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ON PARTIALLY PSEUDO SYMMETRIC K-CONTACT RIEMANNIAN MANIFOLDS

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Abstract. A Riemannian manifold (M, g) is semi-symmetric if $(R(X,Y) \circ R)(U,V,W) = 0$. It is called pseudo-symmetric if $R \circ R = \mathfrak{F}$, $\mathfrak{F}$ being a given function of $X, \dots, W$ and g. It is called partially pseudo-symmetric if this last relation is fulfilled by not all values of $X, \dots, W$. Such manifolds were investigated by several mathematicians: I.Z. Szabó, S. Tanno, K. Nomizu, R. Deszcz and others.

In this paper we investigate K-contact Riemannian manifolds. In these manifolds the structure vector field ξ plays a special role, and this motivates our interest in the partial pseudo-symmetry of these manifolds. We also investigate the case when $R \circ R$ is replaced by $R \circ S$ (S being the Ricci tensor). We obtain conditions in order that our manifold be: (1) Sasakian or Sasakian of constant curvature 1 (in case of $R \circ R$); (2) an Einstein manifold (in case of $R \circ S$). – Our investigation is closely related to the results of S. Tanno.

1. Introduction

A Riemannian manifold (M, g) is called *locally symmetric* if its curvature tensor R is parallel: $\nabla R = 0$ (∇ denotes the Levi-Civita connection). As a proper generalization of locally symmetric manifolds the notion of *semi symmetric* manifolds was defined by

$$(R(X,Y) \circ R)(U,V,W) = 0, \quad X, \dots, W \in \mathfrak{X}(M) \tag{1}$$

(or shortly by $R \circ R = 0$) and studied by many authors, e.g. [6], [5], [7], [11], [13]). A complete intrinsic classification of these spaces was given by Z.I. Szabó [9].

It is interesting to investigate the semi-symmetry of special Riemannian manifolds. K. Nomizu proved [6] that if M is a complete, connected hypersurface in a Euclidean space R^{n+1} $(n > 3)$ and it satisfies $R \circ R = 0$, then it is locally symmetric. For the case of a compact Kähler manifold M. Ogawa [7] proved that if it is semi-symmetric, then it must be locally-symmetric, that is there exists no proper semi-symmetric compact Kähler manifold. In the case of contact manifolds S. Tanno [11]–[13] showed among others that there exists no proper semi symmetric (or semi-Ricci-symmetric) K-contact manifold. Namely in K-contact manifolds $R \circ R = 0 \Rightarrow \nabla R = 0$, and $R \circ S = 0 \Rightarrow \nabla S = 0$, where S denotes the Ricci curvature tensor.

R. Deszcz and others [2]–[4] weakened the notion of semi-symmetry (respectively semi-Ricci-symmetry) and introduced the notion of *pseudo-symmetric* spaces by

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requiring in place of $R \circ R = 0$ (respectively $R \circ S = 0$) the relation

(P) $$(R(X,Y) \circ R)(U,V,W) = f[((X \wedge Y) \circ R)(U,V,W)]$$

only, where $f \in C^\infty(M)$. – S. Tanno [11], [13], M. Okumura [8], T. Takahashi [10] and others investigated cases, where (1) is satisfied only by certain special vector fields, e.g. (I) $Y = U = W = \xi$, (II) $Y = U = \xi$ or (III) $Y = \xi$. These spaces need not be semi-symmetric. We call them *partially semi symmetric* or *partially pseudo symmetric*, as the restrictions refer to (1), respectively (P).

