Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
18 (2002), 71–76
www.emis.de/journals

NORMAL STRUCTURE AND FIXED POINTS OF NONEXPANSIVE MAPS IN GENERAL TOPOLOGICAL SPACES

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Abstract. The main purpose of this paper is to define the concept of p-normal structure and give some new fixed point theorems of nonexpansive maps in general topological space (X,τ) by introducing the notion of a τ-symmetric function $p\colon X\times X\to\mathbb{R}^+$. An application to symmetrizable topological spaces has been made.

1. Introduction

The concept of normal structure was introduced by Brodskii and Milman [1] for the case of linear normed spaces. It was frequently used to prove existence theorems in fixed point theory. There were also some attempts to generalize the concept of normal structure to metric spaces [5, 9] and more abstract sets [3, 4].

Let (X,d) be a metric space. A selfmapping T of X is said to be nonexpansive if for each $x,y\in X$, $d(Tx,Ty)\leq d(x,y)$. Although such mappings are natural extension of the contraction mappings, it was clear from the outset that the study of fixed points of nonexpansive mappings required techniques which go far beyond the purely metric approch.The property of normal structure was introduced into fixed point theory for mappings of this class by W.A. Kirk in Banach spaces and since then a number of absract results were discovered, along with important discoveries related both to the structure of the fixed point sets and to techniques for approximating fixed points.

On the other hand, it has been observed that the distance function used in metric fixed point theorems proofs need not satisfy the triangular inequality nor $d(x,x)=0$ for all $x\in X$. Motivated by this idea, Hicks [2] established several important common fixed point theorems for general contractive selfmappings of a symmetrizable (resp. semi-metrizable) topological spaces. Recall that a symmetric function on a set X is a nonnegative real valued function d defined on $X\times X$ by

(1) $d(x,y)=0$ if and only if $x=y$,
(2) $d(x,y)=d(y,x)$

A symmetric function d on a set X is a semi-metric if for each $x\in X$ and each $\epsilon>0$, $B_d(x,\epsilon)=\{y\in X : d(x,y)\leq\epsilon\}$ is a neighborhood of x in the topology $t(d)$ defined as follows

$$\tau=\{U\subseteq X/\ \forall x\in U,\ B_d(x,\epsilon)\subset U,\ \text{for some }\epsilon>0\}$$

A topological space X is said to be symmetrizable (semi-metrizable) if its topology is induced by a symmetric (semi-metric) on X. Moreover, Hicks [2] proved that very general probabilistic structures admit a compatible symmetric or semi-metric.

2000 *Mathematics Subject Classification.* 47H10, 47H09, 54A20.
Key words and phrases. Hausdorff topological spaces, nonexpansive maps, normal structure, symmetrizable topological spaces.

For further details on semi-metric spaces (resp. probabilistic metric spaces), see, for example, [11] (resp. [10]).

In this paper, we follow ideas in [6, 7, 3, 4] to establish a generalization of the well known extension of Kirk's fixed point theorem [4]. Let (X,τ) be a topological space. The paper is structured as follows: We define a new function called τ-symmetric which extend the usual symmetric function and define the concept of p-normal structure and give some new fixed point theorems of nonexpansive maps in general topological space (X,τ) by introducing the notion of a τ-symmetric function $p\colon X\times X\to\mathbb{R}^+$. An application to symmetrizables topological spaces has been made.

2. τ-SYMMETRIC FUNCTION

Let (X,τ) be a topological space and $p\colon X\times X\to\mathbb{R}^+$ be a function. For any $\epsilon>0$ and any $x\in X$, let $B_p(x,\epsilon)=\{y\in X: p(x,y)<\epsilon\}$ and $B'_p(x,\epsilon)=\{y\in X: p(x,y)\le\epsilon\}$. $B'_p(x,\epsilon)$ will be said "band".

Definition 2.1. The function p is said to be a τ-symmetric if

- (τ_1) For all $x,y\in X$, $p(x,y)=p(y,x)$,
- (τ_2) For each $x\in X$ and any neighborhood V of x, there exists $\epsilon>0$ with $B_p(x,\epsilon)\subset V$.

