Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
18 (2002), 67–70
www.emis.de/journals

A SHORT REMARK ON KOLMOGOROFF NORMABILITY THEOREM

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Abstract. Kolmogoroff normability theorem turns to be a characterization for the complete normability of a topological vector space by replacing the convexity hypothesis with the σ-convexity one. In particular, the well known theorem that characterizes completeness of a normed vector spaces by means of absolutely convergent series, is obtained as an easy consequence of the Theorem below.

In what follows E will denote a topological vector space (t.v.s. for short) on a field $K(=\mathbb{R}$ or $\mathbb{C})$. Kolmogoroff theorem about normability asserts that a t.v.s. is normable if and only if E is Hausdorff and there exists a convex bounded neighborhood U of θ_E (θ_E denotes the zero of E). By requiring that U is more than a convex bounded set, i.e. a σ-convex set, we actually obtain a characterization for the completeness; precisely we prove the following theorem.

Theorem. *Let E be a t.v.s. Then E is a Banach space if and only if E is Hausdorff and there exists a σ-convex neighborhood U of θ_E.*

So what we need is the following definition.

Definition. Let E be an Hausdorff t.v.s. and $C \subseteq E$. The set C is said to be σ-convex provided that the following condition hold:

$\forall \{x_n\} \subseteq C$, $\forall \{\lambda_n\} \subseteq [0,1]$ such that $\sum_{n=1}^{+\infty} \lambda_n = 1, \implies \sum_{n=1}^{+\infty} \lambda_n x_n \in C.$

It is apparent that if C is a σ-convex set then also $C+a$ ($a \in E$) and αC ($\alpha \in K$) are σ-convex.

Proposition 1. *Let E be an Hausdorff t.v.s., and $C \subseteq E$. Then*

(i) *if C is σ-convex then is convex and bounded;*
(ii) *if C is σ-convex and $B \subseteq C$ is convex and closed, then B is σ-convex too;*
(iii) *if C is σ-convex then $\operatorname{int}(C)$ is too;*
(iv) *let C be convex and bounded. If E (respectively C) is sequentially complete then $\overline{C}$ (respectively C) is σ-convex;*

Proof. *(i)* C is obviously convex. Observe now that in any t.v.s. the condition $B \subseteq E$ bounded is equivalent to requiring that for any $\{\alpha_n\} \subseteq [0,1]$, $\alpha_n \longrightarrow 0$, $\sum_{n=1}^{\infty} \alpha_n \leq 1$, and for any $\{x_n\} \subseteq B$, then $\alpha_n x_n \longrightarrow \theta_E$. So chosen $\{x_n\} \subseteq C$ and $\{\alpha_n\}$ as above, set $\alpha_0 = 1 - \sum_{n=1}^{\infty} \alpha_n$ and fix $x_0 \in C$ arbitrarily: the convergence of the series $\sum_{n=0}^{\infty} \alpha_n x_n$ concludes the argument.

(ii) Take $\{x_n\} \subseteq B$ and $\{\lambda_n\} \subseteq [0,1]$ with $\sum_{n=1}^{\infty} \lambda_n = 1$. Then $T_n = S_n + (1 - \sum_{i=1}^{n} \lambda_i)x_1 \in B$ $\forall n \in \mathbb{N}$. But $S_n = \sum_{i=1}^{n} \lambda_i x_i \longrightarrow \sum_{i=1}^{\infty} \lambda_i x_i \in C$, $(1 - \sum_{i=1}^{n} \lambda_i)x_1 \longrightarrow \theta_E$, so T_n does converges in B to the same limit of S_n.

2000 *Mathematics Subject Classification.* 46A03, 46B20.
Key words and phrases. Topological Vector Spaces, Normability Criteria, Completeness.

(iii) Suppose that $\mathrm{int}(C) \neq \emptyset$. Take $\{x_n\} \subseteq \mathrm{int}(C)$, $\{\lambda_n\} \subseteq [0,1]$ such that $\sum_{n=1}^{+\infty} \lambda_n = 1$ and set $\sum_{n=1}^{+\infty} \lambda_n x_n = \overline{x} \in C$. If $\overline{x} \notin \mathrm{int}(C)$ then, by a consequence of the Hahn-Banach Theorem (see Theorem 3.4, (a), of [4], p.58) there exist $T \in E^*$ and $\alpha \in \mathbb{R}$ such that $\mathrm{Re}\, T(x_n) < \alpha \leq \mathrm{Re}\, T(\overline{x})$ for each $n \in \mathbb{N}$. It follows that $\alpha \leq \mathrm{Re}\, T(\overline{x}) = \sum_{n=1}^{+\infty} \lambda_n \mathrm{Re}\, T(x_n) < \alpha$. So $\overline{x} \in \mathrm{int}(C)$.

