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QUANTISED $\mathfrak{sl}_2$-DIFFERENTIAL ALGEBRAS

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Abstract. We propose a definition of a quantised $\mathfrak{sl}_2$-differential algebra and show that the quantised exterior algebra (defined by Berenstein and Zwicknagl) and the quantised Clifford algebra (defined by the authors) of $\mathfrak{sl}_2$ are natural examples of such algebras.

1. Introduction

Let $\mathfrak{g}$ be a Lie algebra. H. Cartan introduced the notion of $\mathfrak{g}$-differential algebras as a generalisation of algebras of differential forms on manifolds with $\mathfrak{g}$-action, [8, 9]. Later $\mathfrak{g}$-differential algebras appeared in the study of equivariant cohomology [2, 14], in Chern–Weil theory [3, 24], and in relation to (algebraic) Dirac operators and Vogan's conjecture [2, 16, 17].

We develop a new approach to quantisation of the notion of $\mathfrak{g}$-differential algebras which assumes that the quantum exterior algebra of $\mathfrak{g}$ has classical dimension. This is different from previous attempts to generalise the notion of $\mathfrak{g}$-differential algebras to the setting of quantum groups and noncommutative geometry; for example, see [4, 5, 25]. Some of these works start with a bicovariant calculus on a quantum group, which usually does not have classical dimension. This is in particular the case for $U_q(\mathfrak{sl}_2)$, see [26], see also [18] for the general case. Other works assume the setting of triangular Hopf algebras which is not applicable to the setting of $U_q(\mathfrak{sl}_2)$ since it is only quasitriangular, see [12].

In this paper we propose a definition of quantised $\mathfrak{sl}_2$-differential algebras based on the 3-dimensional quantised adjoint $U_q(\mathfrak{sl}_2)$-module and give first examples, certain quantised Clifford and exterior algebras. The advantage of our approach is that we start with the quantum exterior algebra defined by Berenstein and Zwicknagl [6] of the classical dimension instead of a bicovariant calculus. We use the coboundary structure on the category of $U_q(\mathfrak{sl}_2)$-modules, see [11]. (As it was shown in [15] such coboundary structure is related to the category of crystals.)

The paper is organised as follows. In §2 we recall necessary facts about the Drinfeld–Jimbo quantum group $U_q(\mathfrak{sl}_2)$, the quantised adjoint representation and its quantum exterior algebra. In §3 we recall the definition of the q-deformed

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Clifford algebra of $\mathfrak{sl}_2$ introduced in [21] and define Lie derivatives, contraction operators and the differential on it. We show that the defined operations enjoy many features of their classical counterparts, in particular, Cartan's magic formula holds for them. In §4 we propose a definition of a quantised $\mathfrak{sl}_2$-differential algebra and show that the quantised exterior and Clifford algebras of $\mathfrak{sl}_2$ are examples of such algebras.

2. Preliminaries

2.1. **$\mathfrak{g}$-differential algebras.** Let $\mathfrak{g}$ be a complex Lie algebra. Let $\bigwedge[\xi]$ be the Grassmann algebra with one generator ξ, and let $\mathrm{d} := \partial_\xi \in \operatorname{Der}\bigwedge[\xi]$ be the derivation with respect to ξ. Set $\widehat{\mathfrak{g}} := \mathfrak{g} \otimes \bigwedge[\xi] \Subset \mathbb{C}\mathrm{d}$. Then $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_{-1} \oplus \widehat{\mathfrak{g}}_0 \oplus \widehat{\mathfrak{g}}_1$ is a $\mathbb{Z}$-graded Lie superalgebra where

$$\widehat{\mathfrak{g}}_{-1} = \mathfrak{g} \otimes \xi\,, \qquad \widehat{\mathfrak{g}}_0 = \mathfrak{g} \otimes 1\,, \qquad \widehat{\mathfrak{g}}_1 = \mathbb{C}\mathrm{d}\,.$$

For $x \in \mathfrak{g}$, let $L_x := x \otimes 1 \in \widehat{\mathfrak{g}}_0$, $\iota_x := x \otimes \xi \in \widehat{\mathfrak{g}}_{-1}$. The non-zero bracket relations in $\widehat{\mathfrak{g}}$ are defined as

$$(1) \quad [L_x, \iota_y] = \iota_{[x,y]}\,, \qquad [L_x, L_y] = L_{[x,y]}\,, \qquad [\iota_x, \mathrm{d}] = L_x \qquad \text{for all } x, y \in \mathfrak{g}.$$

2.1.1. **Digression: semisimple Lie superalgebras.** Assume that $\mathfrak{g}$ is simple. Let $\bigwedge(n)$ denote the Grassmann algebra with n generators $\xi_1, \ldots, \xi_n$. Then $\bigwedge(n)$ has a natural $\mathbb{Z}$-grading given by $\deg \xi_i = 1$. Let $\mathfrak{vect}(0|n) := \operatorname{Der}\bigwedge(n)$. Clearly, $\mathfrak{vect}(0|n)$ is a $\mathbb{Z}$-graded Lie superalgebra where $\deg \partial_{\xi_i} = -1$. Let $\mathfrak{vect}(0|n)_{-1}$ denotes the homogeneous component of degree -1. As it was shown in [10], any semisimple Lie superalgebra is the direct sum of the following summands

$$\tilde{\mathfrak{s}} \otimes \textstyle\bigwedge(n) \Subset \mathfrak{v},$$

where $\mathfrak{s}$ is a simple Lie superalgebra, $\mathfrak{s} \subseteq \tilde{\mathfrak{s}} \subseteq \operatorname{Der}\mathfrak{s}$, and $\mathfrak{v} \subset \mathfrak{vect}(0|n)$ is such that the projection $\mathfrak{v} \to \mathfrak{vect}(0|n)_{-1}$ is onto. In our case (for $\widehat{\mathfrak{g}}$) we have that $n = 1$, $\mathfrak{v} = \operatorname{Span}_{\mathbb{C}}(\partial_\xi)$, $\tilde{\mathfrak{s}} = \mathfrak{s} = \mathfrak{g}$.

