Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
17 (2001), 117–120
www.emis.de/journals

DISCRETE LAGUERRE FUNCTIONS AND EQUILIBRIUM CONDITIONS

SCHIPP FERENC

Dedicated to the 60th birthday of Professor Árpád Varecza

Abstract. The discrete Laguerre functions L_n^a $(n \in \mathbb{N})$ forms an orthonormal system on the unite circle $\mathbb{T}$ and the finite set of functions L_n^a $(n = 0, 1, \cdots, N-1)$ is orthonormal with respect to a discrete scalar product defined by the discrete subset $\mathbb{T}_N^a$ of $\mathbb{T}$. It is showed that the set $\mathbb{T}_N^a$ can be interpreted as a solution of an electrostatic equilibrium problem.

1. Introduction

In the control theory the discrete Laguerre functions and their generalizations are often used to identify the transfer function of the system [1], [2], [3]. The discrete Laguerre functions L_n^a $(n \in \mathbb{N})$ contain a complex parameter $a \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and can be expressed by the Blaschke functions

$$B_a(z) := \frac{z-a}{1-\bar{a}z} \quad (z \in \mathbb{C}).$$

Namely (see [2])

$$L_n^a(z) := \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} B_a^n(z) = \sqrt{1-|a|^2} \frac{(z-a)^n}{(1-\bar{a}z)^{n+1}} \quad (z \in \mathbb{C}).$$

The Laguerre functions form an orthonormal system on the unite circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, i.e.

$$\langle L_n^a, L_m^a \rangle := \frac{1}{2\pi} \int_0^{2\pi} L_n^a(e^{it}) \overline{L_m^a(e^{it})} \, dt = \delta_{mn} \quad (m, n \in \mathbb{N}),$$

where δ_{mn} is the Kronecker symbol (see [2]). A generalization of this system is the Malmquist–Takenaka system (see [4]).

If a belongs to $\mathbb{D}$ then B_a is a $1-1$ map on $\mathbb{D}$ and on $\mathbb{T}$, respectively. Moreover (see [2]) B_a can be written in the form

$$B_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R}, a = re^{i\alpha} \in \mathbb{D}), \tag{1.1}$$

where

$$\beta_a(t) := \alpha + \gamma_s(t-\alpha), \; \gamma_s(t) := 2\arctan(s \tan\frac{t}{2}) \;\; (t \in [-\pi, \pi)), \; s := \frac{1+r}{1-r}$$

2000 *Mathematics Subject Classification.* 42C05, 33C52.

Key words and phrases. Discrete Laguerre system, discrete orthogonal systems, electrostatic equilibrium.

This research was supported by the grant OTKA/T 032719 and FKFP/0198/1999.

and γ_s is extended to $\mathbb{R}$ by $\gamma_s(t+2\pi) = 2\pi + \gamma_s(t)$ $(t \in \mathbb{R})$. The finite collection of the functions L_n^a $(0 \le n < N)$ form a discrete orthonormal system with respect to the scalar product

$$[f,g]_N := \frac{1}{N} \sum_{z \in \mathbb{T}_N^a} f(z)\overline{g(z)}\rho_a(z),$$

where

$$\mathbb{T}_N^a := \left\{ e^{i\beta_a^{-1}(t)} : t = \frac{2k\pi}{N}, k = 0, 1, \cdots, N-1 \right\},$$
$$\rho_a(z) := \frac{|1-\bar{a}z|^2}{1-|a|^2} \quad (z \in \mathbb{T}) \quad \text{and} \quad \beta_a^{-1}(t) = \alpha + \gamma_{s^{-1}}(t-\alpha) \ (t \in \mathbb{R})$$

is the inverse function of β_a (see [2]). Namely, it is easy to check that

$$[L_n^a, L_m^a] = \delta_{mn} \quad (0 \le m, n < N).$$

Indeed, by the definition of L_n^a, $\mathbb{T}_N^a$ and (1.1) and by the orthogonality of the discrete trigonometric system, for $0 \le m, n < N$ we have

$$[L_n^a, L_m^a]_N := \frac{1}{N} \sum_{z \in \mathbb{T}_N^a} L_n^a(z)\overline{L_m^a(z)}\rho_a(z) =$$
$$= \frac{1}{N} \sum_{z \in \mathbb{T}_N^a} B_a^{n-m}(z) = \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i(n-m)k/N} = \delta_{mn}.$$

In this paper we show that the points of $\mathbb{T}_N^a$ are the solution of the following problem of electrostatic equilibrium:

