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ON THE PROLONGATION OF VERTICAL CONNECTIONS

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Abstract. With the help of a horizontal endomorphism we construct the Abate – Patrizio prolongation of a regular vertical connection, and show that the linear connection obtained coincides with the Grifone-prolongation.

1. Horizontal endomorphisms and horizontal mappings

1.1. Conventions.

(a) In what follows, we are going to work throughout on an n-dimensional ($n \geq 2$), paracompact smooth manifold M. By $C^\infty(M)$ and $\mathfrak{X}(M)$ we denote the ring of smooth functions on M and the $C^\infty(M)$-module of vector fields respectively. As to general conventions concerning manifolds, tensors, ... we are going to use the notations and conventions of the text [4].

(b) $\tau_M = (TM, \pi, M)$ denotes the tangent bundle of the manifold M. The canonical objects of τ_M, the *Liouville vector field* C and the *vertical endomorphism* J are supposed to be known. A detailed discussion of these can be found e.g. in the monograph [3].

(c) Without any special mention, we are going to use for the $(1,1)$ tensor fields of the tangent manifold TM the interpretation, according which an arbitrary tensor $F \in \mathcal{T}^1_1(TM)$ is being identified with a mapping $\tilde{F}\colon TTM \to TTM$ (satisfying suitable smoothness conditions), for which

$$\forall v \in TM : \tilde{F} \restriction T_vTM \in \operatorname{End} T_vTM.$$

(d) Let $\xi = (E, \pi, M)$ be a vector bundle over the base manifold M. $V\xi = (VE, \pi_v, E)$ denotes the vertical bundle over E; in particular – if we start with the tangent bundle τ_{TM} – then $\tau^v_{TM} = (VTM, \pi_v, TM)$ is the vertical bundle over TM; $\mathfrak{X}^v(TM)$ is the $C^\infty(TM)$-module of the vertical vector fields given on TM, i.e. of the sections of τ^v_{TM}.

1.2. Definition ([5])**.** By a *horizontal endomorphism* given on M we mean a $(1,1)$-tensor field $h \in \tau^1_1(TM)$, which is smooth over $\overset{\circ}{T}M := \cup_{p\in M}(T_pM \setminus \{0\})$, is not necessarily of class C^1 on TM, and has the following properties:

(H1) *it is a projector, i.e.* $h^2 = h$;

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(H2) $$\operatorname{Ker} h = \mathfrak{X}^v(\overset{\circ}{T}M).$$

1.3. Remark. It is known (see [2]) that any horizontal endomorphism $h \in \mathcal{T}_1^1(TM)$ uniquely determines a $(1,1)$-tensor $F \in \mathcal{T}_1^1(TM)$ with the following properties:

(1) $$F \circ h = J,$$

(2) $$F \circ J = h,$$

(3) $$F^2 = -1_{TTM}.$$

This $(1,1)$-tensor will be called the *almost complex structure* linked with h.

1.4. Definition and proposition. Let a horizontal endomorphism $h \in \mathcal{T}_1^1(TM)$ be given, and let us consider the almost complex structure F linked with h.

(a) The mapping

$$\theta := F \upharpoonright \mathfrak{X}^v(TM): \quad \mathfrak{X}^v(TM) \to \mathfrak{X}(TM)$$

will be called the *horizontal mapping* belonging to h. It has the following properties:

(4) $\operatorname{Im}\theta = \operatorname{Im} h =: \mathfrak{X}^h(TM)$, and θ is an isomorphism between the $C^\infty(TM)$-modules $\mathfrak{X}^v(TM)$ and $\mathfrak{X}^h(TM)$;

(5) $J \circ \theta = 1_{\mathfrak{X}^v(TM)}$.

(b) Conversely, if $\theta\colon \mathfrak{X}^v(TM) \to \mathfrak{X}(TM)$ is a $C^\infty(TM)$-linear mapping satisfying (5), then $h := \theta \circ J$ is a horizontal endomorphism.

Proof.

