Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
17 (2001), 131–135
www.emis.de/journals

ON THE CHERN–WEIL HOMOMORPHISM IN FINSLER SPACES

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Dedicated to Professor Árpád Varecza on the occasion of his 60th birthday

Abstract. The aim of this paper is to devise a Chern – Weil-type construction for a Finsler manifold $(M, \mathcal{L})$ which is determined only by the manifold M and by the Finslerian fundamental function $\mathcal{L}$.

1. Introduction

The focus in this paper is to set up a framework in which the famous Chern–Weil homomorphism can be formulated on a Finsler manifold. Most of the basic notations in this paper are the same as in [GHV73]. Background information on Finsler geometry can be found e.g. in [Mat86] and [AP94].

Let $(M, \mathcal{L})$ be a Finsler space, the horizontal projection determined by the Finslerian fundamental function $\mathcal{L}$ is h. h can be interpreted as a τ_{TM}-valued 1-form on TM: $h \in \mathrm{A}^1(TM; \tau_{TM}) \cong \mathrm{Hom}(\tau_{TM}; \tau_{TM})$. The horizontal subbundle of τ_{TM} will be denoted by HM, $\mathrm{Sec}\, HM = \mathfrak{X}_h(TM)$.

Denote by $(\mathrm{A}(TM), \wedge)$ the graded algebra of differential forms on TM. From h one can derive a first order graded derivative $d_h \colon \mathrm{A}(TM) \to \mathrm{A}(TM)$

$$\begin{aligned} d_h\omega(X_0, \dots, X_p) = \\ = \; & \sum_{i=0}^{p} (-1)^i hX_i\omega(X_0, \dots, \widehat{X}_i, \dots, X_p) + \\ & + \sum_{i<j} (-1)^{i+j} \omega\left([X_i, X_j]_h, X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_p\right) \end{aligned}$$

where $\omega \in \mathrm{A}^p(TM)$ $(p \geq 1)$ is a p-form, $X_i \in \mathfrak{X}(TM)$ $(i = 0 \dots p)$, $[X, Y]_h = [hX, Y] + [X, hY] - h[X, Y]$, furthermore $d_h f(X) = (hX)f$ $(f \in \mathrm{A}^0(TM) \equiv C^\infty(TM))$ ([FN56] or [Mic87]). It is easy to see that $d_h^2 = 0$ iff the Frölicher–Nijenhuis bracket of the operator pair (h, h) is zero: $[h, h] = 0$. In the Finslerian case this condition means that the torsion R^1 of the unique Cartan connection vanishes, i.e. the horizontal distribution is integrable.

In the Finslerian case this special situation was studied in [ACD87] and their main result is the following:

Theorem. *If $R^1 = 0$ then the cohomology groups of d_h are isomorphic to the de Rham cohomology groups of an integral manifold of the nonlinear connection associated to $\mathcal{L}$.*

2000 *Mathematics Subject Classification.* 53C05.

Key words and phrases. Chern–Weil homomorphism, twisted cohomology, Frölicher–Nijenhuis theory.

This research was supported by the grant FKFP 0690/99.

In this paper we do not suppose the integrability of the horizontal distribution.

2. Tools

Forms. Let $\nabla^C \colon \mathfrak{X}(TM) \times \mathfrak{X}_h TM \to \mathfrak{X}_h TM$ be the Cartan connection of the Finsler space $(M, \mathcal{L})$. Then (∇, h), where $\nabla \colon \mathfrak{X}(TM) \times \mathfrak{X}_h TM \to \mathfrak{X}_h TM$, $\nabla_X Y = \nabla^C{}_{hX} Y$, is the so called *h-connection* of the Finsler space.

