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REMARKS ON THE NUMBER OF NON-ZERO COEFFICIENTS OF POLYNOMIALS

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Dedicated to Professor Árpád Varecza on his 60th birthday

Abstract. We give sharp lower bounds for the number of non-zero coefficients of polynomials given respectively their number of distinct zeros or their number of distinct intersections with polynomials of low degree.

We give sharp lower bounds for the number of non-zero coefficients of polynomials given, respectively, their number of distinct zeros or (in the real case) the number of distinct intersections of their graph with that of polynomials of low degree.

Throughout f denotes a polynomial

$$f(X) = f_n X^{k_n} + \cdots + f_2 X^{k_2} + f_1 X^{k_1},$$

where $f_n \cdots f_2 f_1 \neq 0$ and, as suggested by the presentation, $k_n > \cdots > k_1$. Let $n(f)$ denote the number of nonzero coefficients of f and $z(f)$ its number of distinct zeros in $\mathbb{C}$. It will be convenient to set

$$f^\star(X) = f_n X^{k_n - k_1} + \cdots + f_2 X^{k_2 - k_1} + f_1.$$

Obviously $n(f) = n(f^\star)$ and it will be useful to work with $f^\star$. Finally, if g is a second polynomial, we denote by $i(f,g)$ the number of intersections (counted simplistically) of the graphs of f and g, that is, the naive cardinality of the set

$$\{(x, f(x)) : x \in \mathbb{R}\} \cap \{(x, g(x)) : x \in \mathbb{R}\}.$$

Simplistically, we take $i(f,f) = 1$. Our results are

Theorem 1. *Given a polynomial $f \in \mathbb{R}[X]$*

$$\text{(i)} \qquad n(f) \geq \frac{\deg f^\star}{z(f^\star)} + 1;$$

$$\text{(ii)} \qquad n(f) \geq \max_{g \in \mathbb{R}[X]} \left\{ \left[\frac{1}{2}(i(f,g) + 1 \right] - n(g) \right\}.$$

Set $r(f)$ to denote the number of distinct real zeros of f. A particular case of the theorem is, on taking $g(x) = 0$,

$$n(f) \geq \max \left\{ \frac{\deg f^\star}{z(f^\star)} + 1, \left[\frac{1}{2}(r(f) + 1) \right] \right\},$$

and, surprisingly perhaps, this is sharp.

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To see that, let a be a non-zero real number. The polynomial $g(X) = (X-a)^{n-1}$ has n non-zero real coefficients and, of course,

$$\frac{\deg g^\star}{z(g^\star)} + 1 = n.$$

For the other inequality it is enough to take an even polynomial, say,

$$h(X) = X^2(X^2 - 1^2)\cdots(X^2 - (n-1)^2),$$

and to note that $r(h) = 2n$ whilst $n(h) = n$. The theorem is two results, the first follows from the following

Lemma 1. *If a polynomial f in $\mathbb{C}[X]$ has a non-zero zero of order s then it has at least $s+1$ non-zero coefficients.*

Proof. Of course $(X-\alpha)^s | f(X)$ for some $\alpha \neq 0$ entails $(X-\alpha)^s | f^\star(X)$. Note that $n > s$ for $s = 1$. By formal differentiation

$$(X-\alpha)^s | f_n X^{k_n - k_1} + \cdots + f_2 X^{k_2 - k_1} + f_1$$

implies

$$(X-\alpha)^{s-1} | f_n(k_n - k_1) X^{k_n - k_2} + \cdots + f_2(k_2 - k_1),$$

and the claim follows by induction on s. □

Since $f^\star$ has only $z(f^\star)$ different zeros, at least one of the non-zero zeros of f has multiplicity at least

$$\frac{\deg f^\star}{z(f^\star)},$$

so the lemma includes part (i) of the theorem.

This first result came to our attention in a rather more sophisticated context. Namely, an identity

$$f_n X^{k_n} + \cdots + f_2 X^{k_2} + f_1 X^{k_1} - f_n X^{k_1} \prod_{i=1}^{z} (X - \alpha_i)^{s_i} = 0$$

is an additive relation in S-units of a function field of genus $g = 0$. Here S is the set of primes vanishing at $\infty, 0$ and the α_i and thus has cardinality $z + 2$. According to an inequality of Brownawell and Masser [1] the height, here $\deg f^\star$, of a solution is bounded by

$$\frac{1}{2} n(n+1)(|S| + 2g - 2) = \frac{1}{2} n(n+1) z(f^\star),$$

yielding, for $\frac{1}{2}n(n+1)$, the lower bound we obtain for $n-1$. However, it is popularly believed that the bound of [1] should $O(n)$ rather than $O(n^2)$.

Lemma 2. *If a polynomial f in $\mathbb{R}[X]$ has r distinct real zeros then it has at least $\left[\frac{1}{2}(r+1)\right]$ non-zero coefficients.*

Proof. By the data $f^\star$ has at least $r-1$ distinct real zeros and so, by Rolle's Theorem. its derivative has at least $r-2$ distinct real zeros (which may include a zero at 0). Thus

$$f_n(k_n - k_1) X^{k_n - k_2} + \cdots + f_2(k_2 - k_1)$$

has at least $r-3$ distinct non-zero real zeros. Since the constant polynomial has no zeros it is plain that $r \leq 2n - 1$, which is the assertion. □

If g is some other polynomial in $\mathbb{R}[X]$ then $f - g$ has $i(f,g)$ distinct real zeros, so $i(f,g) + 1 \leq 2n(f-g)$. Since, certainly, $n(f) \geq n(f-g) - n(g)$, assertion (ii) of the theorem follows immediately.

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References

[1] W. D. Brownawell and D. W. Masser, *Vanishing sums in function fields,* Proc. Camb. Phil. Soc. **100** (1986), 427–434

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