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MIKUSIŃSKI FUNCTIONAL EQUATION ON A HEXAGON

KÁROLY LAJKÓ

Dedicated to Professor Árpád Varecza on the occasion of his 60th birthday

Abstract. The general solution of the conditional functional equation (M) is described for functions $f\colon (-r,r)\to\mathbb{R}$, where (M) is satisfied for all $(x,y)\in H$, where $H=\{(x,y)\mid x,y,x+y\in(-r,r)\}$ is a hexagon.

1. Introduction

J. Mikusiński (in 1971) mentioned the functional equation

$$f(x+y)\big[f(x+y)-f(x)-f(y)\big]=0 \tag{M}$$

which since has been named after him.

The authors of [2] find the general solution of (M) for functions $f\colon X\to Y$ where $(X,+)$ and $(Y,+)$ are (not necessarily commutative) groups. In case $X=Y=\mathbb{R}$ they proved the following

Theorem 1. *The only solutions of equation* (M) *for functions* $f\colon\mathbb{R}\to\mathbb{R}$ *are additive functions, i.e. the solutions of Cauchy functional equation*

$$f(x+y)=f(x)+f(y)\qquad(x,y\in\mathbb{R}). \tag{1}$$

The aim of this paper is to present the general solution of (M) for functions $f\colon(-r,r)\to\mathbb{R}$, where (M) is satisfied for all $(x,y)\in H=\{(x,y)\mid x,y,x+y\in(-r,r)\}$ and $(-r,r)$ is an open interval in $\mathbb{R}$.

2. An extension theorem for (M)

Following the ideas of Aczél [1] and Kuczma [3] we prove the following extension theorem for the Mikusiński functional equation (M).

Theorem 2. *If the function* $f\colon(-r,r)\to\mathbb{R}$ *satisfies the Mikusiński functional equation* (M) *for all* $(x,y)\in H$, *where* H *is a hexagon given above, then there exists a unique function* $F\colon\mathbb{R}\to\mathbb{R}$ *satisfying* (M) *for any* $x,y\in\mathbb{R}$ *and*

$$f(x)=F(x),\qquad x\in(-r,r).$$

Proof. a) First we show that

$$f\left(\frac{x}{2^n}\right)=\frac{1}{2^n}f(x),\qquad x\in(-r,r),\ n\in N. \tag{2}$$

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If $f(x)=0$, then it is easy to see that $f(2^n x)=0$ $(2^n x\in(-r,r))$, which implies (2). If $f(x)\neq 0$, then replacing both x and y by $\frac{x}{2}$, we get from (M)

$$f\left(\frac{x}{2}\right)=\frac{1}{2}f(x)\neq 0,\qquad x\in(-r,r). \tag{3}$$

Thus (2) holds for $n=1$.
Using (3) repeatedly completes the statement.

b) On the other hand, for every $u\in\mathbb{R}$ there exists an $n\in N\cup\{0\}$ such that $x=\frac{u}{2^n}\in(-r,r)$. We define the function F by

$$F\colon\mathbb{R}\to\mathbb{R},\quad F(u)=2^n f\left(\frac{u}{2^n}\right)\quad\left(\frac{u}{2^n}\in(-r,r)\right). \tag{4}$$

This definition is correct and (4) gives

$$f(x)=F(x),\qquad x\in(-r,r).$$

c) We must verify that F satisfies (M) for all $x,y\in\mathbb{R}$.
If $x,y\in\mathbb{R}$ are arbitrary, then there exists an $n\in N\cup\{0\}$ such that

$$\frac{x}{2^n},\ \frac{y}{2^n},\ \frac{x+y}{2^n}\in(-r,r).$$

Now

$$f\left(\frac{x}{2^n}+\frac{y}{2^n}\right)\left[f\left(\frac{x}{2^n}\right)+f\left(\frac{y}{2^n}\right)-f\left(\frac{x}{2^n}+\frac{y}{2^n}\right)\right]=0,$$

that is

$$2^n f\left(\frac{x+y}{2^n}\right)\left[2^n f\left(\frac{x}{2^n}\right)+2^n f\left(\frac{y}{2^n}\right)-2^n f\left(\frac{x+y}{2^n}\right)\right]=0.$$

This implies that the function F, defined by (4) satisfies (M) for all $x,y\in\mathbb{R}$.

d) To prove the uniqueness, suppose that a function $G\colon\mathbb{R}\to\mathbb{R}$ satisfies (M) in $\mathbb{R}$ and fulfills the condition

$$G(x)=f(x),\qquad x\in(-r,r). \tag{5}$$

Similarly as in a) one can get that

$$G\left(\frac{x}{2^n}\right)=\frac{1}{2^n}G(x),\qquad x\in\mathbb{R},\ n\in N\cup\{0\}. \tag{6}$$

Take an arbitrary $x\in\mathbb{R}$. There exists an $n\in N\cup\{0\}$ such that $\frac{x}{2^n}\in(-r,r)$. Thus we have by (4), (5) and (6)

$$G(x)=2^n G\left(\frac{x}{2^n}\right)=2^n f\left(\frac{x}{2^n}\right)=F(x).$$

Consequently $G=F$ in $\mathbb{R}$. □

3. The general solution of (M) on a hexagon

Using Theorems 1 and 2 we obtain

Theorem 3. *If the function $f\colon(-r,r)\to\mathbb{R}$ satisfies the Mikusiński functional equation* (M) *for all $(x,y)\in H$, then there exists a unique additive function $A\colon\mathbb{R}\to\mathbb{R}$ such that*

$$f(x)=A(x),\quad x\in(-r,r). \tag{7}$$

Proof. Theorem 2 shows that there exists a unique function $F\colon\mathbb{R}\to\mathbb{R}$ satisfying (M) for all $x,y\in\mathbb{R}$ and $f(x)=F(x)$ $x\in(-r,r)$.

Because of Theorem 1 F is an additive function.

It is easy to see that all additive functions A fulfill also (M) for all $(x,y)\in H$. □

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Institute of Mathematics and Informatics,
University of Debrecen,
H-4010 Debrecen, Pf. 12, Hungary
E-mail address: lajko@math.klte.hu