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BEST APPROXIMATION BY VILENKIN-LIKE SYSTEMS

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Abstract. In this paper we give a common generalization of the Walsh, Vilenkin system, the character system of the group of 2-adic (m-adic) integers, the product system of normalized coordinate functions of continuous irreducible unitary representations of the coordinate groups of noncommutative Vilenkin groups, the UDMD product systems (defined by F. Schipp) and some other systems. We introduce the notion of the modulus of continuity on Vilenkin spaces, the concept of the best approximation by Vilenkin-like polynomials. We prove a Jackson type theorem.

Denote by $\mathbf{N}$ the set of natural numbers, $\mathbf{P}$ the set of positive integers, respectively. Denote $m := (m_k : k \in \mathbf{N})$ a sequence of positive integers such that $m_k \geq 2$, $k \in \mathbf{N}$ and G_{m_k} a set of cardinality m_k. Suppose that each (coordinate) set has the discrete topology and measure μ_k which maps every singleton of G_{m_k} to $\frac{1}{m_k}$ ($\mu_k(G_{m_k}) = 1$), $k \in \mathbf{N}$. Let G_m be the compact set formed by the complete direct product of G_{m_k} with the product of the topologies and measures (μ). Thus each $x \in G_m$ is a sequence $x := (x_0, x_1, \ldots)$, where $x_k \in G_{m_k}$, $k \in \mathbf{N}$. G_m is called a Vilenkin space. G_m is a compact totally disconnected space, with normalized regular Borel measure μ, $\mu(G_m) = 1$. A base for the neighborhoods of G_m can be given as follows

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbf{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbf{P}$.

$$\mathcal{I} := \{I_n(x) : n \in \mathbf{N}, x \in G_m\}$$

is the set of intervals on G_m.

Denote by $L^p(G_m)$ the usual Lebesgue spaces ($\|.\|_p$ the corresponding norms) ($1 \leq p \leq \infty$), $\mathcal{A}_n$ the σ algebra generated by the sets $I_n(x)\,(x \in G_m)$ and E_n the conditional expectation operator with respect to $\mathcal{A}_n\,(n \in \mathbf{N})$.

If the sequence m is bounded, then we call G_m a bounded Vilenkin space. If this is not the case then we call it an unbounded Vilenkin space.

Let $M_0 := 1$ and $M_{k+1} := m_k M_k$, for $k \in \mathbf{N}$ be the so-called generalized powers. Then every $n \in \mathbf{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, $0 \leq n_k < m_k$, $n_k \in \mathbf{N}$. This allows one to say that the sequence $(n_0, n_1, \ldots)$ is the expansion of n with respect to m. We often use the following notations. Let $|n| := \max\{k \in \mathbf{N} : n_k \neq 0\}$ (that is, $M_{|n|} \leq n < M_{|n|+1}$) and $n^{(k)} = \sum_{j=k}^{\infty} n_j M_j$. Next we introduce on G_m an orthonormal system we call Vilenkin-like system.

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The complex valued functions which we call the generalized Rademacher functions $r_k^n : G_m \to \mathbf{C}$ have these properties:

i. r_k^n is $\mathcal{A}_{k+1}$ measurable (i.e. $r_k^n(x)$ depends only on $x_0, \dots, x_k$ $(x \in G_m)$), $r_k^0 = 1$ for all $k, n \in \mathbf{N}$.

ii. If M_k is a divisor of n and l and if $n^{(k+1)} = l^{(k+1)}$ $(k, l, n \in \mathbf{N})$, then

$$E_k(r_k^n \bar{r}_k^l) = \begin{cases} 1 \text{ if } n_k = l_k, \\ 0 \text{ if } n_k \neq l_k \end{cases}$$

($\bar{z}$ is the complex conjugate of z).

iii. If M_{k+1} is a divisor of n, then

$$\sum_{j=0}^{m_k-1} |r_k^{jM_k+n}(x)|^2 = m_k$$

for all $x \in G_m$.

iv. There exists a $\delta > 1$ for which $\|r_k^n\|_\infty \leq \sqrt{m_k/\delta}$.