Examples 2.1. 1. Let $X=\mathbb{R}^+$ and $\tau=\{X,\emptyset\}$. It is well known that the space (X,τ) is not metrisable. Consider the function p defined on $X\times X$ by $p(x,y)=(x-y)^2$ for all $x,y\in X$. It is easy to see that the function p is a τ-symmetric.

2. Each symmetric function d on a nonempty set X is a τ-symmetric on X where the topology τ is defined as follows: $U\in\tau$ if $\forall x\in U,\ B_d(x,\epsilon)\subset U,$ for some $\epsilon>0$.

3. Let $X=[0,+\infty[$ and $d(x,y)=|x-y|$ the usual metric. Consider the function $p\colon X\times X\to\mathbb{R}^+$ defined by

$$p(x,y)=e^{|x-y|},\ \ \forall x,y\in X$$

It is easy to see the function p is a τ-symmetric on X where τ is the usual toplogy since $\forall x\in X,\ B_p(x,\epsilon)\subset B_d(x,\epsilon),\ \ \epsilon>0$. Moreover, (X,p) is not a symmetric space since for all $x\in X$, $p(x,x)=1$.

Lemma 2.1. *Let (X,τ) be a topological space with a τ-symmetric p.*

- (a) *Let (x_n) be arbitrary sequence in X and (α_n) be a sequence in $\mathbb{R}^+$ converging to 0 such that $p(x_n,x)\le\alpha_n$ for all $n\in\mathbb{N}$. Then (x_n) converges to x with respect to the topology τ.*
- (b) *If τ is Hausdorff, then*
  - (b_1) *$p(x,y)=0$ implies $x=y$*
  - (b_2) *Given (x_n) in X, conditions $\lim_{n\to\infty}p(x,x_n)=0$ and $\lim_{n\to\infty}p(x_n,y)=0$, imply $x=y$.*

Proof. (a) Let V be a neighborhood of x. Since $\lim_{n\to\infty}p(x,x_n)=0$, there exists $N\in\mathbb{N}$ such that $\forall n\ge N,\ x_n\in V$. Therefore $\lim_{n\to\infty}x_n=x$ with respect to τ.

(b_1) Since $p(x,y)=0$, then $p(x,y)<\epsilon$ for all $\epsilon>0$. Let V be a neighborhood of x. Then there exists $\epsilon>0$ such that $B_p(x,\epsilon)\subset V$, which implies that $y\in V$. Since V is arbitrary, we conclude $y=x$.

(b_2) From (a), $\lim_{n\to\infty}p(x,x_n)=0$ and $\lim_{n\to\infty}p(y,x_n)=0$ imply $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}x_n=y$ with respect to the topology τ which is Hausdorff. Thus $x=y$. $\square$

Let us recall that each family (A_n) of closed nonempty subsets of a complete metric space (X,d) such that $\lim_{n\to\infty} \delta(A_n) = 0$, where $\delta(A) = \sup\{d(x,y) : x, y \in A\}$, has a nonempty intersection. It will be helpfull in the sequal to generalize this result to our setting. First, we give the following definition

Definition 2.2. Let (X,τ) be a topological space with a τ-symmetric p.

(1) We say that a nonempty subset A of X is p-closed iff

$$\overline{A}^p = \{x \in X : p(x,A) = 0\} \subset A$$

where $p(x,A) = \inf\{p(x,y)/y \in A\}$.

(2) A sequence in X is said p-Cauchy sequence if it satisfies the usual metric condition. There are several concepts of completeness in this setting.

(2.1) X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim_{n\to\infty} p(x_n, x) = 0$

(2.2) X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim_{n\to\infty} x_n = x$ with respect to the topology τ

(3) We say that X is sequentially p-compact if each sequence (x_n) of X has a p-convergente subsequence $(x_{n'})$, i-e. there exists $x \in X$ with $\lim_{n\to\infty} p(x, x_{n'}) = 0$.

Remark 2.1. Let (X,τ) be a topological space with a τ-symmetric p and let (x_n) be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x \in X$ such that $\lim_{n\to\infty} p(x_n, x) = 0$. Lemma 1$(a)$ then gives $\lim_{n\to\infty} x_n = x$ with respect to the topology τ. Therefore S-completeness implies p-Cauchy completeness. Moreover, it is easy to see that sequentially p-compactness implies that (X,τ) is sequentially compact.