2.2. **$\mathfrak{g}$-differential spaces and algebras.** A $\mathfrak{g}$-*differential space* is a superspace V, together with a $\widehat{\mathfrak{g}}$-module structure $\rho\colon \widehat{\mathfrak{g}} \to \operatorname{End}(V)$. A $\mathfrak{g}$-*differential algebra* is a superalgebra A, equipped with a structure of $\mathfrak{g}$-differential space such that $\rho(x) \in \operatorname{Der}A$ for all $x \in \widehat{\mathfrak{g}}$. Observe that if A is a $\mathfrak{g}$-differential algebra then the contraction operators ι define a $\mathfrak{g}$-equivariant representation of $U(\widehat{\mathfrak{g}}_{-1}) \cong \bigwedge\mathfrak{g}$ on A, where $U(\widehat{\mathfrak{g}}_{-1})$ is the universal enveloping algebra of the Lie superalgebra $\widehat{\mathfrak{g}}_{-1}$. The idea of a $\mathfrak{g}$-differential algebra is due to H. Cartan [8, 9]. We follow the terminology and notation from [24].

2.2.1. **Example.** Take $A = \bigwedge\mathfrak{g}^*$, equipped with the coadjoint action of $\mathfrak{g}$ denoted by L_x for $x \in \mathfrak{g}$. For $x \in \mathfrak{g}$ and $f \in \mathfrak{g}^* = \bigwedge^1\mathfrak{g}^*$ define the contraction operator by $\iota_x f = f(x)$. The odd map ι_x is extended to $\bigwedge\mathfrak{g}^*$ by the super Leibniz rule. Let e_a be a basis of $\mathfrak{g}$ and f_a be the corresponding dual basis in $\mathfrak{g}^*$. The Lie algebra differential on $\bigwedge\mathfrak{g}^*$ may be written as

$$\mathrm{d}_\wedge = \frac{1}{2}\sum_a f_a \circ L_{e_a}\,,$$

with f_a acting by the exterior multiplication. Then $\bigwedge\mathfrak{g}^*$ is a $\mathfrak{g}$-differential algebra. One can show that $H(\bigwedge\mathfrak{g}^*, \mathrm{d}_\wedge) \cong (\bigwedge\mathfrak{g}^*)^{\mathfrak{g}}$.

2.2.2. **Example.** Suppose that $\mathfrak{g}$ has a nondegenerate invariant symmetric bilinear form B (for example, see review in [7]), used to identify $\mathfrak{g} \cong \mathfrak{g}^*$. Let $\mathrm{Cl}(\mathfrak{g})$ be the Clifford algebra of $\mathfrak{g}$ with respect to B defined by

$$\mathrm{Cl}(\mathfrak{g}) = T(\mathfrak{g})/\left\langle x \otimes y + y \otimes x - 2B(x,y) \mid x, y \in \mathfrak{g} \right\rangle .$$

Let z_i be an orthonormal basis of $\mathfrak{g}$, then the Chevalley map (or quantisation) $q_{\mathrm{Cl}} \colon \bigwedge(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$ is defined by

$$z_{i_1} \wedge \ldots \wedge z_{i_k} \mapsto z_{i_1} \ldots z_{i_k} \qquad (\text{and } 1 \mapsto 1)\,,$$

where $1 \leq i_1 < \cdots < i_k \leq \dim \mathfrak{g}$. Set

$$\gamma = -\frac{1}{12} \sum_{a,b,c=1}^{\dim \mathfrak{g}} B([z_a, z_b], z_c) z_a \wedge z_b \wedge z_c \in (\textstyle\bigwedge^3 \mathfrak{g})^{\mathfrak{g}}\,.$$

Define the map $\alpha \colon \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ by

$$\alpha(x) = -\frac{1}{4} \sum_{a,b=1}^{\dim \mathfrak{g}} B(x, [z_a, z_b]) z_a z_b \quad \text{for } x \in \mathfrak{g}\,.$$

The map α extends to an algebra homomorphism $\alpha \colon U(\mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g})$.

The Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ is a filtered $\mathfrak{g}$-differential algebra with differential, Lie derivatives and contractions given as

$$\mathrm{d}_{\mathrm{Cl}} = [q_{\mathrm{Cl}}(\gamma), -]_{\mathrm{Cl}}\,, \qquad L_x = [\alpha(x), -]_{\mathrm{Cl}}\,, \qquad \iota_x = \frac{1}{2}[x, -]_{\mathrm{Cl}}, \quad \text{for } x \in \mathfrak{g}\,,$$

where $[-,-]_{\mathrm{Cl}}$ denotes the supercommutator in $\mathrm{Cl}(\mathfrak{g})$. The quantisation map $q_{\mathrm{Cl}} \colon \bigwedge\mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ intertwines the Lie derivatives and contractions, but does not intertwine the differential. The cohomology of $(\mathrm{Cl}(\mathfrak{g}), \mathrm{d}_{\mathrm{Cl}})$ is trivial in all filtration degrees (except if $\mathfrak{g}$ is abelian, in which case $\mathrm{d}_{\mathrm{Cl}} = 0$); for example, see [24, §7.1].

2.3. $\boldsymbol{U_q(\mathfrak{sl}_2)}$. Fix a nonzero $q \in \mathbb{C}$ which is not a root of unity. The quantised enveloping algebra $U_q(\mathfrak{sl}_2)$ is the associative algebra with unit generated by the elements E, F, K, and K^{-1} subject to the relations

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad KK^{-1} = K^{-1}K = 1\,, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}\,.$$

A Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ is given by

$$\Delta E = E \otimes K + 1 \otimes E\,, \quad \Delta F = F \otimes 1 + K^{-1} \otimes F\,, \quad \Delta K = K \otimes K\,,$$
$$\Delta K^{-1} = K^{-1} \otimes K^{-1}\,, \quad S(E) = -EK^{-1}\,, \quad S(F) = -KF\,, \quad S(K^{-1}) = K\,,$$
$$S(K) = K^{-1}\,, \quad \varepsilon(E) = \varepsilon(F) = 0\,, \quad \varepsilon(K) = \varepsilon(K^{-1}) = 1\,,$$

where Δ is the coproduct, S is the antipode, and ε is the counit. In what follows we use Sweedler notation for the coproduct $\Delta x = \sum x_{(1)} \otimes x_{(2)}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{sl}_2$, $\mathcal{P} \subset \mathfrak{h}^*$ be the weight lattice of $\mathfrak{sl}_2$, and $\mathcal{P}_+$ be the sublattice of dominant weights generated by the fundamental weight π.