Define the Vilenkin-like system $\psi = (\psi_n : n \in \mathbf{N})$ as follows.

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}}, \quad n \in \mathbf{N}.$$

(Since $r_k^0 = 1$, then $\psi_n = \prod_{k=0}^{|n|} r_k^{n^{(k)}}$).

Example A, the Vilenkin and the Walsh system. Let $G_{m_k} := Z_{m_k}$ be the m_k-th ($2 \leq m_k \in \mathbf{N}$) discrete cyclic group ($k \in \mathbf{N}$). That is Z_{m_k} can be represented by the set $\{0, 1, \dots, m_k - 1\}$, where the group operation is the mod m_k addition and every subset is open. The group operation on G_m $(+)$ is the coordinate-wise addition. G_m is called a Vilenkin group. The Vilenkin group for which $m_k = 2$ for all $k \in \mathbf{N}$ is the Walsh-Paley group. In this case let $r_k^n(x) := (\exp(2\pi \imath x_k/m_k))^{n_k}$, where $\imath := \sqrt{-1}, x \in G_m$. The system $\psi := (\psi_n : n \in \mathbf{N})$ is the Vilenkin system, where $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k M_k}$. In the case of the Vilenkin group, $m_k = 2$ for all $k \in \mathbf{N}$, we get the Walsh-Paley system. Since $|r_k^n| = 1$, *iii* and *iv* are trivial and so are *i* and *ii*. For more on Vilenkin and Walsh system and group see e.g. [SWS, AVD].

Example B, the group of 2-adic (m-adic) integers. Let $G_{m_k} := \{0, 1, \dots, m_k - 1\}$ for all $k \in \mathbf{N}$. Define on G_m the following (commutative) addition: Let $x, y \in G_m$. Then $x + y = z \in G_m$ is defined in a recursive way. $x_0 + y_0 = t_0 m_0 + z_0$, where (of course) $z_0 \in \{0, 1, \dots, m_0 - 1\}$ and $t_0 \in \mathbf{N}$. Suppose that $z_0, \dots, z_k$ and $t_0, \dots, t_k$ have been defined. Then write $x_{k+1} + y_{k+1} + t_k = t_{k+1} m_{k+1} + z_{k+1}$, where $z_{k+1} \in \{0, 1, \dots, m_{k+1} - 1\}$ and $t_{k+1} \in \mathbf{N}$. Then G_m is called the group of m-adic integers (if $m_k = 2$ for all $k \in \mathbf{N}$, then 2-adic integers). In this case let

$$r_k^n(x) := \left(\exp\left(2\pi\imath\left(\frac{x_k}{m_k} + \frac{x_{k-1}}{m_k m_{k-1}} + \dots + \frac{x_0}{m_k m_{k-1} \dots m_0}\right)\right)\right)^{n_k}.$$

Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k M_k}$. Then the system $\psi := (\psi_n : n \in \mathbf{N})$ is the character system of the group of m-adic (if $m_k = 2$ for each $k \in \mathbf{N}$ then 2-adic) integers. Since $|r_k^n| = 1$, *i*, *iii* and *iv* are trivial. *ii* is also easy to see and well-known [SW2, p. 91]. For more on the group of m-adic (if $m_k = 2$ for each $k \in \mathbf{N}$ then 2-adic) integers see e.g. [HR, SW2, T].

