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HYPERSTABILITY OF A CLASS OF LINEAR FUNCTIONAL EQUATIONS

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Dedicated to the 60th birthday of Professor Árpád Varecza

ABSTRACT. The aim of this note is to offer hyperstability results for linear functional equations of the form

$$f(s) + f(t) = \frac{1}{n}\sum_{i=1}^{n} f(s\varphi_i(t)) \qquad (s, t \in S),$$

where S is a semigroup and where $\varphi_1, \ldots, \varphi_n \colon S \to S$ are pairwise distinct automorphisms of S such that the set $\{\varphi_1, \ldots, \varphi_n\}$ is a group equipped with the composition as the group operation. The main results state that if f satisfies a stability inequality related to the above equation then it is also a solution of this equation.

1. INTRODUCTION

In a recent paper of Kocsis and Maksa [KM98], the stability problem of a sum form functional equation from information theory led to the investigation of the stability of the equation

$$\varphi(xy) = x^\alpha \varphi(y) + y^\alpha \varphi(x) \qquad (x, y \in]0, 1]), \tag{1}$$

where $\alpha \in \mathbb{R}$ is a fixed power and $\varphi \colon]0, 1] \to \mathbb{R}$. It is well-known and easy to see that the general solution of (1) is of the form

$$\varphi(x) = x^\alpha \ell(x) \qquad (x \in]0, 1]),$$

where $\ell \colon]0, 1] \to \mathbb{R}$ satisfies the Cauchy equation

$$\ell(xy) = \ell(x) + \ell(y) \qquad (x, y \in]0, 1]). \tag{2}$$

The stability problem of (1) can now be formulated as follows:

$$(\mathcal{P}) \quad \begin{cases} \textit{Assume that a function } \psi \colon]0,1] \to \mathbb{R} \textit{ satisfies the stability inequality} \\ \qquad |\psi(xy) - x^\alpha \psi(y) - y^\alpha \psi(x)| \le \varepsilon \qquad (x, y \in]0, 1]) \qquad (3) \\ \textit{for some constant } \varepsilon \ge 0. \textit{ Does there exist a solution } \varphi \textit{ of } (1) \textit{ such that the} \\ \textit{difference function } \psi - \varphi \textit{ is bounded?} \end{cases}$$

In the case $\alpha = 0$ it follows from the Hyers-Ulam stability theorem for the Cauchy functional equation that there exists a solution φ of (1) such that $|\psi - \varphi| \le \varepsilon$ (see [Hye41]). The discussion of the case $\alpha = 1$ was proposed by Maksa [Mak97]

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at the 34th ISFE and an affirmative solution to $(\mathcal{P})$ was found by Jacek Tabor [Tab97a] (see also [Bad00], [Pál97], [Tab97b] for related or more general results). The case $\alpha > 0$ can easily be reduced to the case $\alpha = 1$ by considering the function $]0,1] \ni x \mapsto \psi(x^{1/\alpha})$ instead of ψ. Thus, it follows from Tabor's result that (1) is stable for $\alpha > 0$.

For the sake of completeness now we consider the case $\alpha < 0$, or more generally, we replace the power function $t \mapsto t^\alpha$ in (1) by a function $M\colon]0,1] \to \mathbb{R}$ satisfying

$$M(xy) = M(x)M(y) \qquad (x, y \in]0,1]) \tag{4}$$

and we also suppose that

$$M(x_0) > 1 \qquad \text{for some} \quad x_0 \in]0,1]. \tag{5}$$

Thus, (3) can be rewritten as

$$|\psi(xy) - M(x)\psi(y) - M(y)\psi(x)| \le \varepsilon \qquad (x, y \in]0,1]). \tag{6}$$

Due to (5), M is positive-valued (see Aczél and Dhombres [AD89]). Therefore, we can introduce the functions

$$\ell(x) = \frac{\psi(x)}{M(x)} \qquad (x \in]0,1]) \tag{7}$$

and

$$F(x, y) = \ell(xy) - \ell(x) - \ell(y) \qquad (x, y \in]0,1]). \tag{8}$$

With these notations, the stability inequality (6) reduces to

$$|F(x, y)| \le \frac{\varepsilon}{M(xy)} \qquad (x, y \in]0,1]). \tag{9}$$

It can easily be checked that the function F defined in (8) satisfies the so-called *cocycle functional equation*

$$F(x, y) + F(xy, z) = F(x, yz) + F(y, z) \qquad (x, y, z \in]0,1]). \tag{10}$$

With the substitution $z = x_0^k$, (10) implies that

$$F(x, y) + F(xy, x_0^k) = F(x, yx_0^k) + F(y, x_0^k) \qquad (x, y \in]0,1],\ k \in \mathbb{N}). \tag{11}$$

Using the estimate (9) and equation (4), we have that

$$|F(s, tx_0^k)| \le \frac{\varepsilon}{M(st)[M(x_0)]^k} \qquad (s, t \in]0,1]).$$

Hence, by (5), we obtain

$$\lim_{k \to \infty} F(s, tx_0^k) = 0 \qquad (s, t \in]0,1]).$$

Thus, taking the limit $n \to \infty$ in (11), we get that

$$F(x, y) = 0 \qquad (x, y \in]0,1]),$$

that is, ℓ is a solution of (2). By (7),

$$\psi(x) = M(x)\ell(x) \qquad (x \in]0,1])$$

and an easy calculation yields that ψ satisfies the functional equation

$$\psi(xy) = M(x)\psi(y) + M(y)\psi(x) \qquad (x, y \in]0,1]), \tag{12}$$

which is analogous to (1).