The category of finite dimensional type 1 modules over $U_q(\mathfrak{sl}_2)$ is equivalent to the category of finite dimensional $\mathfrak{sl}_2$ modules; for example, see [13, §5.8] or [19, §3]. For $\lambda \in \mathcal{P}_+$ we denote the corresponding type 1 finite dimensional $U_q(\mathfrak{sl}_2)$-module with highest weight λ by V_λ.

Let $\mathfrak{sl}_q(2)$ denote the vector subspace of $U_q(\mathfrak{sl}_2)$ spanned by the elements

$$X = E\,, \qquad Z = q^{-2}EF - FE\,, \qquad Y = KF\,.$$

The space $\mathfrak{sl}_q(2)$ is closed with respect to the left adjoint action of $U_q(\mathfrak{sl}_2)$ on itself defined by

$$\mathrm{ad}_x\, y = \sum x_{(1)} y S(x_{(2)}) \qquad \text{for } x, y \in U_q(\mathfrak{sl}_2).$$

It is easy to see that as a $U_q(\mathfrak{sl}_2)$-module, $\mathfrak{sl}_q(2)$ is isomorphic to the quantised adjoint representation $V_{2\pi}$ of $\mathfrak{sl}_2$. In what follows we will use notation $\mathfrak{sl}_q(2)$ to emphasise that elements X, Z, and Y belong to $\mathfrak{sl}_q(2) \subset U_q(\mathfrak{sl}_2)$. In the case when $V_{2\pi}$ is treated as an abstract $U_q(\mathfrak{sl}_2)$-module and in the case when we will construct quantum exterior and Clifford algebras, we will use the following notation for basis elements in $V_{2\pi}$:

$$v_2 = X\,, \qquad v_0 = Z\,, \qquad v_{-2} = Y\,.$$

2.4. **Normalised braiding.** The following construction is due to Drinfeld [11]. Let C be a braided monoidal category linear over $\mathbb{C}[[\hbar]]$ and assume that the braiding satisfies $\sigma_{W,V} \circ \sigma_{V,W} = \mathrm{id}_{V\otimes W} + O(\hbar)$. Then the map

$$\tilde{\sigma}_{V,W} = \sigma_{V,W} \circ (\sigma_{W,V} \circ \sigma_{V,W})^{-1/2}\,,$$

is called a normalised braiding and defines a coboundary structure on C in the sense of [11]. For details see [13, Exercise 8.3.25 on p. 202]. In particular, we have that $\tilde{\sigma}^2 = \mathrm{id}$.

The category of type one finite-dimensional $U_q(\mathfrak{sl}_2)$-module is a braided monoidal category where the braiding σ is given by the universal R-matrix; see [13, §8.3] for details. The R-matrix braiding σ satisfies the above condition. In what follows we denote by $\tilde{\sigma}$ the corresponding normalised braiding.

2.5. **Quantum exterior algebras.** Following [6] define the quantum exterior algebra ${\textstyle\bigwedge_q} V_{2\pi}$ of $V_{2\pi}$ as

$${\textstyle\bigwedge_q} V_{2\pi} = T(V_{2\pi}) / \langle v \otimes w + \tilde{\sigma}(v \otimes w) \mid v, w \in V_{2\pi} \rangle$$

The algebra ${\textstyle\bigwedge_q} V_{2\pi}$ is generated by v_2, v_0, v_{-2} subject to the following relations

$$\begin{aligned} v_2 \wedge v_2 &= 0\,, & v_{-2} \wedge v_{-2} &= 0\,,\\ v_0 \wedge v_2 &= -q^{-2} v_2 \wedge v_0\,, & v_{-2} \wedge v_0 &= -q^{-2} v_0 \wedge v_{-2}\,,\\ v_0 \wedge v_0 &= \frac{(1-q^4)}{q^3} v_2 \wedge v_{-2}\,, & v_{-2} \wedge v_2 &= -v_2 \wedge v_{-2}\,. \end{aligned}$$

We note that ${\textstyle\bigwedge_q} V_{2\pi}$ is a $\mathbb{Z}$-graded super algebra in the braided monoidal category of type 1 finite-dimensional $U_q(\mathfrak{sl}_2)$-modules. The $\mathbb{Z}_2$-grading corresponding to a super algebra structure is given by setting $p(v_2) = p(v_0) = p(v_{-2}) = \bar{1}$, where $p(v) \in \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ denotes the parity of the element v. The algebra ${\textstyle\bigwedge_q} V_{2\pi}$ is (super)commutative with respect to normalised braiding, i.e. $v \wedge w = (-1)^{p(v)p(w)} \wedge \circ \tilde{\sigma}(v \otimes w)$ for all parity homogeneous $v, w \in \bigwedge_q V_{2\pi}$.

3. Motivating example: $\mathrm{Cl}_q(\mathfrak{sl}_2)$

Fix a non-zero constant $c \in \mathbb{C}[q, q^{-1}]$. First recall from [21, §2.7] that $V_{2\pi}$ admits a nondegenerate $U_q(\mathfrak{sl}_2)$-invariant bilinear form given by

$$\langle v_2, v_{-2} \rangle = c\,, \qquad \langle v_0, v_0 \rangle = q^{-3}(1+q^2)c\,, \qquad \langle v_{-2}, v_2 \rangle = cq^{-2}\,.$$

Note that the form $\langle \cdot, \cdot \rangle$ is symmetric with respect to the normalised braiding $\tilde{\sigma}$, i.e., $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle \circ \tilde{\sigma}$. The q-deformed Clifford algebra of $\mathfrak{sl}_2$ was defined in [21, §3] as filtered deformation of $\bigwedge_q V_{2\pi}$ by the bilinear form $\langle \cdot, \cdot \rangle$:

$$\mathrm{Cl}_q(\mathfrak{sl}_2) = T(V_{2\pi}) / \langle v \otimes w + \tilde{\sigma}(v \otimes w) - 2\langle v, w \rangle \mid v, w \in V_{2\pi} \rangle\,,$$

As it was shown in [21, Lemma 3.3], the algebra $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is generated by v_2, v_0, v_{-2} satisfying the following relations

$$\begin{aligned} v_2 v_2 &= 0\,, & v_{-2} v_{-2} &= 0\,,\\ v_0 v_2 &= -\,q^{-2} v_2 v_0\,, & v_{-2} v_0 &= -\,q^{-2} v_0 v_{-2}\,,\\ v_0 v_0 &= \frac{1-q^4}{q^3} v_2 v_{-2} + \frac{q^2+1}{q} c1\,, & v_{-2} v_2 &= -\,v_2 v_{-2} + \frac{q^2+1}{q^2} c1\,. \end{aligned}$$

It is easy to see that $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is a filtered super algebra in the (braided) monoidal category of $U_q(\mathfrak{sl}_2)$-modules. We note that $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is a filtered $U_q(\mathfrak{sl}_2)$-module where the elements of $U_q(\mathfrak{sl}_2)$ act by operators of degree 0.

