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ON THE FEJÉR KERNEL FUNCTIONS WITH RESPECT TO THE WALSH–KACZMARZ SYSTEM

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Dedicated to Professor Árpád Varecza on the occasion of his 60th birthday

Abstract. Let G be the Walsh group. In this paper we prove that the integral of the maximal function of the Walsh–Kaczmarz–Fejér kernels is infinite on every interval. This is a sharp contrast with the Walsh–Paley system.

The Walsh system in the Kaczmarz enumeration was studied by a lot of authors (see [Sch1], [Sch2], [Sk1], [Sk2], [Bal], [SWS], [Wy]). In [Sne] it has been pointed out that the behavior of the Dirichlet kernel of the Walsh–Kaczmarz system is worse than of the kernel of the Walsh–Paley system considered more often. Namely, it is proved [Sne] that for the Dirichlet kernel $D_n(x)$ of the Walsh-Kaczmarz system the inequality $\limsup_{n\to\infty}\frac{D_n(x)}{\log n} \geq C > 0$ holds a.e. This "spreadness" of this system makes easier to construct examples of divergent Fourier series [Bal]. A number of pathological properties is due to this "spreadness" property of the kernel. For example, for Fourier series with respect to the Walsh–Kaczmarz system it is impossible to establish any local test for convergence at a point or on an interval, since the principle of localization does not hold for this system.

On the other hand, the global behavior of the Fourier series with respect to this system is similar in many aspects to the case of the Walsh–Paley system. Schipp [Sch2] and Wo-Sang Young [Wy] proved that the Walsh–Kaczmarz system is a convergence system. Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$ the set of nonnegative integers and Z_2 the discrete cyclic group of order 2, respectively. That is, $Z_2 = \{0, 1\}$ the group operation is the mod 2 addition and every subset is open. Haar measure is given in a way that the measure of a singleton is 1/2. Set

$$G := \mathop{\times}_{k=0}^{\infty} Z_2$$

complete direct product. Thus, every $x \in G$ can be represented by a sequence $x = (x_i, i \in \mathbf{N})$, where $x_i \in \{0, 1\}\ (i \in \mathbf{N})$. The group operation on G is the coordinate-wise addition, (which is the so-called logical addition) the measure (denoted by μ) and the topology are the product measure and topology. The compact Abelian group G is called the Walsh group. Set $e_i := (0, 0, \dots , 1, 0, 0, \dots) \in G$ the i-th coordinate of which is 1, the rest are zeros. A base for the neighborhoods of G can be given as follows

$$I_0(x) := G, \quad I_n(x) := \{y = (y_i, i \in \mathbf{N}) \in G : y_i = x_i \text{ for } i < n\}$$

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for $x \in G, n \in \mathbf{P}$. Let $0 = (0, i \in \mathbf{N}) \in G$ denote the null element of $G, I_n := I_n(0)\,(n \in \mathbf{N})$. Let $\mathcal{I} := \{I_n(x) : x \in G, n \in \mathbf{N}\}$. The elements of $\mathcal{I}$ are called the dyadic intervals on G. Furthermore, let $L^p(G)\,(1 \le p \le \infty)$ denote the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G, $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbf{N})\,(f \in L^1.)$ Define the Hardy space H^1 as follows. Let $f^* := \sup_{n\in\mathbf{N}} |E_n f|$ be the maximal function of the integrable function $f \in L^1(G)$. Then,

$$H^1(G) := \{f \in L^1(G) : f^* \in L^1(G)\},$$

moreover H^1 is a Banach space endowed with the norm $\|f\|_{H^1} := \|f^*\|_1$. Another definition is come: $a \in L^\infty(G)$ is called an atom, if either $a = 1$ or a has the following properties: $\operatorname{supp} a \subseteq I_a, \|a\|_\infty \le 1/\mu(I_a)$, $\int_I a = 0$, for some $I_a \in \mathcal{I}$. We say that the function f belongs to Hardy space $H(G)$, if f can be represented as $f = \sum_{i=0}^\infty \lambda_i a_i$, where a_i 's are atoms and for the coefficients $\lambda_i (i \in \mathbf{N})$ $\sum_{i=0}^\infty |\lambda_i| < \infty$ is true. It is known that $H(G)$ is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^\infty |\lambda_i|,$$

where the infimum is taken over all decompositions $f = \sum_{i=0}^\infty \lambda_i a_i \in H(G)$. Moreover, (cf. Theorem 3.6 in [SWS]), $H^1(G) = H(G)$ and

