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ON THE RECTIFIABILITY CONDITION OF A SECOND ORDER ORDINARY DIFFERENTIAL EQUATION

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Dedicated to Professor Árpád Varecza on his 60th birthday

ABSTRACT. In this paper we wish to survey the rectifiability conditions of a second order differential equation, and we give some examples for a projectively flat two-dimensional Finsler space.

1. INTRODUCTION

In a famous book of Arnold [2] we can find the following theorem: "An equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$ can be reduced to the form $d^2\overline{y}/d\overline{x}^2 = 0$ if and only if the right-hand side is a polynomial in the derivative of order not greater 3 both for the equation and for its dual."

This theorem can be formulated in the following form on the basis of [4]: "An equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$ can be reduced to the form $d^2\overline{y}/d\overline{x}^2 = 0$ if and only if the path space P^2 (determined by the equation $d^2y/dx^2 = \Phi(x, y, dy/dx)$) is projectively related to a two-dimensional projectively flat Finsler space F^2."

The aim of this paper is to give some projectively flat two-dimensional Finsler spaces using this latter theorem.

2. NOTATIONS AND THEOREMS

Proposition 2.1. [1] *The second order differential equations*

$$d^2x^i/dt^2 = -2G^i(x, \dot{x});\ \dot{x}^i = dx^i/dt\ (i = 1, 2, \ldots, n)$$

give a path space P^n, where the functions $G^i(x, \dot{x})$ are positively homogeneous of degree two in $\dot{x}$.

Definition 2.2. [2] The integral curves of this second order differential equation are called paths.

Definition 2.3. [1] A Finsler space F^n is a pair (M^n, L), where M^n is a connected differentiable manifold of dimension n, $L(x, \dot{x})$ is the metrical function defined on the manifold TM/O of nonzero tangent vectors, and $L(x, \dot{x})$ is positively homogeneous of degree one in $\dot{x}$.

The differential equations of geodesic curves of F^n:

$$d^2x^i/dt^2 = -2G^i(x, \dot{x})\ ,$$

where

$$G^i = g^{ij}\left[\dot{x}^r \partial^2 L^2/\partial\dot{x}^j \partial x^r - \partial L^2/\partial x^j\right]\ ,$$

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and

$$g^{ij} = (g_{ij})^{-1} \quad ; \quad g_{ij} = \frac{1}{2}\partial^2 L^2/\partial\dot{x}^i\partial\dot{x}^j .$$

Definition 2.4. [1]A path space P^n and a Finsler space F^n are called projectively related to each other, if any path of P^n is a geodesic curve of F^n and vice versa.

Theorem 2.5. [5] *In two-dimensions any path space is projectively related to a two dimensional Finsler space.*

Definition 2.6. [1] A Finsler space is called projectively flat, if it is covered by coordinate neighborhoods in which any geodesic is represented by linear equations.

Theorem 2.7. [1] *If a path space P^n projectively is related to a Finsler space F^n, then we have two invariant tensors, called the Weyl and the Douglas tensor respectively.*

Definition 2.8. [3] A Finsler space is said to be a Douglas space, if the Douglas tensor D^h_{ijk} vanishes identically, where

$$D^h_{ijk} = \partial^3 Q^h/\partial\dot{x}^k\partial\dot{x}^j\partial\dot{x}^i$$

with $Q^h = G^h - \dot{x}^h G\dfrac{1}{n+1} \quad ; \quad G = G^r_r \quad ; \quad G^i_j = \partial G^i/\partial\dot{x}^j .$

Theorem 2.9. [3] *A two-dimensional Finsler space F^2 is a Douglas space if and only if (in a local coordinate system (x,y)) the right-hand side $\Phi(x,y,dy/dx)$ of the equation of geodesics $y'' = \Phi(x,y,dy/dx)$ is a polynomial in $dy/dx = y'$ of degree at most three.*

From the previous Theorems and Definitions we obtain

THEOREM 1. *An equation $d^2y/dx^2 = \Phi(x,y,y')$ can be reduced to the form $d^2\overline{y}/d\overline{x}^2 = 0$ if the pathspace P^2 (determined by the equation $d^2y/dx^2 = \Phi(x,y,y')$) is projective related to a two-dimensional Douglas space.*

Some examples can be found in the papers [3] and [4] for the Douglas spaces.