Example C, noncommutative Vilenkin groups. Let σ be an equivalence class of continuous irreducible unitary representations of a compact group G. Denote by Σ the set of all such σ. Σ is called the dual object of G. The dimension of a representation $U^{(\sigma)}$, $\sigma \in \Sigma$, is denoted by d_σ and let

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle \qquad i,j \in \{1,\dots,d_\sigma\}$$

be the coordinate functions for $U^{(\sigma)}$, where $\xi_1,\dots,\xi_{d_\sigma}$ is an orthonormal basis in the representation space of $U^{(\sigma)}$. (For the notations see [HR, vol 2, p. 3].) According to the Weyl-Peter's theorem (see e.g. [HR, vol 2, p. 24]), the system of functions $\sqrt{d_\sigma} u_{i,j}^{(\sigma)}$, $\sigma \in \Sigma$, $i,j \in \{1,\dots,d_\sigma\}$ is an orthonormal base for $L^2(G)$. If G is a finite group, then Σ is finite too. If $\Sigma := \{\sigma_1,\dots,\sigma_s\}$, then $|G| = d_{\sigma_1}^2 + \dots + d_{\sigma_s}^2$.

Let G_{m_k} be a finite group with order m_k, $k \in \mathbf{N}$. Let $\{r_k^{sM_k} : 0 \le s < m_k\}$ be the set of all normalized coordinate functions of the group G_{m_k} and suppose that $r_k^0 \equiv 1$. Thus for every $0 \le s < m_k$ there exist a $\sigma \in \Sigma_k$, $i,j \in \{1,\dots,d_\sigma\}$ such that

$$r_k^{sM_k} = \sqrt{d_\sigma} u_{i,j}^{(\sigma)}(x) \qquad (x \in G_{m_k}),$$

$r_k^n := r_k^{n_k M_k}$. Let ψ be the product system of r_k^j, namely

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n^{(k)}}(x_k) \qquad (x \in G_m),$$

where n is of the form $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, \dots)$. In [GT, Gát4] it is proved that the system ψ satisfies the properties i, ii, iii. If $\sup_k m_k < \infty$, then iv is satisfied [Gát4]. For more on this system and noncommutative Vilenkin groups see [GT, Gát2, Gát4].

Example D, a system in the field of number theory. Let

$$r_k^n(x) := \exp\left(2\pi\imath \sum_{j=k}^{\infty} \frac{n_j}{M_{j+1}} \sum_{i=0}^{k} x_i M_i\right)$$

for $k,n \in \mathbf{N}$ and $x \in G_m$. Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}}$, $n \in \mathbf{N}$.

Then, $\psi := (\psi_n : n \in \mathbf{N})$ is a Vilenkin-like system (introduced in [Gát3]) which is a useful tool in the approximation theory of limit periodic, almost even arithmetical functions [Gát3]. This system (on Vilenkin groups) was a new tool in order to investigate limit periodic arithmetical functions. For the definition of these arithmetical functions see also the book of Mauclaire [Mau, p. 25].

Example E, the UDMD product system. The notion of the UDMD product system is introduced by F. Schipp [SW2, p. 88] on the Walsh-Paley group. Let functions $\alpha_k : G_m \to \mathbf{C}$ satisfy: $|\alpha_k| = 1$ and α_k is $\mathcal{A}_k$ measurable. Let $r_k^n(x) := (-1)^{x_k n_k} \alpha_k(x)$. i is trivial and since $|r_k^n| = 1$, so are iii and iv. To prove ii is simple. Let $\psi_n := \prod_{k=0}^{\infty} r_k^{n^{(k)}} = \prod_{k=0}^{\infty} r_k^{n_k M_k}$ $(n \in \mathbf{N})$. The system $\psi := (\psi_n : n \in \mathbf{N})$ is called an UDMD product system. For more on UDMD product systems see [SW, SW2].