(iv) Let E be sequentially complete. Set $\{x_n\} \subseteq \overline{C}$, $\{\lambda_n\} \subseteq [0,1]$ such that $\sum_{n=1}^{+\infty} \lambda_n = 1$. If $S_n = \lambda_1 x_1 + \cdots + \lambda_n x_n = (\sum_{i=1}^{n} \lambda_i x_i + \sum_{i=n+1}^{+\infty} \lambda_i x_1) - \sum_{i=n+1}^{+\infty} \lambda_i x_1$ does converge, its limit must belong to $\overline{C}$ because $\sum_{i=n+1}^{+\infty} \lambda_i x_1$ goes to zero and $(\sum_{i=1}^{n} \lambda_i x_i + \sum_{i=n+1}^{+\infty} \lambda_i x_1) \in C$ converges too. So it is enough to verify that $\{S_n\}$ is a Cauchy sequence. Fixed a neighborhood U of θ_E, choose $V \subseteq U$ balanced neighborhood of θ_E and $\alpha > 0$ such that $\overline{C} \subseteq \alpha V$. Moreover choose k large enough such that $(\lambda_{k+1} + \lambda_{k+2} + \cdots)\alpha \leq 1$. Then for each $n > m \geq k$ we have $S_n - S_m = \lambda_{m+1} x_{m+1} + \cdots + \lambda_n x_n = (\lambda_{m+1} + \cdots + \lambda_n)\big(\frac{\lambda_{m+1}}{(\lambda_{m+1} + \cdots + \lambda_n)} x_{m+1} + \cdots + \frac{\lambda_n}{(\lambda_{m+1} + \cdots + \lambda_n)} x_n\big) \in (\lambda_{m+1} + \cdots + \lambda_n)\overline{C} \subseteq (\lambda_{m+1} + \cdots + \lambda_n)\alpha V \subseteq V$. Let now C be sequentially complete: in a similar way one can prove that $\sum_{i=1}^{n} \lambda_i x_i + \sum_{i=n+1}^{+\infty} \lambda_i x_1$ is a Cauchy sequence in C so that S_n does converge to some element of C. $\square$

Remark. Relatively to *(i)* note that not every convex bounded set is σ-convex: the set C of all complex sequences $\{x_n\}$ in the unit ball of l_∞ such that $|x_n| \leq 1$ for each $n \in \mathbb{N}$ and $x_n = 0$ for all but finitely many $n \in \mathbb{N}$, is convex and bounded, but evidently C is not σ-convex in l_∞. Moreover, in the assertion *(ii)*, the assumption B closed cannot be removed: the set $C = C_0^\infty(\mathbb{R}) \cap B$, being B the closed unit ball in $L^1(\mathbb{R})$, is convex bounded and dense in B, but C is not σ-convex in $L^1(\mathbb{R})$.

Proof of Theorem. By *(iv)* the closed unit ball is a σ-convex neighborhood of θ_E whenever E is Banach space. Let assume now that U is a σ-convex neighborhood of θ_E. By *(i)* of Proposition 1 and Kolmogoroff theorem we can find a norm $||\cdot||$ on E whose topology coincides with the given one. Let $B \subseteq U$ be a closed ball. By *(ii)* of Proposition 1 B is σ-convex, consequently by *(iii)* and translations argument, so is any other ball in E. Let $\{x_n\}$ be a Cauchy sequence and $\{x_{n_k}\}$ a subsequence such that $\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}$ $\forall k \in \mathbb{N}$: it is enough to prove that such a sequence converges to some element of E. Set $y_k = 2^k(x_{n_{k+1}} - x_{n_k})$ $\forall k \in \mathbb{N}$. It results $\{y_k\} \subseteq B$, B the open unit ball in E. By construction there exists $y \in B$ such that $y = \sum_{k=1}^{+\infty} \frac{y_k}{2^k} = \lim_k \sum_{i=1}^{k} (x_{n_{i+1}} - x_{n_i})$. This implies that $x_{n_k} \longrightarrow y + x_{n_1} \in E$, and the proof is complete. $\square$

It is known that if E is a finite dimensional vector space a set C is σ-convex if and only if it is convex and bounded (the if part follows by Example 1.6, *iv)*, of [2], p.84). This is false in general as showed in the Remark. If E is a normed space, next Proposition 2 will give us a similar characterization. Recall that a set $B \subseteq E$ containing θ_E is said to be *absorbing* if for any $x \in E$ is possible to find a number $t > 0$, depending on x, such that $x \in tB$; moreover B is said to be *radial at* θ_E if for any $x \in E$ there is a number $\delta > 0$, depending on x, such that $\lambda x \in B$ for any $\lambda \in [0, \delta]$. Clearly any set B radial at θ_E is an absorbing set and it is easy to verify that any neighborhood of θ_E is radial at θ_E. Finally recall that to any absorbing set B we can associate the *Minkowski functional* p_B defined by the position $p_B(x) = \inf\{t > 0 : x \in tB\}$.