(a) By (3) F is invertible, hence θ is injective. Any vector field $X \in \mathfrak{X}^v(TM)$ can be given in the form $X = JY$ $(Y \in \mathfrak{X}(TM))$, thus $\theta X = F \circ J(Y) \overset{(2)}{=} hY \in \operatorname{Im} h$ and consequently $\operatorname{Im}\theta \subset \operatorname{Im} h$. On the other hand, for each vector field $Z \in \operatorname{Im} h$, $\quad Z = hY \overset{(2)}{=} \theta(JY) \in \operatorname{Im}\theta$ and this means that the inclusion $\operatorname{Im} h \subset \operatorname{Im}\theta$ is also valid. Thus (4) has been established.

Let $\nu := 1_{TTM} - h$. Then $\nu \upharpoonright \mathfrak{X}^v(TM) = 1_{\mathfrak{X}^v(TM)}$, and as can easily be verified, we also have $J \circ F = \nu$. From these remarks it follows that

$$\forall X \in \mathfrak{X}^v(TM): \quad J \circ \theta(X) = J \circ F(X) = \nu(X) = X$$

and so (5) holds.

(b) We verify that $h := \theta \circ J$ satisfies conditions (H1) and (H2).

$$h^2 = \theta \circ (J \circ \theta) \circ J \overset{(5)}{=} \theta \circ J = h,$$
$$hX = 0 :\iff \theta(JX) = 0 \iff JX = 0 \iff X \in \mathfrak{X}^v(TM);$$

here we use the fact that θ has a left inverse by (5) and therefore it is injective. □

2. Regular connections and prolongation

2.1. Definition.

(a) We start with a vector bundle $\xi = (E, \pi, M)$ (of finite rank, real) over M, and let $\operatorname{Sec}\xi$ denote the $C^\infty(M)$-module of the sections of ξ. By a *linear connection* given in ξ we mean a mapping

$$D : \mathfrak{X}(M) \times \operatorname{Sec}\xi \to \operatorname{Sec}\xi, \ (X, \sigma) \mapsto D_X\sigma$$

which has the following properties:

(D1) for a fixed $\sigma \in \operatorname{Sec}\xi$ the mapping $X \in \mathfrak{X}(M) \mapsto D_X\sigma$ is $C^\infty(M)$-linear;

(D2) for a fixed $X \in \mathfrak{X}(M)$ the mapping $\sigma \in \operatorname{Sec}\xi \mapsto D_X\sigma$ is additive;

(D3) $\forall f \in C^\infty(M): \quad D_X f\sigma = (Xf)\sigma + fD_X\sigma$ (rule of Leibniz).

The linear connections given in the vertical bundle $V\xi = (VE, \pi_v, E)$ will be called *vertical connections.*

(b) Let us suppose that D is a linear connection in the bundle τ_{TM}, i.e. on the tangent manifold TM. We say that D is *regular*, if it satisfies the following conditions:

(R1) $\boxed{DJ = 0}$, i.e. for each vector fields $X, Y \in \mathfrak{X}(TM)$, $DJ(X,Y) := (D_Y J)(X) = D_Y JX - JD_Y X = 0$; hence $D_Y JX = JD_Y X$.

(R2) The restriction ϕ to $\mathfrak{X}^v(TM)$ of the mapping $\tilde{\phi} := DC$ is an automorphism of the $C^\infty(TM)$-module $\mathfrak{X}^v(TM)$.

(c) A vertical connection $\bar{D}\colon \mathfrak{X}(TM) \times \mathfrak{X}^v(TM) \to \mathfrak{X}^v(TM)$ will be said to be *regular*, if $\phi := \bar{D}C \restriction \mathfrak{X}^v(TM)$ is an automorphism of $\mathfrak{X}^v(TM)$.

2.2. Proposition. (See [2], p. 297) *If D is a regular linear connection on the tangent manifold TM, then (with the notations of* 2.1. (R2)*)*

$$h := 1_{\mathfrak{X}(TM)} - \phi^{-1} \circ \tilde{\phi}$$

is a horizontal endomorphism on M.