Example F, the universal contractive projections. Let $\phi_n : G_m \to \mathbf{C}$ $(n \in \mathbf{N})$ be measurable functions with $|\phi_n| = 1$ $(n \in \mathbf{N})$ and $\phi_0 = 1$. The notion of universal contractive projection system (UCP) is introduced by F. Schipp [Sch4] as follows. Let $f \in L^1(G_m)$ and $P_{n^{(s)}} f := \phi_{n^{(s)}} E_s(f \bar\phi_{n^{(s)}})$ for $n,s \in \mathbf{N}$. Then let [Sch4] $P_{n^{(s)}} = P_{n^{(s)}} P_{n^{(s+j)}} = P_{n^{(s+j)}} P_{n^{(s)}}$ for all $j \in \mathbf{N}$. Moreover, if $n^{(s)}$ and $k^{(t)}$ are incomparable, that is, there are no $j \in \mathbf{N}$ such that $n^{(s+j)} = k^{(t)}$ or $k^{(t+j)} = n^{(s)}$, then let $P_{n^{(s)}} P_{k^{(t)}} = P_{k^{(t)}} P_{n^{(s)}} = 0$. In [Gát4] it is proved that the system $(\phi_n : n \in \mathbf{N})$ is also a Vilenkin-like system.

For $f \in L^1(G_m)$ we define the Fourier coefficients and partial sums by

$$\widehat{f}(k) := \int_{G_m} f \overline{\psi}_k d\mu \qquad (k \in \mathbf{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k)\psi_k \qquad (n \in \mathbf{P},\ S_0 f := 0).$$

The Dirichlet kernels:

$$D_n(y,x) := \sum_{k=0}^{n-1} \psi_k(y)\overline{\psi}_k(x) \qquad (n \in \mathbf{P},\ D_0 := 0).$$

It is clear that

$$S_n f(y) = \int_{G_m} f(x) D_n(y,x) d\mu(x).$$

Denote by

$$\sigma_n f = \frac{1}{n}\sum_{k=0}^{n-1} S_k f \qquad (n \in \mathbf{P},\ \sigma_0 f := 0)$$

Proposition 1 (Gát4)**.** *The Vilenkin-like system ψ is orthonormal.*

The Dirichlet kernels play a prominent role in the convergence of Fourier series. The following two lemmas will be useful in this regard. They can be find in [Gát4].

Lemma 2. *Let $M_{n+1}|k$, $y \in I_n(x)$ $(n,k \in \mathbf{N},\ x,y \in G_m)$. Then*

$$\sum_{j=0}^{m_n-1} r_n^{k+jM_n}(y)\bar{r}_n^{k+jM_n}(x) = \begin{cases} 0, & \textit{if } y \notin I_{n+1}(x), \\ m_n, & \textit{if } y \in I_{n+1}(x). \end{cases}$$

Lemma 3.

$$D_{M_n}(y,x) = \begin{cases} M_n, & \textit{if } y \in I_n(x), \\ 0, & \textit{if } y \notin I_n(x). \end{cases}$$

Define the maximal operator $S^* f := \sup_{n\in\mathbf{N}} |S_{M_n} f|$.

Proposition 4 (Gát4)**.** *The operator S^* is of type (p,p) for all $1 < p \le \infty$ and of weak type* $(1,1)$.

Set $\mathcal{P}_n := \{\sum_{k=0}^{n-1} b_k\psi_k : b_0,\dots,b_{n-1} \in \mathbf{C}\}$ $(n \in \mathbf{P})$ the set of polynomials the degree of which is less then n, $\mathcal{P} := \cup_{n=1}^{\infty}\mathcal{P}_n$ the set of polynomials (with respect to the system ψ). $\mathcal{P}$ is dense in $C(G_m)$ (the set of functions continuous on G_m) [Gát4] and we also have that the set of polynomials is dense in $L^p(G_m)$ $(1 \le p < \infty)$ [Gát4].