Lemma 2.2. *Let (X,τ) be a Hausdorff topological space with a τ-symmetric p. Suppose that for each $x \in X$, the function $p(x,.) \colon X \to \mathbb{R}^+$ is lower semi-continuous. Then for each $x \in X$, the band $B'_p(x,r)$ is p-closed.*

Proof. Let $y \in \overline{B'_p(x,r)}^p$. Then $p(y, B'_p(x,r)) = 0$ and therefore, for all $n \in \mathbb{N}^*$, there exists a sequence (y_n) in $B'_p(x,r)$ such that $\lim_{n\to\infty} p(y, y_n) = 0$, which implies that $\lim_{n\to\infty} y_n = y$ with respect to the topology τ (lemma 2.1(a)). Since $p(x, y_n) \le r$ and $p(x,.)$ is lower semi-continuous, we get, by letting n to infty, $p(x,y) \le r$. Hence $y \in B'_p(x,r)$ and therefore $B'_p(x,r)$ is p-closed. ☐

Proposition 2.1. *Let (X,τ) be a Hausdorff topological space with a τ-symmetric p. Suppose that X is S-complete and p-bounded. Let (A_n) be a family of p-closed nonempty subsets of a X such that $\lim_{n\to\infty} \delta_p(A_n) = 0$. Then $\cap_{n\in\mathbb{N}} A_n = \{a\}$ for some $a \in X$.*

Proof. As in metric case, we can show that there exists $a \in X$ with $a \in A_n$ for all $n \in \mathbb{N}$. Lemma 1.(b_1) then assures the uniqueness of a. ☐

Definition 2.3. Let $\mathcal{F}$ be a nonempty family of subset of X. We say that $\mathcal{F}$ defines a convexity structure on X if and only if it is stable by intersection.

Example 2.1. Let (X,τ) be a Hausdorff topological space with a τ-symmetric p.

An admissible subset of X is any intersection of "bands". Let us denote the family of admissible subsets of X by $\mathcal{A}(X)$. It is obvious that $\mathcal{A}(X)$ defines a convexity structure on X.

Remark 2.2. 1. In view of lemma 2.2, if for each x in X, the function $p(x,.)\colon X \to \mathbb{R}^+$ is lower semi-continuous, then each admissible subset of X is p-closed.

2. In this work, we suppose that any other convexity structure $\mathcal{F}$ on X, contains $\mathcal{A}(X)$.

Definition 2.4. We say that $\mathcal{F}$ has the property (R) if and only if any decreasing sequence (A_n) of nonempty p-bounded and p-closed subsets of X with $A_n \in \mathcal{F}$, has a nonempty intersection.

Proposition 2.2. *Let (X,τ) be a Hausdorff topological space with a τ-symmetric p. Assume that X is S-complete and sequentially p-compact. Then*

(1) *Let C be a nonempty p-closed subset of X. Let $a \in X$ be such that $p(a,C) < \infty$. Then, there exists $b \in C$ such that $p(a,b) = p(a,C)$, where $p(a,C) = \inf\{p(a,c) : c \in C\}$.*
(2) *Let (C_n) be a decreasing family of p-closed nonempty subsets of a X. Then $\cap_{n\in\mathbb{N}} C_n \neq \emptyset$.*
(3) *X has the property (R).*

Proof. (1) It is not hard to see that a exists. Let us denote $\alpha = p(a,C) < \infty$. We can assume that $\alpha > 0$ (otherwise, $a \in C$ since C is p-closed). By the definition of α, there exists a sequence (x_n) in C such that $\lim\limits_{n\to\infty} p(a,x_n) = \alpha$. Since X is sequentially p-compact, there exists a subsequence $(x_{n'})$ of (x_n) and $b \in X$ such that $\lim\limits_{n\to\infty} p(b,x_{n'}) = 0$ which implies that $(x_{n'})$ converges to b with respect to the topology τ. Since C is p-closed, we have $b \in C$. Moreover, by using the lower semi-continuity of the function $p(a,.)$, we get $p(a,b) \leq \lim\limits_{n\to\infty} \inf p(a,x_{n'}) = \alpha$. Hence $p(a,b) = \alpha = p(a,C)$.