The quantum exterior algebra $\bigwedge_q V_{2\pi}$ and the q-deformed Clifford algebra $\mathrm{Cl}_q(\mathfrak{sl}_2)$ can be considered as results of the deformation quantisation of the classical exterior algebra $\bigwedge \mathfrak{sl}_2$. The corresponding Poisson structures on $\bigwedge \mathfrak{sl}_2$ are studied in [1].

3.1. **$\tilde{\sigma}$-commutators.** For $x, y \in \mathrm{Cl}_q(\mathfrak{sl}_2)$ homogeneous with respect to parity set

$$[x, y]_{\tilde{\sigma}} := \left(m_{\mathrm{Cl}_q} - (-1)^{p(x)p(y)} m_{\mathrm{Cl}_q} \circ \tilde{\sigma} \right) (x \otimes y)\,,$$

where m_{Cl_q} denotes the multiplication map in $\mathrm{Cl}_q(\mathfrak{sl}_2)$. The map $[-,-]_{\tilde{\sigma}}$ is extended to $\mathrm{Cl}_q(\mathfrak{sl}_2)$ by linearity. By construction $[-,-]_{\tilde{\sigma}}$ is $U_q(\mathfrak{sl}_2)$-equivariant since it is composed from equivariant maps.

3.1.1. **Lemma.** *The bracket $[-,-]_{\tilde{\sigma}}$ is $\tilde{\sigma}$-skew-symmetric:*

$$[\omega, \mu]_{\tilde{\sigma}} = -(-1)^{p(\omega)p(\mu)} [-,-]_{\tilde{\sigma}} \circ \tilde{\sigma}(\omega \otimes \mu) \qquad \textit{for } \omega, \mu \in \mathrm{Cl}_q(\mathfrak{sl}_2),$$

and has the filtration degree -1.

Proof. The $\tilde{\sigma}$-skew-symmetricity follows form the definition of $[-,-]_{\tilde{\sigma}}$. Since $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is a filtered deformation of the $\tilde{\sigma}$-supercommutative algebra $\bigwedge_q V_{2\pi}$, it follows from the definition that the bracket $[-,-]_{\tilde{\sigma}}$ has the filtration degree -1. $\square$

3.2. **The α and β maps and Lie derivatives.** The quantum moment map (in the sense of [23]) is the algebra map $\alpha_q\colon U_q(\mathfrak{sl}_2)\to \mathrm{Cl}_q(\mathfrak{sl}_2)$ defined in [21, §3.7]. It is given on generators by

$$\alpha_q(E) = -\frac{q}{(1+q^2)c}v_2v_0\,, \qquad \alpha_q(F) = -\frac{q^2}{(1+q^2)c}v_0v_{-2}\,,$$
$$\alpha_q(K) = \frac{q^3-q}{(1+q^2)c}v_2v_{-2}+q^{-1}\,, \qquad \alpha_q(K^{-1}) = -\frac{q^3-q}{(1+q^2)c}v_2v_{-2}+q\,.$$

Since α_q is an algebra map it follows that

$$\alpha_q(Z) = \frac{1}{c}v_2v_{-2}-1\,, \qquad \alpha_q(Y) = -\frac{q}{(1+q^2)c}v_0v_{-2}\,.$$

As it was shown in [21, Lemma 3.7.1] the inner $U_q(\mathfrak{sl}_2)$-action defined by α_q coincides with the natural one:

$$x\triangleright\omega = \sum \alpha_q(x_{(1)})\omega\alpha_q(S(x_{(2)})) \qquad \text{for } x\in U_q(\mathfrak{sl}_2),\ \omega\in \mathrm{Cl}_q(\mathfrak{sl}_2),$$

where $x\triangleright\omega$ denotes the $U_q(\mathfrak{sl}_2)$-action on $\mathrm{Cl}_q(\mathfrak{sl}_2)$.

Following the classical situation we define Lie derivatives on $\mathrm{Cl}_q(\mathfrak{sl}_2)$ with respect to elements of $U_q(\mathfrak{sl}_2)$ by

$$L_x\omega := x\triangleright\omega \qquad \text{for } x\in U_q(\mathfrak{sl}_2),\ \omega\in \mathrm{Cl}_q(\mathfrak{sl}_2).$$

Classically, the map α defined the (adjoint) action of $\mathfrak{g}$ by taking the (super)commutator; see Example 2.2.2. This is no longer true in the quantum case for $\tilde\sigma$-commutators. Define a linear map $\beta_q\colon \mathfrak{sl}_q(2)\to \mathrm{Cl}_q(\mathfrak{sl}_2)$ by

$$\beta_q(X) = \frac{1+q^2}{q}\alpha_q(X)\,, \qquad \beta_q(Y) = \frac{1+q^2}{q}\alpha_q(Y)\,, \qquad \beta_q(Z) = \frac{1+q^2}{q}\alpha_q(Z)\,.$$

The definition of β_q is motivated by the following lemma.

3.2.1. **Proposition.** *The β_q-map defines the quantum Hamiltonian with respect to the $\tilde\sigma$-commutator for the action of elements of $\mathfrak{sl}_q(2)\subset U_q(\mathfrak{sl}_2)$ on $\mathrm{Cl}_q(\mathfrak{sl}_2)$. Namely, we have that*

$$L_x\omega = [\beta_q(x),\omega]_{\tilde\sigma} \qquad \textit{for } x\in \mathfrak{sl}_q(2),\ \omega\in \mathrm{Cl}_q(\mathfrak{sl}_2).$$

Proof. For $X\in\mathfrak{sl}_q(2)$ and $v_2\in \mathrm{Cl}_q(\mathfrak{sl}_2)$, we have that

$$[\beta_q(X),v_2]_{\tilde\sigma} = -\frac{1}{2c}v_2v_0v_2+\frac{1}{2c}v_2v_2v_0 = \frac{1}{2q^2c}v_2v_2v_0 = 0 = X\triangleright v_2\,.$$

The computations for other elements of $\mathfrak{sl}_q(2)$ and $\mathrm{Cl}_q(\mathfrak{sl}_2)$ are analogous. □

3.2.2. **Remark.** One can check that β_q defines a morphism of "quantum brackets" in the following sense: $[\beta_q(x),\beta_q(y)]_{\tilde\sigma} = \beta_q(\mathrm{ad}_x\, y)$ for all $x,y\in\mathfrak{sl}_q(2)$. Moreover, it follows from Lemma 3.1.1 that $\mathrm{ad}\circ\tilde\sigma = -\,\mathrm{ad}$ on $\mathfrak{sl}_q(2)$.