Then by the usual density argument (see e.g. [SWS, p. 81]) we have

Proposition 5 (Gát4)**.** *$S_{M_n} f \to f$ a.e. for each $f \in L^1(G_m)$.* □

It is simple to prove the convergence theorem above (that is, Proposition 5) with respect to convergence in norm. We mean convergence in the L^p-norm $(1 < p < \infty$ and with respect to the supremum norm for continuous functions as well. The situation with respect to the whole sequence of the partial sums of the Fourier series changes. For Vilenkin systems (also on unbounded G_m groups) it is known (see e.g. [Sch2]) that $f \in L^p(G_m)$ $(1 < p < \infty)$ implies that $S_n f \to f$ in the L^p norm and on bounded Vilenkin groups $S_n f \to f$ almost everywhere (see e.g. [Sch3]) for $1 < p < \infty$. On the other hand, this is not the case on nonabelian Vilenkin groups ([GT, Gát2, Gát4]). It seems to be interesting to try to give a control sequence for the norm of the difference of a function and the M_nth partial seum of the Fourier series of the function. In the Walsh, Vilenkin and even in the nonabelian Vilenkin case this control sequence is the sequence of modulus of continuity. The problem is that we do not have a group or any other operation on G_m and consequently the natural way to define the sequence of modulus of continuity does not work. However, we do define it in the following way. In the sequel let $\lambda_i : G_{m_i} \to G_{m_i}$ be an $1-1$ function for $i \in \mathbf{N}$. Set $\lambda := (\lambda_0, \lambda_1, \dots) \in \Lambda$

$$\Lambda_n := \{\lambda : \lambda_i(x_i) = x_i \text{ for } x_i \in G_{m_i}, i < n\}$$

for $n \in \mathbf{P}$. For any $\lambda \in \Lambda$ and $x \in G_m$ set $\lambda(x) := (\lambda_0(x_0), \lambda_1(x_1), \ldots)$.

Definition 6. Let $1 < p < \infty$, $f \in L^p(G_m)$ and $n \in \mathbf{P}$. The nth $L^p(G_m)$ modulus of continuity of function f is

$$\omega_n^{(p)}(f) := \sup_{\lambda \in \Lambda_n} \|f(.) - f(\lambda(.))\|_p.$$

Definition 7. Let $f \in C(G_m)$ and $n \in \mathbf{P}$. The nth $C(G_m)$ modulus of continuity of function f is

$$\omega_n(f) := \sup_{\lambda \in \Lambda_n} \|f(.) - f(\lambda(.))\|_\infty.$$

For functions in L^p and $n \in \mathbf{P}$ set

$$E_n^{(p)}(f) := \inf_{P \in \mathcal{P}_{M_n}} \|P - f\|_p$$

and for continuous functions set

$$E_n(f) := \inf_{P \in \mathcal{P}_{M_n}} \|P - f\|_\infty$$

the best approximation by Vilenkin-like polynomials. It is obvious that sequences $\omega_n^{(p)}(f)$ (for functions in L^p) and $\omega_n(f)$ (for continuous functions) are decreasing. We prove that they converge to zero. We prove even more, namely:

Theorem 8. *Let $1 < p < \infty$, $f \in L^p(G_m)$ and $n \in \mathbf{P}$. Then*

$$\frac{1}{2}\omega_n^{(p)}(f) \le \|S_{M_n} f - f\|_p \le \omega_n^{(p)}(f).$$

and

$$\frac{1}{2}\omega_n^{(p)}(f) \le E_n^{(p)}(f) \le \omega_n^{(p)}(f).$$

Theorem 8 with respect to the Walsh-Paley system is the result of Watari [SWS] and with respect to the Vilenkin system is the result of Efimov [AVD]. With respect to the trigonometric system the fourth inequality is a Jackson type inequality. The other three inequalities have no trigonometric analogue.

Proof. Let $P \in \mathcal{P}_{M_n}$ be a polynomial and $\lambda \in \Lambda_n$. Then we prove

$$P \circ \lambda = P.$$

Let $j \in \{0, 1, \ldots, M_n - 1\}$. Then $\psi_j := \prod_{k=0}^\infty r_k^{j^{(k)}}$ and $r_k^0 = 1$ implies

$$\psi_j := \prod_{k=0}^{n-1} r_k^{j^{(k)}}.$$

That is, since $r_k^{j^{(k)}}$ is $\mathcal{A}_{k+1}$ measurable (i.e. $r_k^{j^{(k)}}(x)$ depends only on $x_0, \ldots, x_k$ $(x \in G_m)$) we get that for any $x \in G_m$