Problem. *Let $a \in \mathbb{D}$ and $N \in \mathbb{N}, N \ge 2$ be given numbers. If N unite "masses" at the variable point $z_1, z_2, \cdots, z_N \in \mathbb{T}$ and two fixed masses $q = -(N-1)/2$ at a and $\bar{a}^{-1}$ are considered, for what position of the points $z_1, z_2, \cdots, z_N$ does the potential function*

$$V(z_1, \cdots, z_N) := -\log \left(\prod_{1 \le i < j \le N} |z_i - z_j| \prod_{1 \le i \le N} |z_i - a|^q |z_i - \bar{a}^{-1}|^q \right) \tag{1.2}$$
$$(z_1, \cdots, z_N \in \mathbb{T})$$

become a minimum ?

The function V can be interpreted as the energy of the electrostatic masses just defined. The minimum position corresponds to the condition of electrostatic equilibrium. Namely we have the following

Theorem. *The numbers $w_k := e^{i\tau_k}$, $\tau_k := \beta_a^{-1}(2\pi(k-1)/N)$ $(k = 1, \cdots, N)$ of $\mathbb{T}_N^a$ are the solution of the equilibrium equations*

$$\sum_{k=1, k \ne n}^{N} \frac{1}{w_n - w_k} + \frac{q}{w_n - a} + \frac{q}{w_n - \bar{a}^{-1}} = 0 \quad (n = 1, 2, \cdots, N). \tag{1.3}$$

Moreover the point $(\tau_1, \tau_2, \cdots, \tau_N) \in \mathbb{R}^N$ is a stationary point of the potential

$$V(e^{it_1}, \cdots, e^{it_N}) \quad ((t_1, \cdots, t_N) \in \mathbb{R}^N),$$

i.e.

$$\frac{\partial V(e^{i\tau_1}, \cdots, e^{i\tau_N})}{\partial t_n} = 0 \quad (n = 1, \cdots, N). \tag{1.4}$$

The system of equations (1.3) can be interpreted as the electrostatic equilibrium condition for the masses in question. We remark that *the zeros of the Jacobi, Laguerre and Hermite polynomials admit a similar interpretation* (see [6], pp. 140, 153).

A generalization of these results with respect to the Malmquist–Takenaka systems will be published in [5].

2. The equilibrium condition

First we show (1.3).

Proof of (1.3). Denote

$$\varphi(z) := B_a^N(z) - 1 = \frac{(z-a)^N - (1-\bar{a}z)^N}{(1-\bar{a}z)^N} \quad (z \in \mathbb{C}).$$

By (1.1) it is clear that $\varphi(z) = 0$ if and inly if $z = w_k := e^{i\beta_a^{-1}(2\pi k/N)}$ $(k = 1, 2, \cdots, N)$. Set

$$f(z) := \prod_{k=1}^{N} (z - w_k) \quad (z \in \mathbb{C}).$$

Since the polynomials f and $g(z) := (z-a)^N - (1-\bar{a}z)^N$ $(z \in \mathbb{C})$ have the same degree and roots, therefore $f(z) = \lambda[(z-a)^N - (1-\bar{a}z)^N]$ $(z \in \mathbb{C})$ with a constant $\lambda \in \mathbb{C}$.

It is easy to see that

$$\frac{1}{2}\frac{g''(w_n)}{g'(w_n)} = \frac{1}{2}\frac{f''(w_n)}{f'(w_n)} = \sum_{k=1}^{N} \frac{1}{w_n - w_k} \quad (n = 1, 2, \cdots, N). \tag{2.1}$$

On the other hand by the definition of the function g and by

$$\left(\frac{w_n - a}{1 - \bar{a}w_n}\right)^N = 1 \quad (n = 1, 2, \cdots, N)$$

we have

$$\begin{aligned} \frac{g''(w_n)}{g'(w_n)} &= \frac{N(N-1)(w_n - a)^{N-2} - \bar{a}^2 N(N-1)(1-\bar{a}w_n)^{N-2}}{N(w_n - a)^{N-1} + \bar{a}N(1-\bar{a}w_n)^{N-1}} = \\ &= \frac{\dfrac{N(N-1)}{(w_n-a)^2} - \dfrac{\bar{a}^2 N(N-1)}{(1-\bar{a}w_n)^2}}{\dfrac{N}{w_n - a} + \dfrac{\bar{a}N}{1-\bar{a}w_n}}. \end{aligned}$$

Hence we get

$$\begin{aligned} \frac{g''(w_n)}{g'(w_n)} &= (N-1)\frac{\dfrac{1}{(w_n-a)^2} - \dfrac{1}{(w_n-\bar{a}^{-1})^2}}{\dfrac{1}{w_n-a} - \dfrac{1}{w_n-\bar{a}^{-1}}} = \\ &= (N-1)\left(\frac{1}{w_n-a} + \frac{1}{w_n-\bar{a}^{-1}}\right). \end{aligned} \tag{2.2}$$

Comparing (2.1) and (2.2) we get (1.3). □ □

Now we show that for the solution of (1.3) the condition (1.4) is satisfied.