Summarizing our observations, we have proved the following hyperstability result for the functional equation (12).

Theorem 1. *Let $M\colon]0,1] \to \mathbb{R}$ be a solution of the functional equation* (4) *and suppose that* (5) *also holds. Assume that the function $\psi\colon]0,1] \to \mathbb{R}$ satisfies the stability inequality* (6) *for some $\varepsilon \geq 0$. Then ψ is a solution of* (12)*, that is,* (6) *is satisfied by $\varepsilon = 0$.*

The above result shows that the solutions of the inequality (6) are just the solutions of the corresponding equation (12). Thus, in the particular case $\alpha < 0$, the solutions of (3) and the solutions of (1) are the same. As we have seen, the basic tool for proving the above result is the cocycle equation (10) which plays an important role in the theory of group extensions (see [JKT68], [Erd59]).

We note that if (5) does not hold, that is, $M(x) \leq 1$ for all $x \in]0,1]$, then, either $M(x) = x^{\alpha}$ $(x \in]0,1])$ for some $\alpha \geq 0$, or $M(x) = 0$ $(x \in]0,1])$, or $M(x) = 0$ $(x \in]0,1[)$ and $M(1) = 1$ (see [Acz66]). In these cases, the stability problem of the functional equation (12) is either solved, or is trivial and uninteresting.

The aim of this paper to extend the above argument to a class of linear functional equations for which a cocycle equation-type identity can be derived.

2. Main Results

Throughout this section, let $S = (S, \cdot)$ denote a semigroup and let X denote a real normed space. In addition, let $\varphi_1, \dots, \varphi_n\colon S \to S$ be pairwise distinct automorphisms of S such that the set $\{\varphi_1, \dots, \varphi_n\}$ is a group with respect to the composition as group operation.

We consider the following functional equation

$$f(s) + f(t) = \frac{1}{n}\sum_{i=1}^{n} f(s\varphi_i(t)) \qquad (s,t \in S), \tag{13}$$

There are two important particular cases of the above equation.

- (PC-1): $n = 1$ and $\varphi_1(t) = t$ $(t \in S)$. In this setting, (13) reduces to the Cauchy equation (2).
- (PC-2): $n = 2$ and $\varphi_1(t) = t$, $\varphi_2(t) = t^{-1}$ $(t \in S)$ and S is an Abelian group. With these assumptions, (13) reduces to the so-called *norm-square equation*

$$f(s) + f(t) = \frac{1}{2}\big(f(st) + f(st^{-1})\big) \qquad (s,t \in S).$$

For further examples and special cases of (13), see [Pál94].

The proof of the main results is based on the following lemma ([Pál94, Theorem 1]) which derives an identity for the two variable function obtained by taking the difference of the left and right hand sides of (13).

Lemma. *Let $f\colon S \to X$ be an arbitrary function. Then the function $F\colon S \times S \to X$ defined by*

$$F(s,t) = f(s) + f(t) - \frac{1}{n}\sum_{i=1}^{n} f(s\varphi_i(t)) \qquad (s,t \in S) \tag{14}$$

satisfies the following functional equation

$$F(x,y) + \frac{1}{n}\sum_{i=1}^{n} F(x\varphi_i(y), z) = \frac{1}{n}\sum_{i=1}^{n} F(x, y\varphi_i(z)) + F(y,z) \qquad (x,y,z \in S). \tag{15}$$

Proof. Let $f\colon S \to X$ be arbitrary and let F given by (14). Evaluating the left hand side of (15), we get

$$F(x,y) + \frac{1}{n}\sum_{i=1}^{n} F(x\varphi_i(y), z) = f(x) + f(y) + f(z) - \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(x\varphi_i(y)\varphi_j(z)).$$

Similarly, for the right hand side, we deduce

$$\begin{aligned} F(y,z) \quad &+ \quad \frac{1}{n}\sum_{i=1}^{n} F(x, y\varphi_i(z)) \\ &= \quad f(x)+f(y)+f(z)-\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} f(x\varphi_j(y\varphi_i(z))) \\ &= \quad f(x)+f(y)+f(z)-\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n} f(x\varphi_j(y)\varphi_j\circ\varphi_i(z))) \\ &= \quad f(x)+f(y)+f(z)-\frac{1}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{n} f(x\varphi_j(y)\varphi_i(z)), \end{aligned}$$

where, in the last steps, we used that φ_j is a homomorphism and $(\varphi_j\circ\varphi_1,\dots,\varphi_j\circ\varphi_n)$ is a permutation of $(\varphi_1,\dots,\varphi_n)$. Thus (15) turns out to be valid. □