In this paper we consider partially pseudo-symmetric K-contact manifolds, where (P) is satisfied with the restrictions (I), (II) or (III) only. Since here f need not be zero, these spaces are more general, than the corresponding partially semi-symmetric spaces. We want to show that even for these spaces – i.e. for spaces with the weaker condition of partial pseudo-symmetry – similar conclusions can be drawn as in the case of partial semi-symmetry. If

(N) $$f + 1 \text{ never vanishes,}$$

then in case of (I) a partially pseudo-symmetric K-contact manifold turns out to be Sasakian (Theorem 1); in case of (II) it is moreover of constant curvature 1, and hence it is locally symmetric (Theorem 3). We also obtain that among the K-contact Riemannian manifolds exactly the partially pseudo-symmetric ones with restriction (I) and (N) are Sasakian (Theorem 2). In the last Section 4 we prove that a partially pseudo-Ricci-symmetric K-contact manifold with the restriction $Y = V = \xi$ and with never vanishing function $f - 1$ is an Einstein manifold (Theorem 4), and we show that the relation $R(X,\xi) \circ S = (X \wedge \xi) \circ S$ holds in every Sasakian manifold (Theorem 5).

2. Preliminaries

Let $(M, \varphi, \xi, \eta, g)$ be an n dimensional contact Riemannian manifold. It satisfies the relations

(2) $$\eta(X) = \langle X, \xi \rangle, \quad \eta(\xi) = 1, \text{ where } \langle X, \xi \rangle \doteq g(X,Y)$$

(We list only those basic relations which will be used in the sequel.) A contact manifold is said to be *K-contact*, if ξ is a Killing vector field [11]. In this case we have [11], [1], [5]:

(3) $$\nabla_X \xi = -\varphi X$$

(4) $$R(X,\xi)\xi = -X + \eta(X)\xi$$

(5) $$\langle R(X,\xi)Y, \xi \rangle = -\langle X, Y \rangle + \eta(X)\eta(Y)$$

(6) $$S(X,\xi) = 2n\eta(X).$$

If a K-contact manifold satisfies the relation

(7) $$R(X,\xi)Y = -\langle X, Y \rangle \xi + \eta(Y)X,$$

then M is called a Sasakian manifold ([5], p. 273, Theorem 5.2).

The endomorphism $X \wedge Y$ and the homeomorphisms $R \circ R$ and $(X \wedge Y) \circ R$ are defined by

(8) $$(X \wedge Y)Z \doteq \langle Y, Z \rangle X - \langle X, Z \rangle Y$$

(9) $$(R(X,Y) \circ R)(U,V,W) \doteq R(X,Y)(R(U,V)W - R(R(X,Y)U,V)W \\ -R(U, R(X,Y)V)W - R(U,V)(R(X,Y)W)$$

(10) $$((X \wedge Y) \circ R)(U,V,W) \doteq (X \wedge Y)(R(U,V)W) - R((X,Y)U,V)W \\ -R(U,(X \wedge Y)V)W - R(U,V)((X \wedge Y)W).$$

(9) and (10) are kind of derivations.

3. Partially pseudo-symmetric K-contact manifolds

We want to investigate partially pseudo-symmetric K-contact manifolds $(M, \varphi, \xi, \eta, g)$ which satisfy (P) with certain restrictions.

A) Let at first the restriction be (I) $Y = U = W = \xi$. Then we get

$$R(X,\xi) \circ R)(\xi, V, \xi) = f[((X \wedge \xi) \circ R)(\xi, V, \xi)]. \tag{11}$$

Such K-contact manifolds with vanishing f were considered by S. Tanno [11]. He proved that such a partially semi-symmetric space is Sasakian. – Our condition is weaker than that of Tanno, for with us f in (11) need not be zero (our space is partially pseudo-symmetric, and not partially semi-symmetric). We want to show that also from this weaker condition we can draw similar conclusions.