Proof. We must verify that (H1) and (H2) are satisfied. First we remark that in view of (R1) any covariant derivative of an arbitrary vertical vector field is a vertical vector field. Thus

$$\forall X \in \mathfrak{X}(TM): \quad DC(X) = D_X C \in \mathfrak{X}^v(TM).$$

Using this

$$\begin{aligned} \forall X \in \mathfrak{X}(TM): \quad h(X) &:= X - \phi^{-1}\left[\tilde{\phi}(X)\right] = \\ &= X - \phi^{-1}\left[DC(X)\right] = X - \phi^{-1}\left(D_X C\right). \end{aligned}$$

For brevity sake we now put

$$Y := \phi^{-1}\left(D_X C\right).$$

Then

$$\begin{aligned} D_X C &= \phi(Y) = DC(Y) = D_Y C, \\ h(Y) &= Y - \phi^{-1}\left[DC(Y)\right] = Y - \phi^{-1}\left(D_Y C\right) = \\ &= Y - \phi^{-1}\left(D_X C\right) = Y - Y = 0, \end{aligned}$$

hence

$$h^2(X) = h\left[h(X)\right] = h(X - Y) = h(X) - h(Y) = h(X)$$

and so $h^2 = h$ is indeed valid.

If $hX = 0$, then we obtain $X = \phi^{-1}\left(D_X C\right)$ and here, as has already been pointed out, $D_X C \in \mathfrak{X}^v(TM)$ hence by (R2) $X \in \mathfrak{X}^v(TM)$. Conversely, if $X \in \mathfrak{X}^v(TM)$, then X can be written in the form $X = JY$ $(Y \in \mathfrak{X}(TM))$, and we obtain

$$hX = JY - \phi^{-1}\left[\tilde{\phi}(JY)\right] = JY - \phi^{-1} \circ \phi(JY) = 0$$

by what has been said, it is clear that $\operatorname{Ker} h = \mathfrak{X}^v(TM)$ too is valid. □

2.3. Corollary. *If* $\bar{D}\colon \mathfrak{X}(TM) \times \mathfrak{X}^v(TM) \to \mathfrak{X}^v(TM)$ *is a regular vertical connection,*

$$\bar{\phi} := \bar{D}C, \quad \phi := \bar{\phi} \restriction \mathfrak{X}^v(TM)$$

then

$$h := 1_{\mathfrak{X}(TM)} - \phi^{-1} \circ \bar{\phi}$$

is a horizontal endomorphism on M.

Proof. We use reasoning employed in establishing 2.2. □

2.4. Proposition. (M. Abate – G. Patrizio) *Let us suppose that $\bar{D}$ is a regular vertical connection in the vertical bundle τ^v_{TM}, and let us consider the horizontal endomorphism h derived from it by 2.3. Let $\nu := 1_{\mathfrak{X}(TM)} - h$, and let θ denote the horizontal mapping belonging to h. Then the mapping*

$$D\colon \mathfrak{X}(TM) \times \mathfrak{X}(TM) \to \mathfrak{X}(TM),$$
$$(X, Y) \mapsto D_X Y := \bar{D}_X \nu Y + \theta\left[\bar{D}_X \theta^{-1}(hY)\right]$$

is a linear connection in the bundle τ_{TM}, the restriction of which to $\mathfrak{X}(TM) \times \mathfrak{X}^v(TM)$ coincides with the given vertical connection $\bar{D}$.

Proof. A trivial calculation will verify that D is a linear connection, and the validity of $\bar{D} = D \restriction \mathfrak{X}(TM) \times \mathfrak{X}^v(TM)$ can immediately be seen from the definition. □

2.5. Remark. The linear connection D constructed in the Proposition will be said to be the *Abate – Patrizio prolongation* of the regular vertical connection $\bar{D}$.

2.6. Definition. Let us suppose that h is a horizontal endomorphism on the manifold M, and D a linear connection in the tangent bundle τ_{TM}. D is said to be *reducible* with respect to h if $Dh = 0$.