3.3. **Differential.** Recall from [21, §3.4] that the element

$$\gamma_q = -\frac{1}{2c^2}(cv_0 + v_2 v_0 v_{-2}) \in \mathrm{Cl}_q^{(3)}(\mathfrak{sl}_2)$$

squares to a scalar. Therefore, we can define a differential on $\mathrm{Cl}_q(\mathfrak{sl}_2)$ by

$$\mathrm{d}_{\mathrm{Cl}_q}\omega_q = [\gamma_q, \omega]_{\tilde{\sigma}} = \gamma_q \omega - (-1)^{p(\omega)}\omega\gamma_q\,, \qquad \omega \in \mathrm{Cl}_q(\mathfrak{sl}_2)\,.$$

3.3.1. **Proposition.** *We have that*

(1) $\mathrm{d}_{\mathrm{Cl}_q} x = 2\beta_q(x)$ *for* $x \in \mathfrak{sl}_q(2)$*;*

(2) *the differential* $\mathrm{d}_{\mathrm{Cl}_q}$ *is* $U_q(\mathfrak{sl}_2)$*-equivariant.*

Proof. (1) We have that

$$\begin{aligned}
\mathrm{d}_{\mathrm{Cl}_q} v_2 &= -\frac{1}{2c^2}(v_2(cv_0 + v_2v_0v_{-2}) + (cv_0 + v_2v_0v_{-2})v_2) \\
&= -\frac{1}{2c^2}(cv_2v_0 - q^{-2}cv_2v_0 - v_2v_0v_2v_{-2} + \frac{q^2+1}{q^2}cv_2v_0) \\
&= -\frac{1}{2c^2}\frac{cq^2 - c + q^2c + c}{q^2}v_2v_2 = -\frac{1}{c}v_2v_0 = 2\beta_q(X)\,, \\
\mathrm{d}_{\mathrm{Cl}_q} v_0 &= -\frac{1}{2c^2}(v_0(cv_0 + v_2v_0v_{-2}) + (cv_0 + v_2v_0v_{-2})v_0) \\
&= -\frac{1}{2c^2}\left(\frac{2c(1-q^4)}{q^3}v_2v_{-2} + \frac{2c^2(q^2+1)}{q} - 2q^{-2}v_2v_0v_0v_{-2}\right) \\
&= -\frac{1}{2c^2}\left(\frac{2c(1-q^4)}{q^3}v_2v_{-2} + \frac{2c^2(q^2+1)}{q} - \frac{2c(q^2+1)}{q^3}v_2v_{-2}\right) \\
&= \frac{q^2+q}{2cq}(v_2v_{-2} - c) = 2\beta_q(Z)\,, \\
\mathrm{d}_{\mathrm{Cl}_q} v_0 &= -\frac{1}{2c^2}(v_{-2}(cv_0 + v_2v_0v_{-2}) + (cv_0 + v_2v_0v_{-2})v_{-2}) \\
&= -\frac{1}{2c^2}(-q^{-2}cv_0v_{-2} + q^{-2}v_2v_{-2}v_{-2}v_0 + \frac{q^2+1}{q^2}cv_0v_{-2} + cv_0v_{-2}) \\
&= -\frac{1}{c}v_0v_{-2} = 2\beta_q(Y)\,.
\end{aligned}$$

(2) The equivariance of $\mathrm{d}_{\mathrm{Cl}_q}$ follows from the equivariance of the bracket $[-,-]_{\tilde{\sigma}}$. □

3.3.2. **Remark.** We note that the differential is dual to the quantised adjoint action of $\mathfrak{sl}_q(2)$ on itself (a version of "quantum Lie bracket") in the following sense: $\iota_x\iota_y\mathrm{d}_{\mathrm{Cl}_q} z = \langle \mathrm{ad}_x\, y, z\rangle$ for $x, y, z \in \mathfrak{sl}_q(2)$.

3.4. **Contractions.** Following the classical case, see Example 2.2.2, define

$$\iota_x\omega = \frac{1}{2}[x,\omega]_{\tilde{\sigma}}\,, \qquad x \in V_{2\pi}, \omega \in \mathrm{Cl}_q(\mathfrak{sl}_2)\,.$$

This definition is motivated by the fact that for linear $v \in V_{2\pi} \subset \mathrm{Cl}_q^{(1)}(\mathfrak{sl}_2)$ we have that

$$\iota_x v = \frac{1}{2}[x,v]_{\tilde\sigma} = \frac{1}{2}(xv + m_{\mathrm{Cl}_q} \circ \tilde\sigma(x \otimes v)) = \frac{1}{2}(2\langle x,v\rangle) = \langle x,v\rangle\,,$$

where m_{Cl_q} is the multiplication map in $\mathrm{Cl}_q(\mathfrak{sl}_2)$. Furthermore, ι_x has the filtration degree -1.

3.4.1. **Proposition.** *For $x, y \in V_{2\pi}$ let $x_i, y_i \in V_{2\pi}$ be defined by $\tilde\sigma(x \otimes y) = \sum_i y_i \otimes x_i$, then for all $\omega \in \mathrm{Cl}_q(\mathfrak{sl}_2)$ the contraction operators satisfy*

$$\iota_x\iota_y\omega + \sum_i \iota_{y_i}\iota_{x_i}\omega = 0\,.$$

Hence the map $\iota\colon V_{2\pi} \to \mathrm{End}(\mathrm{Cl}_q(\mathfrak{sl}_2))$ extends to a $U_q(\mathfrak{sl}_2)$-equivariant morphism of superalgebras $\iota\colon \bigwedge_q V_{2\pi} \to \mathrm{End}(\mathrm{Cl}_q(\mathfrak{sl}_2))$.