Theorem 1. *A partially pseudo-symmetric K-contact manifold, where* (P) *is restricted by* (I) $Y = U = W = \xi$ *and* $f + 1$ *never vanishes (i.e. condition (N) is satisfied) is a Sasakian manifold.*

Proof. We denote the expression in the bracket on the right-hand side of (11) by A, and we calculate it. By (10)

$$\begin{aligned} A = (X \wedge \xi)(R(\xi, V)\xi) - R((X \wedge \xi)\xi, V)\xi - R(\xi, (X \wedge \xi)V)\xi \\ -R(\xi, V)((X \wedge \xi)\xi). \end{aligned}$$

Using for the individual terms of A the relations of Section 2 we obtain

$$\begin{aligned} (X \wedge \xi)(R(\xi, V)\xi) &\overset{(4)}{=} (X \wedge \xi)(V - \eta(V)\xi) \overset{(8)}{=} \langle \xi, V\rangle X - \langle X, V\rangle\xi \\ &-\eta(V)(\langle \xi, \xi\rangle X - \langle X, \xi\rangle\xi) \overset{(2)}{=} -\langle X, V\rangle\xi + \eta(V)\eta(X)\xi, \\ R((X \wedge \xi)\xi, V)\xi &\overset{(2),(8)}{=} R(X - \eta(X)\xi, V)\xi \overset{(4)}{=} R(X, V)\xi - \eta(X)(V - \eta(V)\xi), \\ R(\xi, (X \wedge \xi)V)\xi &\overset{(4)}{=} (X \wedge \xi)V - \eta((X \wedge \xi)V)\xi \overset{(8)}{=} \eta(V)X - \langle X, V\rangle\xi \\ &-\eta(\eta(V)X - \langle X, V\rangle\xi)\xi \overset{(2)}{=} \eta(V)X - \eta(X)\eta(V)\xi, \\ R(\xi, V)((X \wedge \xi)\xi) &\overset{(8)}{=} R(\xi, V)(X - \eta(X)\xi) \\ &\overset{(4)}{=} R(\xi, V)X - \eta(X)(V - \eta(V)\xi). \end{aligned}$$

From these

$$A = -(R(X, V)\xi + R(\xi, V)X + \eta(V)X - 2\eta(X)V + \langle X, V\rangle\xi).$$

However, after similar considerations, the left-hand side of (11) leads to the same expression with opposite sign (see also Tanno [11] (2.8)). Thus (11) reduces to

$$-A = fA. \tag{12}$$

With respect to our condition (N) this can hold only if $A = 0$. However $A = 0$ yields

$$R(\xi, V)X = \eta(X)V - \langle X, V\rangle\xi, \tag{13}$$

and this means (see Tanno [11] p. 451 (2.11) that M is a Sasakian manifold. $\square$

In consequence of (12) every K-contact Riemannian manifold with restriction (I) and with $f = -1$ is partially pseudo-symmetric. So we have proved the

Proposition 1. *In every K-contact Riemannian manifold*

$$R(X, \xi) \circ R = (\xi \wedge X) \circ R \quad \textit{over } (\xi, V, \xi).$$

Now we easily obtain

Theorem 2. *Among the K-contact Riemannian manifolds exactly the partially pseudo-symmetric ones with restriction (I) $Y = U = W = \xi$ are Sasakian if in* (P) *$f + 1$ never vanishes.*

Proof. If a K-contact Riemannian manifold M is partially pseudo-symmetric with restriction (I) and (N), then according to our Theorem 1 M is Sasakian.

Conversely, let M be a Sasakian manifold. Tanno deduces (13) from $A = 0$ by making use of (5) ([11], p. 451). Now (13) holds, since M is Sasakian. Also (5) is true, since any Sasakian manifold is K-contact. Then one can invert this consideration, and from (13) obtain $A = 0$. Thus (12) is true also with (N), and this, by repeated application of (2), (4) and (8), leads to (11), i.e. a Sasakian manifold is partially pseudo-symmetric with restriction (I) and (N). ☐