Theorem 2.10. [4] *A Finsler space F^2 (a two-dimensional Finsler space) is a projectively flat space if and only if F^2 is a Douglas space and satisfies $\Pi_{ijk} = 0$, where $\Pi_{ijk} = \partial Q_{ij}/\partial x^k + Q^r_{ij}Q_{rk} - [ij]$. The tensor $Q_{ij} = Q^r_{ijr}$, where in a Douglas space*

$$Q^h_{ijk} = \partial Q^h_{ij}/\partial x^k + Q^r_{ij}Q^h_{rk} - [ij] \quad , \quad Q^h_{ij} = \frac{\partial^2\left[G^h - \dot{x}^h G\frac{1}{n+1}\right]}{\partial\dot{x}^i\partial\dot{x}^j} .$$

Theorem 2.11. [4] *A two-dimensional Finsler space F^2 is a Douglas space if and only if the differential equation of F^2 has the form*

$$\begin{aligned} y'' &= k(x,y)(y')^3 + h(x,y)(y')^2 + g(x,y)y' + f(x,y) = \\ &= Q^1_{22}(y')^3 - (Q^2_{22} - 2Q^1_{12})(y')^2 - (2Q^2_{12} - Q^1_{11})y' + Q^2_{11} . \end{aligned}$$

Theorem 2.12. [4] *A two dimensional Douglas space is projectively flat if and only if $\Pi_{112} = 0$ and $\Pi_{212} = 0$.*

THEOREM 2. *In a Douglas space*

$$\Pi_{112} = -f_{yy}+\frac{2}{3}g_{xy}-\frac{1}{3}gg_y+fh_y+hf_y-\frac{1}{3}h_{xx}+\frac{1}{3}h_xg+kf_x-\frac{2}{27}g^2h+\frac{2}{3}gkh-\frac{2}{3}fhk ,$$

$$\begin{aligned} \Pi_{212} = &- \frac{1}{3}g_{yy} + \frac{2}{3}h_{xy} - \frac{1}{3}g_yh + k_yf + 2kf_y - k_{xx} + \frac{2}{3}hh_x + \frac{2}{3}h_xk - \frac{4}{27}gh^2 + \\ &+ \frac{2}{3}hk_x - \frac{1}{3}gk_x - \frac{1}{3}g_xk - \frac{2}{9}hgk + \frac{2}{9}g^2k , \end{aligned}$$

where $f_x = \partial f/\partial x$, $f_y = \partial f/\partial y$, ...

3. Some examples

Assume that a two-dimensional Finsler space F^2 on a domain of the (x, y)-plane has the geodesics given by the equations:

1. $y'' = f(x, y)$,
2. $y'' = g(x, y)y' + f(x, y)$,
3. $y'' = k(x, y)(y')^3$,
4. $y'' = h(x, y)(y')^2$,
5. $y'' = g(x, y)y'$.

The components of the tensor Π are the following in these cases:

1. $\Pi_{112} = -f_{yy}$; $\Pi_{212} = 0$,
2. $\Pi_{112} = -f_{yy} + \frac{2}{3}g_{xy} - \frac{1}{3}gg_y$; $\Pi_{212} = -\frac{1}{3}g_{yy}$,
3. $\Pi_{112} = 0$; $\Pi_{212} = -k_{xx}$,
4. $\Pi_{112} = -\frac{1}{3}h_{xx}$; $\Pi_{212} = h_{xy} + hh_x$,
5. $\Pi_{112} = \frac{2}{3}g_{xy} - \frac{1}{3}gg_y$; $\Pi_{212} = -\frac{1}{3}g_{yy}$.

Consequently, F^2 is projectively flat, if and only if

1. $f(x, y) = A(x)y + B(x)$,
2. $f(x, y) = \sigma_1(x)y^3 + \sigma_2(x)y^2 + \sigma_3(x)y + \sigma_4(x)$; $g(x, y) = \alpha(x)y + \beta(x)$,
3. $k(x, y) = C(y)x + D(y)$,
4. $h(x, y) = E(y)x + F(y)$, where $dE/dy + (Ex + F)E = 0$,
5. $g(x, y) = \gamma(x)y + \delta(x)$, where $\frac{2}{3}dy/dx - \frac{1}{3}(\gamma y + \delta)\gamma = 0$.

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