By easy calculations, one can show the following statement:

Proposition 1. *Let* (∇, h) *be the h-connection of the Finsler space. The map*

$$\nabla \colon A(TM; HM) \to A(TM; HM)\,,$$

$$(\nabla\Psi)(X_0, \dots, X_p) = \sum_{i=0}^{p} (-1)^i \nabla_{X_i} \Psi(X_0, \dots, \widehat{X}_i, \dots, X_p) + \\ + \sum_{i<j} (-1)^{i+j} \Psi([X_i, X_j]_h, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_p)$$

($\Psi \in A^p(TM; HM)$ $(p > 0)$*; for* $p = 0:$ $(\nabla\sigma)(X) = \nabla_X \sigma$*) is a first order graded derivation of the graded algebra of* HM*-valued forms on* TM *in the sense of* [Mic87] *i.e.* $\nabla(\omega \wedge \Psi) = d_h\omega \wedge \Psi + (-1)^p \omega \wedge \nabla\Psi;\ \omega \in A^p(TM),\ \Psi \in A(TM; HM)$.

We will use the following construction in the next section. Let $\xi_0, \xi_1, \dots, \xi_m$ be vector bundles with the same base B and let $\phi \in \mathrm{Hom}(\xi_1, \dots, \xi_m; \xi_0)$. ϕ determines a map $\phi_* \in \mathrm{Hom}(\mathrm{A}(B;\xi_1), \dots, \mathrm{A}(B;\xi_m); \mathrm{A}(B;\xi_0))$ as follows.

$$\phi_*(\sigma_1, \dots, \sigma_m) = \phi(\sigma_1, \dots, \sigma_m)$$

for $\sigma_i \in \mathrm{A}^0(B;\xi_i) \equiv \mathrm{Sec}\xi_i$ and for elements in $\mathrm{A}^{p_i}(B;\xi_i)$ this map is determined by

$$\phi_*(\omega_1 \wedge \sigma_1, \dots, \omega_m \wedge \sigma_m) = (\omega_1 \wedge \dots \wedge \omega_m) \wedge \phi_*(\sigma_1, \dots, \sigma_m)$$

where $\omega_i \in \mathrm{A}(B)$, $\sigma_i \in \mathrm{A}^0(B;\xi_i)$ $(i = 1 \dots m)$. ϕ_* satisfies the following identity:

$$\phi_*(\Psi_1, \dots, \omega \wedge \Psi_i, \dots, \Psi_m) = (-1)^{q_i q} \omega \wedge \phi_*(\Psi_1, \dots, \Psi_m) \tag{1}$$

where $\Psi_i \in \mathrm{A}^{p_i}(B;\xi_i)$, $q_i = p_1 + \dots + p_{i-1}$ $(i \geq 2)$, $q_1 = 0$, $\omega \in \mathrm{A}^q(B)$, and moreover,

$$\phi_*(\Psi_1, \dots, \Psi_m)(X_1, \dots, X_p) = \\ = \frac{1}{p_1! \cdots p_m!} \sum_{\sigma \in \mathfrak{S}_p} \varepsilon(\sigma) \phi(\Psi_1(X_{\sigma(1)}, \dots), \dots, \Psi_m(\dots, X_{\sigma(p)})) \tag{2}$$

where $\Psi_i \in \mathrm{A}^{p_i}(B;\xi_i)$, $X_i \in \mathfrak{X}(B)$ $(i = 1 \dots p)$, $p = p_1 + \dots + p_m$.

Invariant polynomials. Let F be a real vector space. An *invariant polynomial of degree* p is a symmetric map

$$f_F^p \in \mathrm{Hom}(\mathrm{L}(\overset{1}{\smile}\!\!\!\!\!F; F), \dots, \mathrm{L}(\overset{p}{\smile}\!\!\!\!\!F; F); \mathbb{R})$$

such that for all $a \in \mathrm{GL}(F)$

$$f_F^p(\mathrm{Ad}\,(a)\alpha_1, \dots, \mathrm{Ad}\,(a)\alpha_p) \quad = \quad f_F^p(\alpha_1, \dots, \alpha_p) \tag{3}$$

where $\alpha_i \in \mathrm{L}(F;F)$ $(i = 1 \dots p)$ is a linear operator and $\mathrm{Ad} : \mathrm{Gl}(F) \to \mathrm{Gl}(\mathrm{L}(F;F))$ is the adjoint representation. By the invariance condition (3) one can extend f_F^p to the bundle of linear operators over the vector bundle ξ with base manifold B and the typical fiber F.

$$f^p \in \mathrm{Hom}(\overset{1}{\smile}\!\!\!\!\!\mathrm{L}_\xi, \dots, \overset{p}{\smile}\!\!\!\!\!\mathrm{L}_\xi; B \times \mathbb{R}) \cong \mathrm{Sec}(\overset{1}{\smile}\!\!\!\!\!\mathrm{L}_\xi \otimes \dots \otimes \overset{p}{\smile}\!\!\!\!\!\mathrm{L}_\xi)^*.$$

This f^p is called invariant polynomial in ξ of degree p.