B) Let now the restriction on (P) be (II) $Y = U = \xi$. Then we get

$$(R(X,\xi)\circ R)(\xi,V,W) = f[((X\wedge\xi)\circ R)(\xi,V,W)]. \tag{14}$$

Such K-contact manifolds with vanishing f were considered by S. Tanno. He proved ([11], Theorem 2.3 or [13]) that for a K-contact manifold M the following four conditions are equivalent (i) M is of constant curvature 1; (ii) $\nabla R = 0$; (iii) $R(X,Y)\circ R = 0\ \forall X,Y$; (iv) $R(X,\xi)\circ R = 0$. His proof runs as follows (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is clear. Then he assumes (iv): $(R(X,\xi)\circ R)(U,V,W) = 0$ and proves (i). At this step, however he uses $(R(X,\xi)\circ R)(\xi,V,W) = 0$ only, which is (14) with $f = 0$. We want to prove that also from our somewhat weaker condition (14), where f is not necessarily zero, one can draw a similar conclusion.

Theorem 3. *A partially pseudo-symmetric K-contact manifold, where* (P) *is restricted by* (II)*: $Y = U = \xi$, and $f + 1$ never vanishes, is a locally symmetric Sasakian manifold with constant curvature 1.*

Proof. Of course the broader is the restriction on the vector fields in (P), the wider is the family of spaces satisfying the restricted (P). Thus the family of the K-contact manifolds satisfying (14) is a part of those satisfying (12). Thus (14) $\Rightarrow$ (11).

Suppose (N). Then, according to our Theorem 1, from (11) follows (13), which means that M is Sasakian.

Let us denote the expression in the bracket on the right-hand side of (14) by C and calculate it. By (10)

$$\begin{aligned} C = (X\wedge\xi)(R(\xi,V)W) - R((X\wedge\xi)\xi,V)W - R(\xi,(X\wedge\xi)V)W \\ -R(\xi,V)((X\wedge\xi)W). \end{aligned}$$

Using for the individual terms of C the relations of Section 2 and (13) we obtain

$$\begin{aligned} &(X\wedge\xi)(R(\xi,V)W) \overset{(13)}{=} (X\wedge\xi)(\eta(W)V - \langle V,W\rangle\xi) \overset{(8)}{=} \\ &= \eta(W)\eta(V)X - \eta(W)\langle X,V\rangle\xi - \langle V,W\rangle X + \langle V,W\rangle\eta(X)\xi, \\ &R((X\wedge\xi)\xi,V)W \overset{(8)}{=} R(X,V)W - \eta(X)R(\xi,V)W \overset{(13)}{=} \\ &= R(X,V)W - \eta(X)[\eta(W)V - \langle V,W\rangle\xi], \\ &R(\xi,(X\wedge\xi)V)W \overset{(13)}{=} \eta(W)((X\wedge\xi)V) - \langle (X\wedge\xi)V,W\rangle\xi \overset{(8)}{=} \\ &= \eta(W)\eta(V)X - \eta(W)\langle X,V\rangle\xi - \langle \eta(V)X - \langle X,V\rangle\xi, W\rangle\xi, \\ &R(\xi,V)((X\wedge\xi)W) \overset{(13)}{=} R(\xi,V)(\eta(W)X - \langle X,W\rangle\xi) \overset{(13)}{=} \\ &= \eta(W)(\eta(X)V - \langle X,V\rangle\xi) - \langle X,W\rangle(V - \eta(V)\xi). \end{aligned}$$

From these

$$C = -R(X,V)W - \langle V,W\rangle X + \langle X,W\rangle V.$$

However the left-hand side of (14) gives, after similar calculations, the same expression with an opposite sign. Thus (14) reduces to

$$-C = f(p)C, \quad \forall p \in M.$$

Since $f(p) = -1$ is now excluded, $C = 0$, i.e.

$$R(X,V)W = \langle X, W\rangle V - \langle V, W\rangle X,$$

which means that M is of constant curvature 1.