(2) As in (1), it is easy to see that there exists $a \in X$ such that for each integer n, $p(a,C_n) < \infty$. Since (C_n) is decreasing, the sequence $(p(a,C_n))$ is increasing and bounded. Hence, there exists $\alpha = \lim\limits_{n\to\infty} p(a,C_n) < \infty$. By (1), for each $n \in \mathbb{N}$, there exists $x_n \in C_n$ such that $p(a,x_n) = p(a,C_n)$. If $\alpha = 0$ then $p(a,C_n) = 0$ and consequently $a \in C_n$ for each $n \in \mathbb{N}$ since (C_n) is decreasing. Assume now that $\alpha > 0$. Repeating the argument from the proof of (1), we can prove that for each integer n, there exists $b \in C_n$ (C_n is p-closed) such that $p(a,b) = p(a,C_n)$. Since C_n are decreasing, it follows then that $b \in C_n$ for any natural n. Hence $\cap_{n\in\mathbb{N}} C_n \neq \emptyset$.

(3) It follows immediately from (2). □

Definition 2.5. Let (X,τ) be a toplogical space with a τ-symmetric p. For a subset A of X, we write

$$r_{p,x}(A) = \sup_{y\in A} p(x,y) \tag{1}$$

$$r_p(A) = \inf_{x\in A} r_{p,x}(A) \tag{2}$$

$$\delta_p(A) = \sup_{x\in A} r_{p,x}(A) \tag{3}$$

$$\operatorname{cov}(A) = \cap_{B'_p\in\mathcal{F}} B'_p \tag{4}$$

$$\operatorname{co}(A) = \cap_{f\in A} B'_p(f, r_{p,f}(A)) \tag{5}$$

where $\mathcal{F}$ is the family of "bands" containing A. Clearly, a subset A of X is admissible if and only if $A = \operatorname{cov}(A)$.

Definition 2.6. We say that X has p-normal structure if there exists a convexity structure $\mathcal{F}$ on X such that

$$r_p(A) < \delta_p(A), \text{ for every } A \in \mathcal{F} \text{ not reduced to a single point}$$

Remark 2.3. It is clear that if the topology τ is Hausdorff, then $\delta_p(A) = 0$ implies that the subset A is reduced to a single point.

Example 2.2. Let (X, d) be a metric space. It is clear that d is a τ-symmetric where τ is the topology induced by the metric d. Recall that X is said to have normal structure if there exists a convexity structure $\mathcal{F}$ on X such that $r_d(A) < \delta_d(A)$, for any nonempty $A \in \mathcal{F}$, which is d-bounded and not reduced to a single point. Hence, (X, d) has d-normal structure.

In [6], Kirk proved the following lemma in metric spaces. The analogue of this lemma in our p-normalsetting can be stated by the following lemma. The details of the proof are essentially the same and we given them for completeness.

Lemma 2.3. *Let (X, τ) be a topological space with a τ-symmetric p. Assume that X is p-bounded and has p-normal structure. Let T be a nonexpansive selfmapping of X. If $D \in \mathcal{A}(X)$ is T-invariant set, then there exists a nonempty admissible subset D^* of D, which is T-invariant, and such that*