$$\|f\|_{H^1} \sim \|f\|_H.$$

Let $n \in \mathbf{N}$. Then $n = \sum_{i=0}^\infty n_i 2^i$, where $n_i \in \{0,1\}\,(n \in \mathbf{N})$, i.e. n is expressed in the number system based 2. Denote by $|n| := \max(j \in \mathbf{N} : n_j \ne 0)$, that is, $2^{|n|} \le n < 2^{|n|+1}$. The Rademacher functions are defined as:

$$r_n(x) := (-1)^{x_n} \quad (x \in G,\, n \in \mathbf{N}).$$

The Walsh–Paley system is defined as the set of Walsh–Paley functions:

$$\omega_n(x) := \prod_{k=0}^\infty (r_k(x))^{n_k} = (-1)^{\sum_{k=0}^{|n|} n_k x_k}, \quad (x \in G,\, n \in \mathbf{N}).$$

That is, $\omega := (\omega_n, n \in \mathbf{N})$. The n-th Walsh–Kaczmarz function is

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} \left(r_{|n|-1-k}(x)\right)^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}},$$

for $n \in \mathbf{P}$, $\kappa_0(x) := 1, x \in G$. The Walsh–Kaczmarz system $\kappa := (\kappa_n, n \in \mathbf{N})$ can be obtained from the Walsh–Paley system by renumbering the functions within the dyadic "block" with indices from the segment $[2^n, 2^{n+1} - 1]$. That is, $\{\kappa_n : 2^k \le n < 2^{k+1}\} = \{\omega_n : 2^k \le n < 2^{k+1}\}$ for all $k \in \mathbf{N}$, $\kappa_0 = \omega_0$. By means of the transformation $\tau_A : G \to G$

$$\tau_A(x) := (x_{A-1}, x_{A-2}, \dots, x_1, x_0, x_A, x_{A+1}, \dots) \in G,$$

which is clearly measure-preserving and such that $\tau_A(\tau_A(x)) = x$ we have

$$\kappa_n(x) = r_{|n|}(x)\omega_n(\tau_{|n|}(x)) \quad (n \in \mathbf{N}).$$

Let us consider the Dirichlet and the Fejér kernel functions:

$$D_n^\alpha := \sum_{k=0}^{n-1} \alpha_k,$$

$$K_n^\alpha := \frac{1}{n} \sum_{k=1}^{n} D_k^\alpha,$$

$K_0^\alpha = D_0^\alpha := 0$, where α is either κ or ω and $n \in \mathbf{P}$. The Fourier coefficients, the n-th partial sum of the Fourier series and the n-th Fejér mean of the Fourier series of $f \in L^1(G)$:

$$\hat{f}^\alpha(n) := \int_G f(x)\alpha_n(x)d\mu(x) \ (n \in \mathbf{N}),$$

$$S_n^\alpha f(y) := \sum_{k=0}^{n-1} \hat{f}^\alpha(k)\alpha_k(y) = \int_G f(x+y)D_n^\alpha(x)d\mu(x)$$

$$\sigma_n^\alpha f(y) := \frac{1}{n}\sum_{k=1}^{n} S_k f^\alpha(y) = \int_G f(x+y)K_n^\alpha(x)d\mu(x)$$

($n \in \mathbf{P}$, $S_0^\alpha f = 0$), where α is either κ or ω.

We say that the operator $T\colon L^1 \to L^0$ is of type (p,p) if $\|Tf\|_p \le c_p\|f\|_p$ for some constant c_p for all $f \in L^p(G)$ $(1 \le p \le \infty)$. T is said to be of type (H^1, L^1) if $\|Tf\|_1 \le c\|f\|_{H^1}$ for all $f \in H^1(G)$. Set $S^{*,\alpha} f := \sup_{n\in\mathbf{P}} |S_n^\alpha f|$ for $f \in L^1$, where α is ω or κ or any piecewise linear rearrangement of the Walsh–Paley system (κ is of this kind) (for the notion of piecewise linear rearrangement see [SWS]). Then, $S^{*,\alpha}$ is of type (p,p) for all $p \ge 2$ and for $f \in L^p$ $(p \ge 2)$ it follows $S_n f \to f$ a.e. [SWS, Theorem 6.10]. Moreover, if $\alpha = \kappa$, $f \in L^1(\log^+ L)^2$ (in particular if $f \in L^p$ for any $p > 1$), then the Walsh–Kaczmarz–Fourier series of f converges to f a.e. on G (cf. Theorem 6.11 in [SWS]).