In the particular case (PC-1), the resulting equation (15) is equivalent to the cocycle equation (10). In the second particular case (PC-2), (15) reduces to the equation

$$F(x,y)+\frac{1}{2}\big(F(xy,z)+F(xy^{-1},z)\big) = \frac{1}{2}\big(F(x,yz)+F(x,yz^{-1})\big)+F(y,z) \; (x,y,z\in S),$$

that was discovered by Székelyhidi [Szék83] and investigated by Ebanks [Eba85], [Eba89] and Székelyhidi [Szék95].

The following theorem is a *hyperstability* result for (13). It states that if the error bound for the difference of the two sides of (13) satisfies a certain asymptotic property then, in fact, the two sides have to be equal to each other.

Theorem 2. *Let $\varepsilon\colon S\times S\to\mathbb{R}$ be a function such that there exists a sequence $u_k\in S$ satisfying*

$$\lim_{k\to\infty}\varepsilon(u_k s,t)=0 \qquad (s,t\in S). \tag{16}$$

Assume that $f\colon S\to X$ satisfies the stability inequality

$$\Big\|f(s)+f(t)-\frac{1}{n}\sum_{i=1}^{n} f(s\varphi_i(t))\Big\| \le \varepsilon(s,t) \qquad (s,t\in S). \tag{17}$$

Then f is a solution of (13).

Proof. Define $F\colon S\times S\to\mathbb{R}$ by (14). Then (15) is satisfied and (17) yields

$$\|F(s,t)\|\le\varepsilon(s,t) \qquad (s,t\in S).$$

Thus, by (16), we have that

$$\lim_{k\to\infty}F(u_k s,t)=0 \qquad (s,t\in S). \tag{18}$$

Let $y,z,s_0\in S$ be fixed. Substituting $x=u_k s_0$ into (15), taking the limit as $k\to\infty$ and applying (18), we deduce from (15) that

$$F(y,z)=0 \qquad (y,z\in S),$$

that is, f is a solution of (13). □

Corollary 1. *Let $\varepsilon\colon S\times S\to\mathbb{R}$ and suppose that there exist $u\in S$ and $0\le q<1$ such that*

$$\varepsilon(us,t)\le q\varepsilon(s,t) \qquad (s,t\in S). \tag{19}$$

Assume that $f\colon S\to X$ satisfies the stability inequality (17). *Then f is a solution of* (13).

Proof. It suffices to show that ε satisfies (16) for some sequence u_k. Then, (19) yields by induction that

$$\varepsilon(u^k s, t) \leq q^k \varepsilon(s,t) \qquad (s,t \in S,\, k \in \mathbb{N}),$$

whence (16) follows with the sequence $u_k = u^k$. Thus the statement is the consequence of Theorem 2. $\square$

Theorem 3. *Let $\varepsilon\colon S \times S \to \mathbb{R}$ be a function such that there exists a sequence $u_k \in S$ satisfying*

$$\lim_{k\to\infty} \varepsilon(s, t\varphi_i(u_k)) = 0 \qquad (s,t \in S,\, i \in \{1,\dots,n\}). \tag{20}$$

Assume that $f\colon S \to X$ satisfies the stability inequality (17). *Then f is a solution of* (13).

Proof. The proof is analogous to that of Theorem 2. Define F by (14). Instead of (18), we now have that

$$\lim_{k\to\infty} F(s, t\varphi_i(u_k)) = 0 \qquad (s,t \in S,\, i \in \{1,\dots,n\}). \tag{21}$$

Let $x, y, t_0 \in S$ be fixed. Substituting $z = t_0 u_k$ into (15), taking the limit as $k \to \infty$ and applying (21), we obtain that

$$F(x,y) = 0 \qquad (x,y \in S).$$

Therefore f is a solution of (13). $\square$

Corollary 2. *Let $\varepsilon\colon S \times S \to \mathbb{R}$ and suppose that there exist $u \in S$ and $0 \leq q < 1$ such that*

$$\varepsilon(s, t\varphi_i(u)) \leq q\varepsilon(s,t) \qquad (s,t \in S,\, i \in \{1,\dots,n\}). \tag{22}$$

Assume that $f\colon S \to X$ satisfies the stability inequality (17). *Then f is a solution of* (13).

Proof. In this case, (22) yields by induction that

$$\varepsilon(s, t\varphi_i(u^k)) \leq q^k \varepsilon(s,t) \qquad (s,t \in S,\, i \in \{1,\dots,n\},\, k \in \mathbb{N}).$$

Therefore (20) is satisfied by $u_k = u^k$ and the statement follows from Theorem 3. $\square$

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