$$r_k^{j^{(k)}}(x) = r_k^{j^{(k)}}(\lambda(x))$$

for $k \le n-1$ (recall that $\lambda_i(x_i) = x_i$ fo $x_i \in G_{m_i}, i < n$). Consequently,

$$\psi_j(x) = \psi_j(\lambda(x)).$$

And since for any complex numbers $c_0, c_1, \ldots, c_{n-1}$ we have

$$\sum_{j=0}^{n-1} \psi_j(x) = \sum_{j=0}^{n-1} \psi_j(\lambda(x)),$$

then

$$P(x) = P(\lambda(x)).$$

Thus,

$$\begin{aligned}\|f - f \circ \lambda\|_p &\le \|f - P\|_p + \|P - P \circ \lambda\|_p + \|P \circ \lambda - f \circ \lambda\|_p \\ &= 2\|f - P\|_p\end{aligned}$$

for all $P \in \mathcal{P}_{M_n}$ which gives

$$\|f - f \circ \lambda\|_p \le 2E_n^{(p)}(f).$$

That is,

$$\omega_n^{(p)}(f) = \sup_{\lambda \in \Lambda_n} \|f(.) - f(\lambda(.))\|_p \le 2E_n^{(p)}(f).$$

Since $S_{M_n} f \in \mathcal{P}_{M_n}$ then we also proved

$$\frac{1}{2}\omega_n^{(p)}(f) \le \|S_{M_n} f - f\|_p.$$

The inequalities on the left sides of Theorem 8 are proved. The rest is to prove

$$\|S_{M_n} f - f\|_p \le \omega_n^{(p)}(f).$$

We prove this for polynomials, first. Let $P \in \mathcal{P}_{M_N}$ for some $n < N \in \mathbf{N}$. This gives (in the same way as above) that $P(x)$ depends only on $x_0, \dots, x_{N-1}$. Therefore, the notation $P(x) = P(x_0, \dots, x_{N-1})$ can be used. Let $\tilde{\lambda}_i : G_{m_i}^2 \to G_{m_i}$ for $n \le i \in \mathbf{N}$ be functions such as that for all $y_i \in G_{m_i}$ the function $\tilde{\lambda}_i(y_i, .) : G_{m_i} \to G_{m_i}$ and for all $x_i \in G_{m_i}$ the function $\tilde{\lambda}_i(., x_i) : G_{m_i} \to G_{m_i}$ is an $1-1$ function ($n \le i \in \mathbf{N}$). Set

$$\tilde{\lambda}(y, x) := (x_0, \dots, x_{n-1}, \tilde{\lambda}_n(y_n, x_n), \tilde{\lambda}_{n+1}(y_{n+1}, x_{n+1}), \dots) \in G_m.$$

Then we have

$$\begin{aligned}&S_{M_n} P(x) = M_n \int_{I_n(x)} P(t) d\mu(t) \\ &= M_n \int_{I_n(x)} P(\tilde{\lambda}(y, x)) d\mu(y) \\ &= \frac{M_n}{M_N} \sum_{y_n \in G_{m_n}} \dots \sum_{y_{N-1} \in G_{m_{N-1}}} P(x_0, x_1, \dots, x_{n-1}, \tilde{\lambda}_n(y_n, x_n), \dots, \tilde{\lambda}_{N-1}(y_{N-1}, x_{N-1})).\end{aligned}$$