Curvature. Let $R^2 \in \mathrm{A}^2(TM; \mathrm{L}_{HM})$ denote the curvature of the Cartan connection.

Proposition 2. *The h-Finsler connection (∇, h) in HM satisfies:*

$$\left[\left(\nabla R^2\right)(X,Y,Z)\right](W) = \underset{(X,Y,Z)}{\mathfrak{S}} \left\{P^2(R^1(X,Y)(hZ)(W))\right\}$$

where P^2 is the hv-curvature, $R^1 = \frac{1}{2}[h,h]$ and $\mathfrak{S}_{(X,Y,Z)}$ is the symbol of the cyclic sum with respect to X, Y, Z.

3. Construction of d_h-closed forms

Theorem. *Let (∇, h) be the h-Finsler connection, f^p an invariant polynomial in HM. If $\nabla R^2 = 0$ then $d_h f^p_*(R^2, \ldots, R^2) = 0$, i.e. $f^p_*(R^2, \ldots, R^2)$ is a d_h-closed $2p$-form.*

Proof. We have found the adequate ideas, so the proof of the theorem is quite easy. First we prove the following statement:

Lemma. *Let $f \in Hom(\overset{1}{\smile}{L_{HM}}, \ldots, \overset{p}{\smile}{L_{HM}}; TM \times \mathbb{R}) \cong Sec(\overset{1}{\smile}{L_{HM}} \otimes \cdots \otimes \overset{p}{\smile}{L_{HM}})^*$. If $\nabla_X f = 0$ for any $X \in \mathfrak{X}(TM)$ then*

$$d_h f_*(\Omega_1, \ldots, \Omega_p) = \sum_{i=1}^{p} (-1)^{q_i} f_*(\Omega_1, \ldots, \nabla\Omega_i, \ldots, \Omega_p), \tag{4}$$

where $\Omega_i \in A^{r_i}(TM; L_{HM})$ $(i = 1 \ldots p)$, $q_i = r_1 + \cdots + r_{i-1}$ $(i = 2 \ldots p)$, $q_1 = 0$. (Concerning $f_* \in Hom(\overset{1}{\smile}{A(TM; L_{HM})}, \ldots, \overset{p}{\smile}{A(TM; L_{HM})}; A(TM))$ see (2)!)

Clearly, $\mathrm{A}^{r_i}(TM; \mathrm{L}_{HM}) \cong \mathrm{A}^{r_i}(TM) \otimes \mathrm{Sec}\mathrm{L}_{HM}$. If $\alpha_i \in \mathrm{Sec}\mathrm{L}_{HM}$ $(i = 1 \ldots p)$ then (4) reduces to:

$$d_h f(\alpha_1, \ldots, \alpha_p) = \sum_{i=1}^{p} f_*(\alpha_1, \ldots, \nabla\alpha_i, \ldots, \alpha_p).$$

We have $(d_h f(\alpha_1, \ldots, \alpha_p))(X) = hXf(\alpha_1, \ldots, \alpha_p)$. On the other hand,

$$\sum_{i=1}^{p} f_*(\alpha_1, \ldots, \nabla\alpha_i, \ldots, \alpha_p)(X) \overset{(2)}{=} \sum_{i=1}^{p} f(\alpha_1, \ldots, (\nabla\alpha_i)(X), \ldots, \alpha_p).$$

Together with the previous line this proves the statement.