Finally the mentioned theorem of Tanno gives that $\nabla R = 0$. i.e. M is locally symmetric. □

Concerning the Sasakian manifolds of constant curvature 1, with the help of the previous results and considerations we easily obtain the following

Proposition 2. *In every Sasakian manifold of constant curvature 1*

$$((\xi \wedge X) \circ R)(\xi, V, \xi) \equiv 0 \tag{15}$$

Proof. Every Sasakian manifold M is K-contact. Hence, by our Proposition 1

$$(R(X,\xi) \circ R)(\xi, V, \xi) = ((\xi \wedge X) \circ R)(\xi, V, \xi).$$

However if M is of constant curvature 1, then $R(X,\xi) \circ R = 0$ according to (iv) of Tanno. So we obtain (15). □

4. Partially pseudo-Ricci-symmetric K-contact manifolds

A K-contact manifold $(M, \varphi, \eta, \xi, g)$ is said to be *pseudo-Ricci-symmetric* if it satisfies

$$\text{(PR)} \qquad \begin{gathered}(R(X,Y) \circ S)(U,V) = f((X \wedge Y) \circ S)(U,V),\\ f \in C^\infty(M),\ U, V \in \mathfrak{X}(M),\end{gathered}$$

where

$$(R(X,Y) \circ S)(U,V) \doteq -S(R(X,Y)U,V) - S(U,R(X,Y)V)$$

and

$$((X \wedge Y) \circ S)(U,V) \doteq -S((X \wedge Y)U,V) - S(U,(X \wedge Y)V).$$

Thus (PR) has the following more developed form

$$\begin{aligned}&S(R(X,Y)U,V) + S(U,R(X,Y)V)\\ &= f[S((X \wedge Y)U,V) + S(U,(X \wedge Y)V)].\end{aligned} \tag{16}$$

We want to investigate partially pseudo-Ricci-symmetric K-contact manifolds which satisfy (PR) with the restriction $Y = V = \xi$. So we have

$$S(R(X,\xi)U,\xi) + S(U,R(X,\xi)\xi) = f[S((X \wedge \xi)U,\xi) + S(U,(X \wedge \xi)\xi)].$$

Applying (4), (6) and (8) we obtain

$$\begin{aligned}&2n\eta(R(X,\xi)U) + S(U,X) - \eta(X)S(U,\xi)\\ &= f[\eta(U)S(X,\xi) - \langle X, U\rangle S(\xi,\xi) + S(U,X) - \eta(X)S(U,\xi)].\end{aligned} \tag{17}$$

The first term of (17) becomes by (2)

$$2n\langle R(X,\xi)U,\xi\rangle \overset{(5)}{=} 2n(-\langle X, U\rangle + \eta(X)\eta(U)).$$

Applying again (6) in four terms of (7), after a reduction we obtain

$$S(U,X) - 2n\langle X, U\rangle = f[S(U,X) - 2n\langle X, U\rangle].$$

This can hold only if either: (a) $f = 1$ or (b) $S(U,X) = 2ng(X,U)$ (i.e $S = 2ng$). However (b) means that M is an Einstein manifold. Thus we have proved

Theorem 4. *A partially pseudo-Ricci-symmetric K-contact manifold with the restriction $Y = V = \xi$ and with never vanishing function $f - 1$ is an Einstein manifold.*

We still want to prove the following

Theorem 5. *The relation*

$$(R(X,\xi) \circ S)(U,V) = ((X \wedge \xi) \circ S)(U,V) \tag{18}$$

is satisfied on every Sasakian manifold M.

This means that every Sasakian manifold is partially pseudo-Ricci-symmetric with the restriction $Y = \xi$ and $f = 1$.

Proof. Putting $Y = \xi$ and $f = 1$ in (16), we obtain a more developed form of (18). Since M is Sasakian we have (13). Applying this, (6) and (8), a direct computation, similar to the previous ones shows that the sides of (18) are identical. □

This also shows that there exist K-contact manifolds satisfying $R(X,\xi) \circ S = (X \wedge \xi) \circ S$.

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