Proof. For $v_2, v_0 \in V_{2\pi} = \bigwedge_q^1 V_{2\pi}$ we have that

$$v_2 \otimes v_0 + \tilde\sigma(v_2 \otimes v_0) = \frac{2}{1+q^4}\left(q^2 v_0 \otimes v_2 + v_2 \otimes v_0\right)\,.$$

For $v_2v_0v_{-2} \in \mathrm{Cl}_q(\mathfrak{sl}_2)$ we have that

$$\begin{aligned}\iota_{v_2}\iota_{v_0}(v_2v_0v_{-2}) &= \iota_{v_2}\left((q^2-1)(q^2+1)q^{-3}c^2 - (1+q^2)q^{-1}v_2v_{-2}\right)\\ &=(1+q^2)q^{-1}c^2v_2\,,\end{aligned}$$

and

$$\iota_{v_0}\iota_{v_2}(v_2v_0v_{-2}) = \iota_{v_0}(cv_2v_0) = -(1+q^2)q^{-3}c^2v_2 = -q^{-2}\iota_{v_2}\iota_{v_0}(v_2v_0v_{-2})\,.$$

The computations for other elements of $\bigwedge_q V_{2\pi}$ and $\mathrm{Cl}_q(\mathfrak{sl}_2)$ are analogous. The $U_q(\mathfrak{sl}_2)$-equivariance follows from the equivariance of the bracket $[-,-]_{\tilde\sigma}$. ☐

3.4.2. **Remark.** It is subject of further research to describe the derivation property for the action of $\bigwedge_q V_{2\pi}$ on $\mathrm{Cl}_q(\mathfrak{sl}_2)$. We show below that several obvious attempts to ensure this property do not work.

We consider the following possible approach: let us assume that for $x \in \mathfrak{sl}_q(2)$ and $\omega, \mu \in \mathrm{Cl}_q(\mathfrak{sl}_2)$,

$$\iota_x(\omega\mu) = \iota_x(\omega)\mu + (-1)^{p(\omega)}\sum_i \omega_i\iota_{x_i}(\mu)\,,$$

where ω_i and x_i are defined by the following options

(1) $\sigma(x \otimes \omega) = \sum \omega_i \otimes x_i$,
(2) $\sigma^{-1}(x \otimes \omega) = \sum \omega_i \otimes x_i$,
(3) $\tilde\sigma(x \otimes \omega) = \sum \omega_i \otimes x_i$,

In what follows we show that none of these options works in the example of $x = v_0$, $\omega = v_2$, and $\mu = v_{-2}$. First note that

$$\iota_{v_0}(v_2v_{-2}) = \frac{c(1-q^2)}{q^2}v_0\,, \qquad \iota_{v_0}(v_2)v_{-2} = 0\,.$$

Case 1. We have that

$$\sigma(v_0 \otimes v_2) = v_2 \otimes v_0 + q^{-2}(q^4 - 1)v_0 \otimes v_2.$$

Therefore,

$$\begin{aligned}\iota_x(\omega)\mu + (-1)^{p(\omega)} \sum_i \omega_i \iota_{x_i}(\mu) &= -v_2 \iota_{v_0}(v_{-2}) - \frac{q^4-1}{q^2} v_0 \iota_{v_2} v_{-2} \\ &= -\frac{c(q^4-1)}{q^2} v_0 \neq \iota_{v_0}(v_2 v_{-2}) .\end{aligned}$$

Case 2. We have that

$$\sigma^{-1}(v_0 \otimes v_2) = v_2 \otimes v_0.$$

Therefore,

$$\iota_x(\omega)\mu + (-1)^{p(\omega)} \sum_i \omega_i \iota_{x_i}(\mu) = -v_2 \iota_{v_0}(v_{-2}) = 0 \neq \iota_{v_0}(v_2 v_{-2}) .$$

Case 3. We have that

$$\tilde{\sigma}(v_0 \otimes v_2) = \frac{2q^2}{1+q^4} v_2 \otimes v_0 + \frac{q^4-1}{1+q^4} v_0 \otimes v_2 .$$

Therefore,

$$\begin{aligned}\iota_x(\omega)\mu + (-1)^{p(\omega)} \sum_i \omega_i \iota_{x_i}(\mu) &= -\frac{2q^2}{1+q^4} v_2 \iota_{v_0}(v_{-2}) - \frac{q^4-1}{1+q^4} v_0 \iota_{v_2} v_{-2} \\ &= -\frac{c(q^4-1)}{1+q^4} v_0 \neq \iota_{v_0}(v_2 v_{-2}) .\end{aligned}$$

Perhaps one could use approach similar to [4, §6] or consider a braided Hopf algebra structure on $\bigwedge_q V_{2\pi}$ similar, for example, to the one constructed [22].

In view of the difficulties explained in Remark 3.4.2, it is surprising that we have the following theorem.

3.5. **Theorem.** *For $x \in \mathfrak{sl}_q(2) = V_{2\pi}$ the operators L_x, ι_x, and d_{Cl} on $\mathrm{Cl}_q(\mathfrak{sl}_2)$ satisfy Cartan's magic formula*

$$L_x = \iota_x \circ \mathrm{d}_{\mathrm{Cl}_q} + \mathrm{d}_{\mathrm{Cl}_q} \circ \iota_x .$$

In particular, cochain maps L_x are homotopic to 0, with ι_x as homotopy operators. Therefore, L_x induces the zero action on cohomology.

Proof. Direct computations. For example, for $v_2 \in \mathfrak{sl}_q(2) = V_{2\pi}$ and $v_{-2} \in \mathrm{Cl}_q(\mathfrak{sl}_2)$ we have that

$$L_{v_2} v_{-2} = v_0$$

and

$$\iota_{v_2} \mathrm{d}_{\mathrm{Cl}_q} v_{-2} + \mathrm{d}_{\mathrm{Cl}_q} \iota_{v_2} v_{-2} = -\frac{1}{c} \iota_{v_2} v_0 v_{-2} + c\,\mathrm{d}_{\mathrm{Cl}_q}(1) = v_0 .$$

The computations for other cases are similar. □

3.6. **Remark.** First note that the element γ_q generates a $U_q(\mathfrak{sl}_2)$-invariant subalgebra in $\mathrm{Cl}_q(\mathfrak{sl}_2)$. Moreover, the element γ_q satisfies $\gamma_q^2 = \frac{1+q^2}{4cq}$. Therefore, by the universal property of Clifford algebras, we have that

$$\mathrm{Cl}_q(\mathfrak{sl}_2)^{U_q(\mathfrak{sl}_2)} = \mathrm{Cl}(P_q(\mathfrak{sl}_2), B_q),$$

where $P_q(\mathfrak{sl}_2)$ is the space of primitive invariants spanned by γ_q equipped with nondegenerate symmetric bilinear form B_q given by $B_q(\gamma_q, \gamma_q) = \frac{1+q^2}{4cq}$. It is now easy to see that $H(\mathrm{Cl}_q(\mathfrak{sl}_2), \mathrm{d}_{\mathrm{Cl}_q}) = 0$.