Let $\Omega_i \in \mathrm{A}^{r_i}(TM; \mathrm{L}_{HM})$ $(i = 1 \ldots p)$, $\Omega_i = \omega_i \wedge \alpha_i$ $(\omega_i \in \mathrm{A}^{r_i}(TM)$ $i = 1 \ldots p)$, and $q = r_1 + \cdots + r_p$. By induction we infer

$$\begin{aligned} & d_h f_*(\omega_1 \wedge \alpha_1, \ldots, \omega_p \wedge \alpha_p) \overset{(1)}{=} \\ = \; & \sum_{i=1}^{p} (-1)^{q_i} \omega_1 \wedge \ldots \wedge d_h\omega_i \wedge \ldots \wedge \omega_p \wedge f_*(\alpha_1, \ldots, \alpha_p) + \\ & + (-1)^q \omega_1 \wedge \ldots \wedge \omega_p \wedge d_h f_*(\alpha_1, \ldots, \alpha_p). \end{aligned}$$

Similarly,

$$\begin{aligned} & f_*(\omega_1 \wedge \alpha_1, \ldots, \nabla(\omega_i \wedge \alpha_i), \ldots, \omega_p \wedge \alpha_p) = \\ = \; & \omega_1 \wedge \ldots \wedge d_h\omega_i \wedge \ldots \wedge \omega_p \wedge f_*(\alpha_1, \ldots, \alpha_p) + \\ & + (-1)^{r_i}(-1)^{r_{i+1}} \cdots (-1)^{r_p} \omega_1 \wedge \ldots \wedge \omega_p \wedge f_*(\alpha_1, \ldots, \nabla\alpha_i, \ldots, \alpha_p). \end{aligned}$$

We proved the lemma.

Now, for an invariant polynomial f^p, $\nabla_X f^p = {\nabla^C}_{hX} f^p = 0$ and applying the lemma for $\Omega_i = R^2$ we get the statement of the theorem. □

4. Remarks

Pseudocomplexes. For d_h we have a sequence of graded vector spaces

$$(\mathsf{PS}) \qquad \cdots \longrightarrow \mathrm{A}^{p-1}(TM) \xrightarrow{d_h} \mathrm{A}^{p}(TM) \xrightarrow{d_h} \mathrm{A}^{p+1}(TM) \longrightarrow \cdots$$

where $d_h \circ d_h$ is not necessarily zero. Following I. Vaisman [Vai68], for (PS) we use the name of *pseudocomplex*. Of course, when $[h,h]=0$ then (PS) is a usual cochain complex.

In the case of non-vanishing d_h^2 the most natural way to define cohomology groups is by

$$H^p(d_h, TM) = {}^{\operatorname{Ker} d_h}\!\big/_{\operatorname{Im} d_h \cap \operatorname{Ker} d_h}.$$

These $H^p(d_h,TM)$ cohomology groups are usual cohomology groups of several cochain complexes. We put

$$(\widetilde{\mathsf{PS}}) \qquad \cdots \longrightarrow \mathrm{A}^{\widetilde{p-1}(TM)} \xrightarrow{\tilde{d}_h} \mathrm{A}^{\widetilde{p}(TM)} \xrightarrow{\tilde{d}_h} \mathrm{A}^{\widetilde{p+1}(TM)} \longrightarrow \cdots$$

where

$$\mathrm{A}^{\widetilde{p}(TM)} = \operatorname{Ker} d_h \circ d_h.$$

and $\tilde{d}_h$ is the restriction of d_h to $\mathrm{A}^{\widetilde{p}(TM)}$. Then it is easy to check that in the case of $(\widetilde{\mathsf{PS}})$ $\tilde{d}_h^2 = 0$ holds and the cochain complex $(\widetilde{\mathsf{PS}})$ has the same cocycles and coboundaries as the pseudocomplex (PS) itself ([HL75], [Vai93]).

Finsler spaces with the condition $\nabla R^2 = 0$. There are several examples for Finsler spaces with vanishing curvature P^2. This condition implies the required identity $\nabla R^2 = 0$, c.f. Proposition 2. These spaces are the so called *Landsberg* spaces ([Koz96], [Mat96]).

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Received November 4, 2000

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