Proof of (1.4). Set $a := re^{i\alpha}$. Then by (1.3) we have

$$\sum_{k=1}^{N} \frac{\cos\tau_n - \cos\tau_k}{|w_n - w_k|^2} + \frac{q(\cos\tau_n - r\cos\alpha)}{|w_n - a|^2} + \frac{q(\cos\tau_n - r^{-1}\cos\alpha)}{|w_n - \bar{a}^{-1}|^2} = 0,$$
$$\sum_{k=1}^{N} \frac{\sin\tau_n - \sin\tau_k}{|w_n - w_k|^2} + \frac{q(\sin\tau_n - r\sin\alpha)}{|w_n - a|^2} + \frac{q(\sin\tau_n - r^{-1}\sin\alpha)}{|w_n - \bar{a}^{-1}|^2} = 0$$
$$(n = 1, 2, \cdots, N).$$

Multiplying the first equality by $\sin\tau_n$ the second by $\cos\tau_n$ and taking the difference we get

$$\sum_{k=1}^{N} \frac{\sin(\tau_n - \tau_k)}{|w_n - w_k|^2} + \frac{qr\sin(\tau_n - \alpha)}{|w_n - a|^2} + \frac{qr^{-1}\sin(\tau_n - \alpha)}{|w_n - \bar{a}^{-1}|^2} = 0 \quad (k = 1, \cdots, N). \tag{2.3}$$

By the definition of V

$$V(e^{it_1}, \cdots, e^{it_N}) = -\log\left(\prod_{1\le j<k\le N} |e^{it_j} - e^{it_k}| \prod_{1\le j\le N} |e^{it_j} - a|^q |e^{it_j} - \bar{a}^{-1}|^q\right) =$$
$$= -\log\left(\prod_{1\le j<k\le N} |e^{i(t_j - t_k)} - 1| \prod_{1\le j\le N} |e^{i(t_j-\alpha)} - r|^q |e^{i(t_j-\alpha)} - r^{-1}|^q\right)$$
$$(0 \le t_1 \le \cdots \le t_N < 2\pi).$$

It is easy to check that

$$\frac{\partial V(e^{i\tau_1}, \cdots, e^{i\tau_N})}{\partial t_n} =$$
$$= \sum_{k=1}^{N} \frac{\sin(\tau_n - \tau_k)}{|w_n - w_k|^2} + \frac{qr\sin(\tau_n - \alpha)}{|w_n - a|^2} + \frac{qr^{-1}\sin(\tau_n - \alpha)}{|w_n - \bar{a}^{-1}|^2}$$
$$(k = 1, \cdots, N).$$

and by (2.3) we get (1.4). □

References

[1] J. Bokor and F. Schipp. L^∞ system approximation algorithms generated by φ summation. *IFAC AUTOMATICA J.*, 33:2019–2024, 1997.

[2] J. Bokor and F. Schipp. Approximate linear H^∞ identification in Laguerre and Kautz basis. *IFAC AUTOMATICA J.*, 34:463–468, 1998.

[3] J. Bokor, F. Schipp, and Z. Szabó. Identification of rational approximate models in H^∞ using generalized orthonormal basis. *IEEE Trans. Autom. Control*, 44(1):153–158, 1999.

[4] G. G. Lorentz, M. Golitschek, and Y. Makovoz. *Constructive Approximation*, volume 304 of *Grundlehren math. Wiss.* Springer, 1996.

[5] M. Papp and F. Schipp. Discrete Malmquist–Takenaka systems and electrostatic equilibrium condition. to appear.

[6] G. Szegő. *Orthogonal polynomials (fourth edition)*, volume 23 of *Colloquium Publications*. Amer. Math. Soc., 1975.

Received November 11, 2000.

Department of Numerical Analysis, Eötvös L. University,
Computer and Automation Research Institute of HAS, University of Pécs,
Pázmány P. sétány I/D, H-1117 Budapest, HUNGARY
E-mail address: schipp@ludens.elte.hu