Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
17 (2001), 179–183
www.emis.de/journals

ON THE AXONOMETRICAL PROJECTION IN THE COMPUTER GRAPHICS

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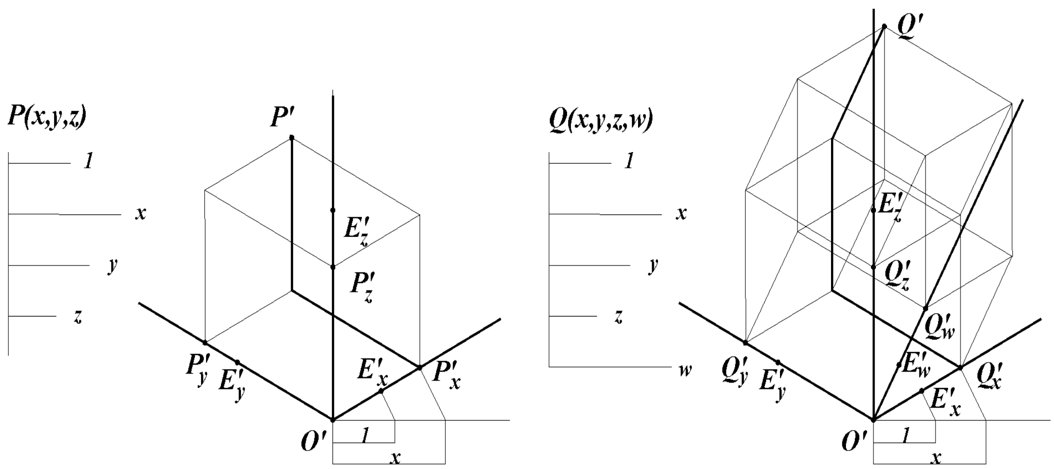
Dedicated to Árpád Varecza on his 60th birthday

Abstract. The axonometrical projection is not projection. It is a degenerated affin or projective mapping.

In the books of computer graphics and in several books of descriptive geometry the axonometry and the parallel projection are considered the same. Though it is simple, this approach doesn't square with the history and science, either. On the other hand it isn't useless because it performs the main aim: the demonstration under certain circumstances. This consideration can't be carried over into the multidimensional case and it can't be generalize to the perspective.



From this can be seen that it isn't absolutely correct the axonometry to be considered as projection. One of its simple forms says: *the axonometrical image of an object is always similar to a parallel projection of that object* [4]. (Pohlke's theorem from 1853.)

This theorem can't be generalized to the multidimensional case [3]. The theorem which is now called the fundamental theorem states that *the axonometrical image of an object is always affin to a parallel projection of that object* [4]. It follows that the axonometrical mapping is straight invariant, divide quotient invariant, parallelism invariant mapping. The axonometrical mapping of the three- and four-dimensional space is illustrated with the axonometrical prisms in the following figures.

The analytical form of the axonometrical mapping follows from the fundamental theorem. If we don't consider motion in the three-dimensional space the system of equations of affinity is

2000 *Mathematics Subject Classification.* 65D18.
Key words and phrases. Computer graphics, descriptive geometry, projective geometry, didactic.
Supported by the grant *Stiftung für Wissenschaft und Technologie zwischen Ungarn-Deutschland* 14/1998.

the following:

$$\begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z \\ y' &= a_{21}x + a_{22}y + a_{23}z \\ z' &= a_{31}x + a_{32}y + a_{33}z \end{aligned}$$

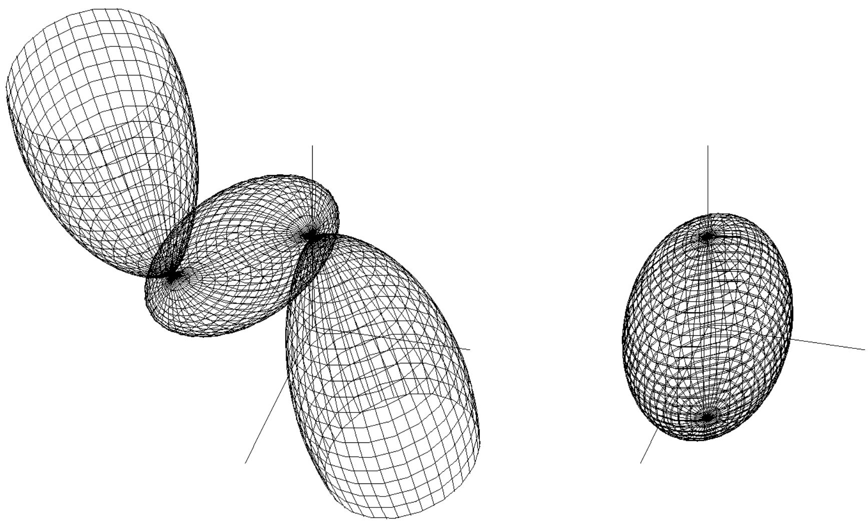
If the rank of the matrix a_{ik} is 2, the three-dimensional space is mapped into the plane. In this case a row becomes unnecessary. So if the co-ordinates of the image point are denoted by u, v then the analytical form of the axonometrical mapping is the following:

$$\begin{aligned} u &= a_{11}x + a_{12}y + a_{13}z \\ v &= a_{21}x + a_{22}y + a_{23}z \end{aligned}$$

If we map the point $Q(x, y, z, w)$ of the four-dimensional Euclidean space and the co-ordinates of the image of Q are denoted by u, v the following system of equations holds for the co-ordinates:

$$\begin{aligned} u &= a_{11}x + a_{12}y + a_{13}z + a_{14}w \\ v &= a_{21}x + a_{22}y + a_{23}z + a_{24}w \end{aligned}$$

where the unit points of the axonometrical axes is the columns of the matrix.



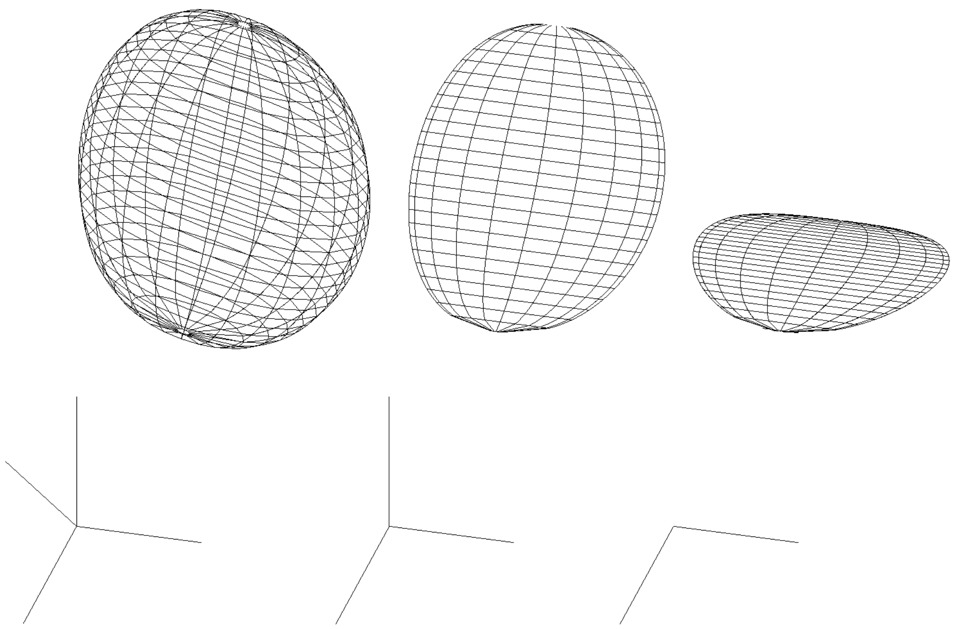
To give an example consider the equation and axonometrical figure of the $\mathbf{r}(u, v)$ surface in the four-dimensional space:

$$\begin{aligned} x' &= \sin u \cos v \\ y' &= \sin u \sin v \\ z' &= \cos u \\ w' &= u \end{aligned}$$

It can be seen because if we intersect it with the subspace x_1, x_2, x_3 all of the intersections will be sphere.

The other surface has positive constant Gauss curvature. The figure in left is the axonometrical image of the surface, the figure in the middle is an intersection with a hyperspace, and the figure in right is an axonometrical image of plain intersection.

The system of equations of the surface:



$$
\begin{aligned}
x' &= -(2\arcsin(u/2) + 4 - u^2 \,/\, \sqrt{4-u^2} \\
y' &= \sqrt{2u-u^2}\sin v \\
z' &= \sqrt{2}(2u+u^2) \,/\, \sqrt{2u+u^2} - \sqrt{2}\ln\left(1+u+\sqrt{2u+u^2}\right) \\
w' &= \sqrt{2u-u^2}\cos v
\end{aligned}
$$

One of the fundamental theorems of the orthogonal axonometry of the three-dimensional space is the theorem of Gauss, which states [4].