Fine [Fin] proved every Walsh–Paley–Fourier series is a.e. (C,β) summable for $\beta > 0$. His argument is an adaptation of the older trigonometric analogue due to Marcinkiewicz [Mar]. Schipp [Sch3] gave a simpler proof for the case $\beta = 1$, i.e. $\sigma_n f \to f$ a.e. $(f \in L^1(G_m))$. He proved that σ^* is of weak type (L^1, L^1). That σ^* is of type (L^1, H^1) was discovered by Fujii [Fuj]. The theorem of Schipp and Fujii with respect to the character system of the group of 2-adic integers is proved by the author [Gát2]. The theorem of Schipp are generalized to the p-series fields by Taibleson [Tai2] and later to bounded Vilenkin systems by Pál and Simon [PS]. The almost everywhere convergence $\sigma_n f \to f$ for integrable function f on noncommutative bounded Vilenkin groups and the (L^1, H^1) typeness of the maximal operator is proved by the author [Gát6].

We remark that the "noncommutative case" differs from the "commutative case" in the view of many aspects. For instance there exsist some bounded noncommutative Vilenkin groups that the partial sums of the Fourier series does not converge to the function either in norm or a.e. for some $f \in L^p$, $p > 1$ [Gát6]. This is a sharp contrast.

Skvorcov proved for continuous functions f, that Fejér means converges uniformly to f. Gát proved [Gát4] for integrable functions that the Fejér means (with respect to the Walsh–Kaczmarz system) converges almost everywhere to the function. The two-dimensional Walsh–Paley and (bounded) Vilenkin case discussed by Weisz [W] and the author [Gát1, BG]. The conception of quasi-locality is introduced by F. Schipp [SWS]. Let $T\colon L^0 \to L^0$ and $f \in L^1(I)$, supp $f \subset I_k(x^0)$ for some $k \in \mathbf{N}, x^0 \in I$ and suppose that the integral of Tf on the set $I \setminus I_k(x^0)$ is bounded by $c\|f\|_1$. Then we call T quasi-local. Behind most of the proof of the pereceding results (one and two-dimension) (except the Walsh–Kaczmarz case) there is the quasi-locality of the maximal function of the Fejér means (i.e. the function $Tf := \sup_{n\in\mathbf{P}} |\sigma_n f|$). The quasi-locality is the consequence of the following lemma

Lemma. *$\int_{G\setminus I_k} \sup_{|n|\ge A} |K_n^\omega(x)|dx \le c\sqrt{2^{k-A}}$, for all $A \ge k \in \mathbf{N}$.*

(Consequently, $\int_{G\setminus I_k} \sup_{n\in\mathbf{N}} |K_n^\omega(x)|dx < \infty$ for all $k \in \mathbf{N}$.) The proof of this Lemma can be found for the Walsh–Paley system in [Gát3], for the Vilenkin system in [Gát5] and for the character system of the group of 2-adic integers in [Gát2]. The main aim of this paper is to prove that this Lemma does not hold for the Walsh–Kaczmarz system. We prove even more:

Theorem. $\int_{I_k(t)} \sup_{n\in\mathbf{N}} |K_n^\kappa(x)|dx = \infty$ *for all* $k \in \mathbf{N}$ *and* $t \in I$.

Theorem gives that the Lemma does not hold for the Walsh–Kaczmarz system. This is a very sharp contrast between the Walsh–Paley and the Walsh–Kaczmarz system. It is surprising a bit because these function systems are rearrangement one another. This also shows that to prove pointwise and norm convergence theorem with respect to the the Walsh–Kaczmarz need different techniques often. On the other hand,

Conjecture. $\sup_{n\in\mathbf{N}} |K_n^\kappa(x)| < \infty$ for a.e. $x \in I$. Moreover, for all $r < 1$ we have

$$\int_G \sup_{n\in\mathbf{N}} |K_n^\kappa(x)|^r dx < \infty.$$

Proof of the Theorem. Skvorcov in [Sk1] proved that for $n \in \mathbf{P}$, $x \in G$

$$\begin{aligned} nK_n^\kappa(x) &= 1 + \sum_{i=0}^{|n|-1} 2^i D_{2^i}(x) + \sum_{i=0}^{|n|-1} 2^i r_i(x) K_{2^i}^\omega(\tau_i(x)) \\ &\quad + (n - 2^{|n|})(D_{2^{|n|}}(x) + r_{|n|}(x) K_{n-2^{|n|}}^\omega(\tau_{|n|}(x))). \end{aligned}$$