Since the element γ_q is $U_q(\mathfrak{sl}_2)$-invariant we have that $[\omega, \gamma_q]_{\tilde{\sigma}} = -(-1)^{p(\omega)}[\gamma_q, \omega]_{\tilde{\sigma}}$ for all parity homogeneous $\omega \in \mathrm{Cl}_q(\mathfrak{sl}_2)$. Therefore, for $x \in \mathfrak{sl}_q(2)$ we have that

$$\iota_x \gamma = [\tfrac{1}{2}x, \gamma_q]_{\tilde{\sigma}} = [\gamma_q, \tfrac{1}{2}x]_{\tilde{\sigma}} = \tfrac{1}{2}\mathrm{d}_{\mathrm{Cl}_q}(x) = \beta_q(x) \in \operatorname{im}\alpha_q\,.$$

Moreover, direct computations show that for $x \in \mathfrak{sl}_q(2)$ we have

$$\iota_x \gamma_q \cdot \gamma_q^* = x\,,$$

where $\gamma_q^* = \frac{4qc}{1+q^2}\gamma_q$ is the dual to γ_q with respect to B_q. This leads to the quantum analogue of the ρ-decomposition from [20]:

$$\mathrm{Cl}_q(\mathfrak{sl}_2) = \mathrm{Cl}(P_q(\mathfrak{sl}_2), B_q) \otimes \operatorname{im}\alpha_q\,.$$

We emphasise that in this case the braided tensor product of algebras in the braided monoidal category of type 1 finite-dimensional $U_q(\mathfrak{sl}_2)$-modules reduces to the usual tensor product of algebras since the elements of $\mathrm{Cl}(P_q(\mathfrak{sl}_2), B_q)$ are $U_q(\mathfrak{sl}_2)$-invariant.

4. The general definition

4.1. **Definition.** A supervector space W is called a quantised $\mathfrak{sl}_2$-differential space if it is equipped with

(1) Lie derivatives $L_x \in \mathrm{End}(W)$ for $x \in U_q(\mathfrak{sl}_2)$ which define a $U_q(\mathfrak{sl}_2)$-module structure on W;
(2) a $U_q(\mathfrak{sl}_2)$-equivariant action $\iota\colon \bigwedge_q V_{2\pi} \otimes W \to W$ of $\bigwedge_q V_{2\pi}$;
(3) a $U_q(\mathfrak{sl}_2)$-equivariant differential $\mathrm{d}_W\colon W \to W$;
(4) such that they satisfy Cartan's magic formula

$$L_x = \iota_x \circ \mathrm{d}_W + \mathrm{d}_W \circ \iota_x \qquad \text{for } x \in \mathfrak{sl}_q(2).$$

A morphism between two quantised $\mathfrak{sl}_2$-differential spaces is a morphism in the category of $U_q(\mathfrak{sl}_2)$-modules which intertwines contractions and differentials (and also Lie derivatives).

4.2. **Definition.** An algebra A is called a quantised $\mathfrak{sl}_2$-differential algebra if it is a quantised $\mathfrak{sl}_2$-differential space such that

(1) the Lie derivatives satisfy

$$L_x(ab) = \sum (L_{x_{(1)}}a)(L_{x_{(2)}}b) \qquad \text{for } a, b \in A,\ x \in U_q(\mathfrak{sl}_2),$$

in other words, A is an algebra in the monoidal category of $U_q(\mathfrak{sl}_2)$-modules;

(2) the differential d_A satisfies the (graded) Leibniz rule.

A morphism between two quantised $\mathfrak{sl}_2$-differential algebras is an algebra morphism in the category of $U_q(\mathfrak{sl}_2)$-modules which intertwines contractions and differentials (and also Lie derivatives).

It is subject of further research to describe the derivation property for the action of $\bigwedge_q V_{2\pi}$ on a quantised $\mathfrak{sl}_2$-differential algebra; see the discussion in Remark 3.4.2.

4.3. **Quantum exterior algebra.** In this subsection we show that the quantum exterior algebra of the 3-dimensional quantised adjoint $U_q(\mathfrak{sl}_2)$-module $V_{2\pi}$ is a quantised $\mathfrak{sl}_2$-differential algebra in the sense of our definition.

First note that the associated graded algebra of $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is the quantum exterior algebra $\bigwedge_q V_{2\pi}$. For $x \in \mathfrak{sl}_q(2)$, the associated graded maps $L_x \colon \bigwedge_q V_{2\pi} \to V_{2\pi}$ to the Lie derivatives $L_x \colon \mathrm{Cl}_q(\mathfrak{sl}_2) \to \mathrm{Cl}_q(\mathfrak{sl}_2)$ define an action of $U_q(\mathfrak{sl}_2)$ since the filtration on $\mathrm{Cl}_q(\mathfrak{sl}_2)$ is compatible with $U_q(\mathfrak{sl}_2)$-action.

The differential $\mathrm{d}_{\mathrm{Cl}_q}$ has filtered degree one. Therefore, we can define the associated graded map $\mathrm{d}_{\wedge_q} \colon \bigwedge_q V_{2\pi} \to \bigwedge_q V_{2\pi}$ which is $U_q(\mathfrak{sl}_2)$-equivariant by construction. It is easy to see, c.f. Example 2.2.1, that $\mathrm{d}_{\wedge_q}$ is nonzero only for

$$\mathrm{d}_{\wedge_q}(v_2) = -\frac{1}{c} v_2 \wedge v_0 \,, \quad \mathrm{d}_{\wedge_q}(v_0) = \frac{1+q^2}{qc} v_2 \wedge v_{-2} \,, \quad \mathrm{d}_{\wedge_q}(v_{-2}) = -\frac{1}{c} v_0 \wedge v_{-2} \,.$$

It is straightforward to check that $\mathrm{d}^2_{\wedge_q} = 0$ and that it satisfies the graded Leibniz rule, so it defines a differential on $\bigwedge_q V_{2\pi}$. Moreover, we have the quantised version of the formula for differential, see Example 2.2.1,

$$\mathrm{d}_{\wedge_q} = \frac{q^2}{1+q^4} \left(\frac{1}{c} v_{-2} L_X + \frac{q^3}{(1+q^2)c} v_0 L_Z + \frac{q^2}{q^2} v_2 L_Y \right) .$$

Note that the formulas for the differential depend on the parameter c since we identify $\bigwedge_q V_{2\pi}^*$ with $\bigwedge_q V_{2\pi}$ via the bilinear form $\langle \cdot, \cdot \rangle$.