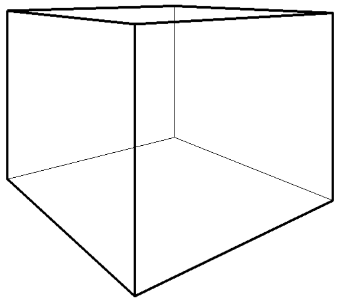
Consider such a Gauss's complex number-plane to the co-ordinate system of the orthogonal axonometry, the origin of which equals to the origin of the axonometrical coordinate system

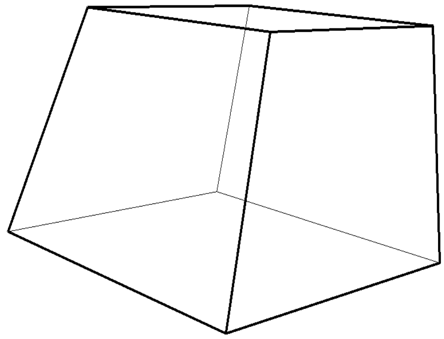
Denote the axonometrical unit points in the complex number-plane with the complex numbers ρ, σ, τ. Then the following relationships hold for them:

$$
\begin{aligned}
\rho^2 + \sigma^2 + \tau^2 &= 0 \\
|\rho|^2 + |\sigma|^2 + |\tau|^2 &= 2
\end{aligned}
$$

On the basis of this theorem we can not only construct orthogonal axonometrical co-ordinate systems to certain given divide quotients but we can state that the axonometrical image created by the command *vpoint* of AutoCAD isn't an orthogonal axonometry though we have to give only the three co-ordinates of a direction vector for determine the image.

A projective generalization of the axonometry shown above can be the central axonometry or perspective Now we use the invariant of the cross ratio along the axes. The following figures show a perspective images of the cube.





These figures look like but not similar to the central projection of the cube. If we determine the intersection points of the parallel edges on the figure, we will get an obtuse-angled triangle. So those can't be the central projection of the points in infinity of three pair-wise rectangle lines.

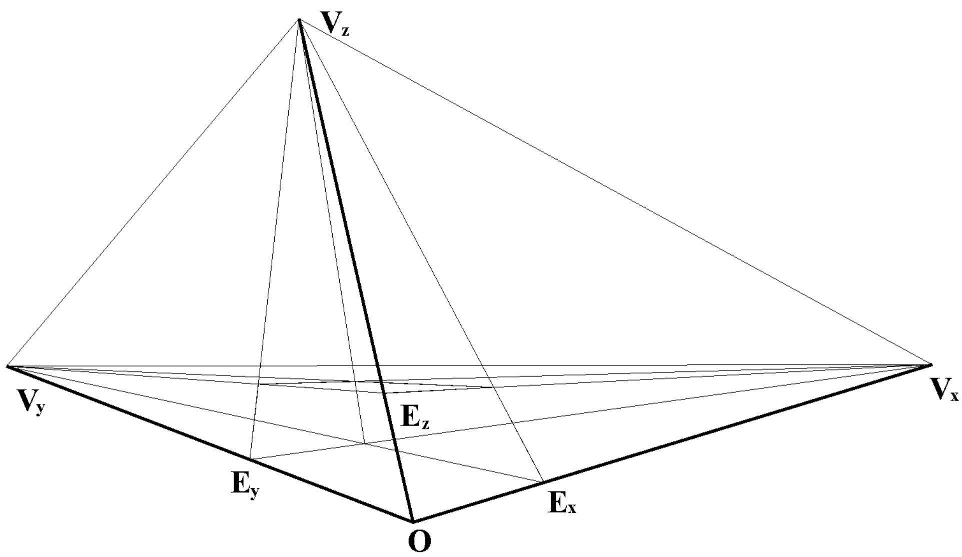
Neither the figure in left can be a central projection of a cube though it can't be seen so clearly as above. Both of them are also in connection with central projection. The fundamental theorem states: *the central axonometric image of an object in the space is always projective to a central projection of that object* [4].

There are theorems which give necessary and sufficient conditions when the figures above are central projection. A synthetic form can be found in [2] and an analytical one which is suitable for computer use in [5].

The theorem states that *the co-ordinate system of the central axonometry is central projection of an orthonormal basis if and only if the following holds:*

$$\left(\frac{OE_x}{E_xV_x}\right)^2 : \left(\frac{OE_y}{E_yV_y}\right)^2 : \left(\frac{OE_z}{E_zV_z}\right)^2 = \mathrm{tg}(V_yV_xV_z) : \mathrm{tg}(V_xV_yV_z) : \mathrm{tg}(V_yV_zV_x).$$

where:



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Received November 21, 2000

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