Lemma. *Let E be an Hausdorff t.v.s. and $A \subseteq E$ an open convex set. Then $A = int(\overline{A})$.*

Proof. It is enough to verify that $\mathrm{int}(\overline{A}) \subseteq A$, i.e., $\overline{A} \setminus A \subseteq \overline{A} \setminus \mathrm{int}(\overline{A})$. Without loss of generality we can suppose that $\theta_E \in A$. A is an open convex neighborhood of θ_E, so the equalities $A = \{x \in E : p_A(x) < 1\}$ and $\overline{A} = \{x \in E : p_A(x) \leq 1\}$ hold (see Lemma 3.5.5, (d), of [1], p.154). Choose $c \in \overline{A} \setminus A$: it results $p_A(c) = 1$. Arguing by contradiction, suppose that $c \in \operatorname{int}(\overline{A})$. So we can find a neighborhood U of c, $U \subseteq \overline{A}$, such that $p_A(x) \leq 1$ for any $x \in U$. Let $g :]0, +\infty[\longrightarrow E$ be the function defined by the formula $g(t) = \frac{c}{t}$. It is $g(1) = c$. By the continuity of g we can find $0 < \epsilon < 1$ such that $g([1-\epsilon, 1+\epsilon]) \subseteq U$. It follows that $c \in (1-\epsilon)\overline{A}$, so $p_{\overline{A}}(c) \leq 1-\epsilon$. But $p_{\overline{A}} = p_A$ (see Theorem 1.35, (d), of [4], p.25): this contradiction concludes the argument. □

Proposition 2. *Let E be a normed space. Then the following facts are equivalent:*

(a) E is a Banach space;
(b) For any open set, convex and bounded means σ-convex;
(c) The open unit ball B, or equivalently any other open ball in E, is σ-convex.

Proof. *(a)$\Longrightarrow$(b)* Let $A \neq \emptyset$ be any open convex bounded set in E. By *(iv)* of Proposition 1, $\overline{A}$ is σ-convex and, by *(iii)* and the previous Lemma, so is $A = \operatorname{int}(\overline{A})$.
(b)$\Longrightarrow$(c) It is trivial.
(c)$\Longrightarrow$(a) Following the notations of the proof of the Theorem (the only if part), if $\{x_n\}$ is Cauchy sequence, then $\{y_k\}$ belongs to the open unit ball B of E. By the hypothesis B is σ-convex: this concludes the proof. □

The following two corollaries to Theorem are immediate consequences, so their easy proofs are left to the reader.

Corollary 1. *Let $(E, \|\cdot\|)$ be a normed space and B its closed unit ball centered in θ_E. Then E is complete if and only if*

$$\forall \{x_n\} \subseteq B,\ \forall \{\lambda_n\} \subseteq [0,1] \text{ such that } \sum_{n=1}^{+\infty} \lambda_n = 1 \implies \sum_{n=1}^{+\infty} \lambda_n x_n \in B.$$

Corollary 2. *Let $(E, \|\cdot\|)$ be a normed space. Then E is complete if and only if every absolutely convergent series is convergent.*

Sketch. Apply Corollary 1. □

We conclude the present note giving some easy application of the previous results.

Example. Let $(\Omega, \mathcal{A}, \mu)$ be an arbitrary measure space. Consider the vector space $L^p(\mu)$, $1 \leq p < \infty$, consisting of all (classes of equivalence of) measurable functions f such that $|f|^p$ is summable over Ω with respect to the measure μ. The formula $\|f\|_p = \left(\int_\Omega |f|^p d\mu\right)^{\frac{1}{p}}$ defines a norm on $L^p(\mu)$ which so becomes an Hausdorff t.v.s.. In order to verify the completeness of the space we can directly apply the Theorem and verify that the closed ball B with radius one, centered at zero is actually σ-convex. Take $\{f_n\} \subset B$ and $\{\lambda_n\} \subseteq [0,1]$ such that $\sum_{n=1}^{+\infty} \lambda_n = 1$. By Minkowski inequality it results $\left(\int_\Omega |\sum_{n=1}^{k} \lambda_n |f_n||^p d\mu\right)^{\frac{1}{p}} \leq \sum_{n=1}^{k} \lambda_n \leq 1\ \forall k \in \mathbb{N}$, so, by the Monotone Convergence theorem, $\left(\int_\Omega |\sum_{n=1}^{\infty} \lambda_n |f_n||^p d\mu\right)^{\frac{1}{p}} \leq 1$. Thus the function $\sum_{n=1}^{+\infty} \lambda_n f_n$, defined a.e. on Ω, belongs to B.

Analogous considerations, based on the convexity property of a Young function M, hold if we want to prove the completeness of the more general Orlicz Spaces L_M (see for instance [3]).

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Received June 18, 2002.

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