$$\delta_p(D^*) \leq \frac{1}{2}(\delta_p(D) + r_p(D))$$

Proof. Set $r = \frac{1}{2}(\delta_p(D) + r_p(D))$. We can assume that $\delta_p(D) > 0$, otherwise we can take $D^* = D$. Since X has p-normal structure, we have $r_p(D) < \delta_p(D)$. Therefore, the set $A = \{f \in D : D \subset B'_p(f, r)\}$ is nonempty subset of X. Moreover, $A = \cap_{f \in D} B'_p(f, r) \cap D$, which implies that A is admissible. Clearly, there is no reason for A to be T-invariant. Put $\vartheta = \{M \in \mathcal{A}(X) : A \subset M \text{and} T(M) \subset M\}$ and $L = \cap_{M \in \vartheta} M$. Note that ϑ is nonempty since $X \in \vartheta$. The set L is T-invariant, admissible subset of X and contains A. Consider $C = A \cup T(L)$, and observe that $\text{co}(C) = L$. Indeed, since $C \subset L$ and $L \in \mathcal{A}(X)$, we have $\text{co}(C) \subset L$. From this we obtain $T(\text{co}(C)) \subset T(L) \subset C$, hence $C \in \mathcal{A}(X)$, and therefore $L \subset \text{co}(C)$. This gives the desired equality. Define $D^* = \{f \in L : L \subset B'_p(f, r)\}$. We claim that D^* is the desired set. Observe that D^* is nonempty since it contains A. Using the same argument we can prove that D^* is an admissible subset of X. On the other hand, it is clear that $\delta_p(D^*) \leq r$. To complete the proof, we have to show that D^* is T-invariant. Let $f \in D^*$. By definition of D^*, we have $L \subset B'_p(f, r)$. Since T is nonexpansive, we have $T(L) \subset B'_p(T(f), r)$. Let $g \in A$. Then $L \subset B'_p(g, r)$. But $T(f) \in L$, so that $T(f) \in B_p(g, r)$, which is equivalent to $g \in B'_p(T(f), r)$. Therefore $A \subset B'_p(T(f), r)$. Since $C = A \cup T(L)$, we deduce that $C \subset B'_p(T(f), r)$. Thus, we have $\text{co}(C) = L \subset B'_p(T(f), r)$. By the definition of D^*, it follows that $T(f) \in D^*$. In other words, D^* is T-invariant. □

Now we are ready to prove the following result

Theorem 2.1. *Let (X, τ) be a topological space with a τ-symmetric p. Assume that X is S-complete, p-bounded, has p-normal structure and satisfies the property (R). Let T be a nonexpansive selfmapping of X. Then T has a fixed point.*

Proof. Let $\mathcal{F} = \{M \in \mathcal{A}(X) : M \neq \emptyset \text{ and } T(M) \subset M\}$. The family $\mathcal{F}$ is stable by intersection and not empty since $X \in \mathcal{F}$. Define the function $\alpha \colon \mathcal{F} \to \mathbb{R}^+$ as follows

$$\alpha(M) = \inf\{\delta_p(A) : A \in \mathcal{F} \text{ and } A \subset M\}$$

Put $M_1 = X$. From the definition of α, we can define $M_2 \in \mathcal{F}$ by $\delta_p(M_2) \leq \alpha(M_1) + \epsilon_1$ and $M_2 \subset M_1$, where (ϵ_n) is a sequence of positive numbers such that

$\lim_{n\to\infty} \epsilon_n = 0$. Assume that M_i have been constructed for $i \leq n$, and define $M_{n+1} \in \mathcal{F}$ by $\delta_p(M_{n+1}) \leq \alpha(M_n) + \epsilon_n$ and $M_{n+1} \subset M_n$. Put $D = \cap_n M_n$. By our previous remarks on $\mathcal{F}$, we deduce that $D \in \mathcal{F}$. Moreover, property (R) implies that $D \neq \emptyset$. Suppose that D is not reduced to a single point. Since D satisfies all hypotheses of lemma 2.3, there exists D^* in $\mathcal{F}$, contained in D, such that

$$\delta_p(D^*) \leq \frac{1}{2}(\delta_p(D) + r_p(D)) \tag{6}$$

We have $\delta_p(D^*) \leq \delta_p(D) \leq \delta_p(M_{n+1}) \leq \alpha(M_n) + \epsilon_n$, for all $n \in \mathbb{N}$. Also, by the definition of α, we have $\alpha(M_n) \leq \delta_p(D^*)$. Since n is arbitrary and $\lim_{n\to\infty} \epsilon_n = 0$, we deduce that $\delta_p(D^*) = \delta_p(D)$. Then the inequality (6) implies that $\delta_p(D) \leq r_p(D)$, which gives a contradiction. Consequently, D is reduced to a single point which is then a fixed point for T. □

Recently, T.L. Hicks [2] established some common fixed point theorems for general contractive maps in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric or semi-metric. Now we look at application of our main results to the setting of symmetric spaces. Note that every metric spaces is a symmetric space.

Corollary 2.1. Let (X, d) be a symmetric space. Assume that X is S-complete, d-bounded, has d-normal structure and satisfies the property (R). Let T be a nonexpansive selfmapping of X. Then T has a fixed point.

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Received December 15, 2001.

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