Similarly, for $x \in \bigwedge_q V_{2\pi}$ we define the contraction operator $\iota_x \colon \bigwedge_q V_{2\pi} \to \bigwedge V_{2\pi}$ as the associated graded map for the contraction operator on $\mathrm{Cl}_q(\mathfrak{sl}_2)$. By construction, the operators ι_x define a $U_q(\mathfrak{sl}_2)$-equivariant representation of $\bigwedge_q V_{2\pi}$. In particular, we have that

$$\begin{aligned}
&\iota_{v_2} v_2 = 0 \,, && \iota_{v_0} v_2 = 0 \,, && \iota_{v_{-2}} v_2 = q^{-2} c \,, \\
&\iota_{v_2} v_0 = 0 \,, && \iota_{v_0} v_0 = q^{-3}(1+q^2) c \,, && \iota_{v_{-2}} v_0 = 0 \,, \\
&\iota_{v_2} v_{-2} = c \,, && \iota_{v_0} v_2 = 0 \,, && \iota_{v_{-2}} v_{-2} = 0 \,, \\
&\iota_{v_2} v_2 \wedge v_0 = 0 \,, && \iota_{v_0} v_2 \wedge v_0 = -\tfrac{1+q^2}{q^3} c v_2 \,, && \iota_{v_{-2}} v_2 \wedge v_0 = c v_0 \,, \\
&\iota_{v_2} v_2 \wedge v_{-2} = -c v_2 \,, && \iota_{v_0} v_2 \wedge v_{-2} = \tfrac{1-q^2}{q^2} c v_0 \,, && \iota_{v_{-2}} v_2 \wedge v_{-2} = q^{-2} c v_{-2} \,, \\
&\iota_{v_2} v_0 \wedge v_{-2} = -c v_0 \,, && \iota_{v_0} v_0 \wedge v_{-2} = \tfrac{1+q^2}{q} c v_{-2} \,, && \iota_{v_{-2}} v_0 \wedge v_{-2} = 0 \,,
\end{aligned}$$

$$\iota_{v_2} v_2 \wedge v_0 \wedge v_{-2} = c v_2 \wedge v_0 \,, \qquad \iota_{v_0} v_2 \wedge v_0 \wedge v_{-2} = -\tfrac{1+q^2}{q} c v_2 \wedge v_{-2} \,,$$
$$\iota_{v_{-2}} v_2 \wedge v_0 \wedge v_{-2} = c v_0 \wedge v_{-2} \,.$$

Cartan's magic formula for associated graded maps L_x, ι_x, and $\mathrm{d}_{\wedge_q}$ on $\bigwedge_q V_{2\pi}$ follows from the fact that Cartan's magic formula on $\mathrm{Cl}_q(\mathfrak{sl}_2)$ has degree 0. It can also be checked by direct computations as follows. First note that for elements of $\bigwedge^0 V_{2\pi}$ and $\bigwedge^3 V_{2\pi}$ Cartan's magic formula holds trivially. The operator $\mathrm{d}_{\wedge_q} \circ \iota_x$ acts by zero on $\bigwedge^1 V_{2\pi}$. We have that

$$\iota_{v_2} \mathrm{d}_{\wedge_q} v_{-2} = -\frac{1}{c} \iota_{v_2}(v_0 \wedge v_{-2}) = v_0 = L_{v_2} v_{-2}\,.$$

The operator $\iota_x \circ \mathrm{d}_{\wedge_q}$ acts by zero on $\bigwedge^2 V_{2\pi}$. We have that

$$\mathrm{d}_{\wedge_q} \iota_{v_2}(v_0 \wedge v_{-2}) = -c\mathrm{d}_{\wedge_q} v_0 = -\frac{1+q^2}{q} v_2 \wedge v_{-2} = L_{v_2}(v_0 \wedge v_{-2})\,.$$

Therefore, we have proved the following theorem.

4.4. **Theorem.** *The algebras* $\mathrm{Cl}_q(\mathfrak{sl}_2)$ *and* $\bigwedge_q V_{2\pi}$ *are quantised* $\mathfrak{sl}_2$*-differential algebras.*

4.5. **Remark.** Similarly to the case of $\mathrm{Cl}_q(\mathfrak{sl}_2)$, see Remark 3.6, we can now compute the cohomology of $\bigwedge_q V_{2\pi}$ using Cartan's magic formula. First note that the subalgebra of $U_q(\mathfrak{sl}_2)$-invariant elements in $\bigwedge_q V_{2\pi}$ is spanned by 1, $v_2 \wedge v_0 \wedge v_{-2}$. Therefore, we have the quantised analogue of Hopf–Koszul–Samelson theorem

$$H(\textstyle\bigwedge_q V_{2\pi}, \mathrm{d}_{\wedge_q}) = (\bigwedge_q V_{2\pi})^{U_q(\mathfrak{sl}_2)} = \bigwedge P_{\wedge_q}(\mathfrak{sl}_2)\,,$$

where $P_{\wedge_q}(\mathfrak{sl}_2)$ is the space of primitive invariants spanned by $v_2 \wedge v_0 \wedge v_{-2}$.

We note that

$$\iota_{v_2\wedge v_0 \wedge v_{-2}}(v_2 \wedge v_0 \wedge v_{-2}) = \iota_{v_2 \wedge v_0}(cv_0 \wedge v_{-2}) = c^2 \tfrac{1+q^2}{q^2} \iota_{v_2} v_{-2} = c^3 \frac{1+q^2}{q^2}\,.$$

Therefore, we can recover the form B_q from Remark 3.6 via contraction operator just as in the classical case of [20].

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