2.7. Lemma. *If D is a reducible connection with respect to the horizontal endomorphism h, then any covariant derivative of a horizontal vector field is horizontal, and any covariant derivative of a vertical vector field is vertical.*

Proof. In view of the reducibility

$$\forall X, Y \in \mathfrak{X}(TM): \quad 0 = Dh(Y, X) = D_X hY - hD_X Y$$

hence $D_X hY = hD_X Y \in \mathfrak{X}^h(TM)$. If $\nu := 1_{\mathfrak{X}(TM)} - h$ is the vertical projector belonging to h, then $D\nu = D1_{\mathfrak{X}(TM)} - Dh = 0$, consequently

$$\forall X, Y \in \mathfrak{X}(TM): \quad 0 = D\nu(Y, X) = D_X \nu Y - \nu D_X Y$$

and this implies $D_X \nu Y = \nu D_X Y \in \mathfrak{X}^v(TM)$. □

2.8. Proposition. (J. Grifone) *Let us suppose that $\bar{D}\colon \mathfrak{X}(TM) \times \mathfrak{X}^v(TM) \to \mathfrak{X}^v(TM)$ is a regular vertical connection. We consider the horizontal endomorphism $h \in \mathcal{T}^1_1(TM)$ derived from $\bar{D}$. Let $\nu = 1_{\mathfrak{X}(TM)} - h$ and let F denote the almost complex structure linked with h. There exists in the vector bundle τ_{TM} one and only one linear connection D reducible with respect to h, for which*

$$D \restriction \mathfrak{X}(TM) \times \mathfrak{X}^v(TM) = \bar{D}$$

is satisfied, i.e. which is a prolongation of $\bar{D}$. A vector field $Y \in \mathfrak{X}(TM)$ is parallel with respect to D if and only if (i.e. $DY = 0$ holds if and only if) JY and νY are parallel with respect to $\bar{D}$. D can be described explicitly by the formula

$$D_X Y = F\bar{D}_X JY + \bar{D}_X \nu Y \qquad (X, Y \in \mathfrak{X}(TM)).$$

2.9. Theorem. *The Abate – Patrizio prolongation of a regular vertical connection*

$$\bar{D}\colon \mathfrak{X}^v(TM) \times \mathfrak{X}(TM) \to \mathfrak{X}^v(TM)$$

coincides with the prolongation characterized by the theorem of Grifone.

Proof. Let us consider the prolongated Abate – Patrizio connection

$$D\colon (X, Y) \in \mathfrak{X}(TM) \times \mathfrak{X}(TM) \mapsto D_X Y := \bar{D}_X \nu Y + \theta\left[\bar{D}_X \theta^{-1}(hY)\right].$$

Since here we have

$$\theta^{-1}(hY) \overset{(2)}{=} \theta^{-1}\left[(F \circ J)(Y)\right] = \left(\theta^{-1} \circ F\right)(JY) =$$
$$= \theta^{-1}\left[F(JY)\right] \overset{1.4.(a)}{=} \theta^{-1}\left[\theta(JY)\right] = JY$$

taking into account repeatedly the definition of θ, we obtain

$$\theta\left[\bar{D}_X \theta^{-1}(hY)\right] = F\bar{D}_X JY.$$

This means that D acts exactly in the manner described in the theorem of Grifone. □

References

[1] M. Abate and G. Patrizio, *Finsler Metrics – A Global Approach.* Springer-Verlag, Berlin, 1994.

[2] J. Grifone, Structure presque tangente et connections. *Ann. Inst. Fourier, Grenoble*, 1, 3:287–334, 291–338, 1972.

[3] M. de Leon and P. R. Rodrigues, *Methods of differential geometry in analytical mechanics.* North-Holland, Amsterdam, 1989.

[4] J. Szilasi, *Introduction to differential geometry.* Kossuth Egyetemi Kiadó, Debrecen, 1998. (Hungarian.)

[5] J. Szilasi, Notable Finsler connections on a Finsler manifold. *Lecturas Matemáticas*, 19:7–34, 1998.

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