Let $A := |n|$ and $n = 2^A + 2^{A-k-1}$. Then by the formula of Skvorcov we have

$$\begin{aligned} nK_n^\kappa(x) &= 1 + \sum_{i=0}^{A-1} 2^i D_{2^i}^\omega(x) + \sum_{i=0}^{A-1} 2^i r_i(x) K_{2^i}^\omega(\tau_i(x)) \\ &\quad + 2^{A-k-1}(D_{2^A}(x) + r_A(x) K_{2^{A-k-1}}^\omega(\tau_A(x))). \end{aligned}$$

Set $t^0 := t_0e_0 + \ldots + t_{k-1}e_{k-1}$. Thus, $I_k(t) = I_k(t^0)$. The author proved [Gát4, Corollary 6.] the following. Let $B, u \in \mathbf{N}$, $B > u$. Suppose that $x \in I_u \setminus I_{u+1}$.Then

$$K_{2^B}^\omega(x) = \begin{cases} 0 & \text{if } x - x_u e_u \notin I_B, \\ 2^{u-1} & \text{if } x - x_u e_u \in I_B. \end{cases}$$

If $x \in I_B$ then $K_{2^B}^\omega(x) = 2^{B-1} + \frac{1}{2}$. Since it is well-known that

$$D_{2^B}^\omega(x) = D_{2^B}^\kappa(x) = \begin{cases} 2^B & if\ x \in I_B, \\ 0 & if\ x \notin I_B. \end{cases}$$

Thus we have for $n = 2^A + 2^{A-k-1}$

$$\begin{aligned} nK_n^\kappa(x) &\geq \sum_{i=0}^{A-1} 2^i r_i(x) K_{2^i}^\omega(\tau_i(x)) \\ &\quad + 2^{A-k-1} r_A(x) K_{2^{A-k-1}}^\omega(\tau_A(x)) \end{aligned}$$

It is easy to prove

$$I_k(t) = \bigcup_{s=k}^{\infty} I_s(t^0) \setminus I_{s+1}(t^0) \cup \{t^0\}.$$

Let $x \in I_s(t^0) \setminus I_{s+1}(t^0)$, $A = s - 1$ and $s > 2k + 3$ (k is fixed). Set $\tau := \{i \in \mathbf{N} : t_i^0 = 1\}$. Then $\tau \subset \{0, 1, \ldots, k-1\}$. Since for $i \notin \tau$, $i \in \{0, 1, \ldots, A-1\}$ we have $r_i(x) = 1$ and consequently

$$2^i r_i(x) K_{2^i}^\omega(\tau_i(x)) \geq 0,$$

thus we have the following lower bound for $nK_n^\kappa(x)$.

$$\begin{aligned} nK_n^\kappa(x) &\geq -\sum_{i\in\tau} 2^i K_{2^i}^\omega(\tau_i(x)) + 2^{A-k-1} r_A(x) K_{2^{A-k-1}}^\omega(\tau_A(x)) \\ &\geq -\sum_{i=0}^{k-1} 2^i(2^{i-1}+\frac{1}{2}) + 2^{s-k-2} r_{s-1}(x) K_{2^{s-k-2}}^\omega(\tau_{s-1}(x)) \\ &\geq -4^k + 2^{s-k-2} K_{2^{s-k-2}}^\omega(\tau_{s-1}(x)). \end{aligned}$$

Since $x_k = x_{k+1} = \ldots = x_{s-1} = 0$ then we have $(\tau_{s-1}(x))_0 = x_{s-2} = 0$, $(\tau_{s-1}(x))_1 = x_{s-3} = 0, \ldots, (\tau_{s-1}(x))_{s-k-2} = x_k = 0$. This implies

$$\tau_{s-1}(x) \in I_{s-k-1}.$$

By this we obtain that

$$K_{2^{s-k-2}}^\omega(\tau_{s-1}(x)) = 2^{s-k-3} + \frac{1}{2}.$$

That is,

$$nK_n^\kappa(x) \geq -4^k + 2^{s-k-2}2^{s-k-3} \geq 2^{2s-2k-6}$$

$2s-2k-5 > 2k+12s-6 > 4ks-3 > 2k$ since $s > 2k+3$. This implies

$$\begin{aligned} \int_{I_k(t)} \sup_{n\in\mathbf{N}} |K_n^\kappa(x)|dx &\geq \sum_{s=2k+4}^{\infty} \int_{I_s(t^0)\setminus I_{s+1}(t^0)} \sup_{n\in\mathbf{N}} |K_n^\kappa(x)|dx \\ &\geq \sum_{s=2k+4}^{\infty} \int_{I_s(t^0)\setminus I_{s+1}(t^0)} 2^{2s-2k-6}/2^s dx \\ &\geq \sum_{s=2k+4}^{\infty} 2^{-2k-7} = \infty. \end{aligned}$$

This completes the proof of the Theorem. □

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