_y, (1 + \cos v_3) e^{2v_2 + 2t} \partial_{v_3}, (1 + \cos v_1) e^{-2v_2 - 2t} \partial_{v_1}, \partial_t\,. \tag{10}$$

It is known [1], that models with growth vector $(2, 3, 5, 6)$ have the dimension of symmetries at most 8, we will show that our model has infinitesimal symmetries only in span of (10). Due to the Jacobi identity, vector fields act on $\mathfrak{g}$ by Lie bracket in the point q as derivations of $\mathfrak{g}$. Infinitesimal symmetries therefore act as derivations preserving the span of the generators X_1', X_2'. One can show, that the only (inner and outer) derivation preserving the subspace spanned by $X_1'(q), X_2'(q) \in \mathfrak{g}$ is the trivial one, hence all infinitesimal symmetries commute with $\mathcal{D}'$ in q.

We will show that they commute everywhere in the neighbourhood of q where G is defined. Consider an infinitesimal symmetry Y such that $[Y, X_1'](q') \neq 0$ or $[Y, X_2'](q') \neq 0$ in some point q' inside the neighbourhood of q. Then we can translate it to q by left multiplication. Because a left multiplication by an element of a Lie group is a diffeomorphism and X_1', X_2' are left-invariant, we obtain an infinitesimal symmetry that does not commute in q which contradicts the previous.

The automorphism of $\mathcal{M}'$ generated by ∂_t preserves $\mathcal{D}'$, because it is an infinitesimal symmetry (10), therefore the t-forgetting projection $p\colon \mathcal{M}' \to \mathcal{M}$ map $\mathcal{D}'$

to $\mathcal{D}$. Moreover, each infinitesimal symmetry of $\mathcal{D}'$ constant with respect to t is projected to an infinitesimal symmetry of $\mathcal{D}$. On the other hand, it is not clear whether any infinitesimal symmetry of $\mathcal{D}$, other than commuting with $\mathcal{D}$, can be lifted to an infinitesimal symmetry of $\mathcal{D}'$. Clearly, if we take a vector field on $\mathcal{M}'$ with the same expression as a vector field on $\mathcal{M}$ commuting with $\mathcal{D}$ it will commute with $\mathcal{D}'$.

One can see that the only t-constant vector fields in the span of (10) are those in the span of the first three. Hence they are also infinitesimal symmetries of the original model $(\mathcal{D}, \mathcal{M})$

$$\partial_x,\ \partial_y,\ \partial_{v_2} - y\,\partial_x + x\,\partial_y\,. \tag{11}$$

On the other hand, we cannot easily deduce that there are no more.

5. Modified models

In order to find a $(2,3,5)$-model with more symmetries, one can try to modify the original model by choosing different lengths (4)

$$\|P_2 - P_1\| = \ell_1\,,\ \|P_3 - P_2\| = \ell_2\,,\ \|P_4 - P_3\| = \ell_3 \tag{12}$$

or shifting the wheels, i.e. modifying (3) to

$$\begin{aligned} a\dot{P}_1 + (1-a)\dot{P}_2 &= \alpha(P_2 - P_1)\\ b\dot{P}_2 + (1-b)\dot{P}_3 &= \beta(P_3 - P_2)\\ c\dot{P}_3 + (1-c)\dot{P}_4 &= \gamma(P_4 - P_3)\,. \end{aligned} \tag{13}$$

Because it does not matter how long the two outer segments are, but how far the outer wheels are from the joints, each generalisation above can be written by putting $a = 1, c = 0$ and choosing parameters $\ell_1, \ell_2, \ell_3, b \in \mathbb{R}$:

$$\begin{aligned} \dot{P}_1 &= \alpha(P_2 - P_1)\\ b\dot{P}_2 + (1-b)\dot{P}_3 &= \beta(P_3 - P_2)\\ \dot{P}_4 &= \gamma(P_4 - P_3) \end{aligned} \tag{14}$$

$$\|P_2 - P_1\| = \ell_1, \|P_3 - P_2\| = \ell_2, \|P_4 - P_3\| = \ell_3\,. \tag{15}$$