Consequently,

$$\|S_{M_n}P - P\|_p^p = \int_{G_m} \left| M_n \int_{I_n(x)} P(t) - P(x) d\mu(t) \right|^p d\mu(x)$$
$$= \frac{1}{M_N} \sum_{x_0 \in G_{M_0}} \cdots \sum_{x_{N-1} \in G_{m_{N-1}}}$$
$$\Bigg| \frac{M_n}{M_N} \sum_{y_n \in G_{m_n}} \cdots \sum_{y_{N-1} \in G_{m_{N-1}}} P(x_0, x_1, \dots, x_{n-1}, \tilde{\lambda}_n(y_n, x_n), \dots, \tilde{\lambda}_{N-1}(y_{N-1}, x_{N-1}))$$
$$- P(x_0, \dots, x_{n-1}, \dots, x_{N-1}) \Bigg|^p$$
$$\leq \frac{M_n}{M_N^2} \sum_{x_0 \in G_{M_0}} \cdots \sum_{x_{N-1} \in G_{m_{N-1}}}$$
$$\sum_{y_n \in G_{m_n}} \cdots \sum_{y_{N-1} \in G_{m_{N-1}}} \Big| P(x_0, x_1, \dots, x_{n-1}, \tilde{\lambda}_n(y_n, x_n), \dots, \tilde{\lambda}_{N-1}(y_{N-1}, x_{N-1}))$$
$$- P(x_0, \dots, x_{n-1}, \dots, x_{N-1}) \Big|^p$$
$$= \frac{M_n}{M_N} \sum_{y_n \in G_{m_n}} \cdots \sum_{y_{N-1} \in G_{m_{N-1}}} \frac{1}{M_N}$$
$$\sum_{x_0 \in G_{M_0}} \cdots \sum_{x_{N-1} \in G_{m_{N-1}}} \Big| P(x_0, x_1, \dots, x_{n-1}, \tilde{\lambda}_n(y_n, x_n), \dots, \tilde{\lambda}_{N-1}(y_{N-1}, x_{N-1}))$$
$$- P(x_0, \dots, x_{n-1}, \dots, x_{N-1}) \Big|^p$$
$$= \frac{M_n}{M_N} \sum_{y_n \in G_{m_n}} \cdots \sum_{y_{N-1} \in G_{m_{N-1}}} \|P(\tilde{\lambda}(y, .) - P(.))\|_p^p.$$

Since for each fixed $y \in G_m$ we have that $\tilde{\lambda}(y, .) \in \Lambda_n$, then it follows

$$\|S_{M_n}P - P\|_p \leq \omega_n^{(p)}(P).$$

That is, the theorem for polynomials is proved. Let $f \in L^p$ and $\epsilon > 0$. Since the set of polynomials is dense in L^p then it follows that there exists a polynomial $P \in \mathcal{P}$ such as that $\|f - P\|_p < \epsilon$. By proposition 4 we have

$$\begin{aligned}
&\|S_{M_n}f - f\|_p \\
&\leq \|S_{M_n}f - S_{M_n}P\|_p + \|S_{M_n}P - P\|_p + \|P - f\|_p \\
&\leq c\|f - P\|_p + \omega_n^{(p)}(P).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\omega_n^{(p)}(P) \\
&= \sup_{\lambda \in \Lambda_n} \|P(.) - P(\lambda(.))\|_p \\
&\leq \sup_{\lambda \in \Lambda_n} \|P(.) - f(.)\|_p + \sup_{\lambda \in \Lambda_n} \|f(.) - f(\lambda(.))\|_p + \sup_{\lambda \in \Lambda_n} \|f(\lambda(.)) - P(\lambda(.))\|_p \\
&\leq 2\|f - P\|_p + \omega_n^{(p)}(f).
\end{aligned}$$

Finally, by the above written

$$\begin{aligned}&\|S_{M_n}f - f\|_p\\ &\leq c\|f-P\|_p + \omega_n^{(p)}(f)\\ &\leq c\epsilon + \omega_n^{(p)}(f)\end{aligned}$$

for each $\epsilon > 0$. That is the proof of theorem 8 is complete. □

Finally, we remark (the proof is left to the reader)

Theorem 9. *Let $f \in C(G_m)$ and $n \in \mathbf{P}$. Then*

$$\frac{1}{2}\omega_n^{(p)}(f) \leq \|S_{M_n}f - f\|_\infty \leq \omega_n(f).$$

and

$$\frac{1}{2}\omega_n(f) \leq E_n(f) \leq \omega_n(f).$$

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