In [6], the above generalisation is attributed to Nurowski together with a question of whether there is such a choice of $(\ell_1, \ell_2, \ell_3, b)$ which has the maximal symmetry algebra $\mathfrak{g}_2$. We are not able to answer the question here, but we use the similar approach as in (5), (6), (7) to gain two generators

$$\begin{aligned} \tilde{X_1} &= \Big(1 + \frac{\ell_2}{\ell_1}\cdot(1-b)\cdot\cos(v_1)\Big)\,\partial_{v_1} - \partial_{v_2} + \Big(1 + \frac{\ell_2}{\ell_3}\cdot b\cdot\cos(v_3)\Big)\,\partial_{v_3}\,,\\ \tilde{X_2} &= -\frac{\ell_2}{\ell_1}\cdot\sin(v_1)\,\partial_{v_1} + \frac{\ell_2}{\ell_3}\cdot\sin(v_3)\,\partial_{v_3} + \ell_2\cdot\cos(v_2)\,\partial_x + \ell_2\cdot\sin(v_2)\,\partial_y\,. \end{aligned} \tag{16}$$

For a general choice of $(\ell_1, \ell_2, \ell_3, b)$, they generate 9-dimensional Lie algebra over reals where the growth vector for generators $\tilde{X_1}, \tilde{X_2}$ is $(2,3,5,7,9)$. For the choice $b = \frac{1}{2}$ and ℓ_1, ℓ_2, ℓ_3 general, the growth vector for generators $\tilde{X_1}, \tilde{X_2}$ is $(2,3,5,6,8,9)$. For $\ell_1 = \ell_2(1-b)$ and ℓ_2, ℓ_3, b general, the Lie algebra is 8-dimensional and growth vector for generators is $(2,3,5,7,8)$ (if we also put $b = \frac{1}{2}$, the Lie algebra is again 9-dimensional with $(2,3,5,6,8,9)$). For the generators (16), it is just a computational problem to find all possible growth vectors for the generators for different choices of parameters $\ell_1, \ell_2, \ell_3, b$.

Notice, the growth vector of the distribution is always $(2,3,5)$, while the discussion in the previous paragraph depends directly on the chosen generators.

There are several problems in imitating the approach of the section 3. The main problem is to find an extension which enjoys the same symmetries as the original model. In order to get an extended model with local Lie group structure, we would need to add four new coordinates now and add new terms to $\tilde{X}_1, \tilde{X}_2$ in order to make $\tilde{X}_6, \dots, \tilde{X}_9$ linearly independent over functions. The problem is that we need to preserve the linear dependence over reals of $[\tilde{X}_i, \tilde{X}_j]$ for $i = 1,2$ and $j = 8,9$, which is almost impossible if they depend on $\tilde{X}_1, \dots, \tilde{X}_5$. As we can see in Table 1, in our original case the only modified vector field X_2 was not inside the table. Thanks to this, it works.

6. Conclusions

We have shown that the particular basis vector fields of the distribution $\mathcal{D}$ form the finite dimensional Lie algebra that allows us to extend the studied model to the Lie group.

One possible advantage of the existence of the extended model can be easier control of the robot thanks to the bigger algebra of symmetries. Clearly all tangent paths to $\mathcal{D}'$ are projected to tangent paths to $\mathcal{D}$.

It is possible that similar Lie algebra structure can be found in many other models. Unfortunately, it is unclear how to generally find such suitable basis of a distribution. It is easy for some very simple models, where one can even find an infinite number of such finite-dimensional Lie algebras.

There are still many unanswered questions. For example, what are all possible such Lie algebras for a model? Does the existence of a finite-dimensional Lie algebra itself suffice to estimate the number of symmetries for such model? Can it help us to find a more symmetric model among the modifications of the snake model outlined in Section 5? It is place for my further research.

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References

[1] Anderson, I., Kruglikov, B., *Rank 2 distributions of monge equations: symmetries, equivalences, extensions*, Adv. Math. **228** (3) (2011), 1435–1465.

[2] Cartan, É., *Les systèmes de pfaff, à cinq variables et les équations aux dérivées partielles du second ordre*, Ann. Sci. Éc. Norm. Supér. (4) **27** (1910), 109–192.

[3] Hrdina, J., Návrat, A., Vašík, P., *Control of 3-link robotic snake based on conformal geometric algebra*, Adv. Appl. Clifford Algebr. **26** (2016), 1069–1080.

[4] Montgomery, R., *A tour of subriemannian geometries, their geodesics and applications*, Amer. Math. Soc., 2002.

[5] Olver, P.J., *Equivalence, Invariants and Symmetry*, London Mathematical Society Lecture Note, Cambridge University Press, 1995.

[6] The, D., *Exceptionally simple PDE [Presentation]*, Pure Math. Colloquium, University of Waterloo, Canada, 2018, January 5, 2018, available online (as of 2024-04-05): `https://math.uit.no/ansatte/dennis/talks/ExcSimpPDE-Waterloo2018.pdf`.

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