*Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*
**18** (2002), 53–56
www.emis.de/journals

# PREORDERS AND EQUIVALENCES GENERATED BY COMMUTING RELATIONS

TAMÁS GLAVOSITS

ABSTRACT. For any relation $R$ on a fixed set $X$, we denote by $R^\star$ and $R^\bigstar$ the smallest preorder and equivalence on $X$ containing $R$, respectively.

We show that if $R$ and $S$ are commuting relations on $X$ in the sense that $R \circ S = S \circ R$, then $R^\star \circ S = S \circ R^\star$, $R \circ S^\star = S^\star \circ R$ and $R^\star \circ S^\star = S^\star \circ R^\star$.

Moreover, we show if in addition to the condition $R \circ S = S \circ R$ we also have $R \circ S^{-1} = S^{-1} \circ R$, then the corresponding equalities hold for the operation $\bigstar$ too.

## 1. A few basic facts on relations

As usual, a subset $R$ of a product set $X^2 = X \times X$ is called a relation on $X$. In particular, the relation $\Delta_X = \big\{(x,x) : \ x \in X\big\}$ is called the identity relation on $X$.

If $R$ is a relation on $X$, and moreover $x \in X$ and $A \subset X$, then the sets $R(x) = \big\{y \in X : (x,y) \in R\big\}$ and $R[A] = \bigcup_{a \in A} R(a)$ are called the images of $x$ and $A$ under $R$, respectively.

If $R$ is a relation on $X$, then the images $R(x)$, where $x \in X$, uniquely determine $R$ since we have $R = \bigcup_{x \in X} \{x\} \times R(x)$. Therefore, the inverse $R^{-1}$ of $R$ can be defined such that $R^{-1}(x) = \big\{y \in X : \ x \in R(y)\big\}$ for all $x \in X$.

Moreover, if $R$ and $S$ are relations on $X$, then the composition $S \circ R$ of $S$ and $R$ can be defined such that $(S \circ R)(x) = S\big[R(x)\big]$ for all $x \in X$. In particular, we write $R^n = R \circ R^{n-1}$ for all $n \in \mathbb{N}$ by agreeing that $R^0 = \Delta_X$.

A relation $R$ on $X$ is called reflexive, symmetric and transitive if $\Delta_X \subset R$, $R^{-1} \subset R$ and $R^2 \subset R$, respectively. Moreover, a reflexive and transitive relation is called a preorder, and a symmetric preorder is called an equivalence.

For any relation $R$ on $X$, we define $R^\star = \bigcup_{n=0}^{\infty} R^n$ and $R^\bigstar = \big(R \cup R^{-1}\big)^\star$. Thus, $R^\star$ and $R^\bigstar$ are the smallest preorder and equivalence on $X$ containing $R$, respectively. Moreover, $\star$ and $\bigstar$ are algebraic closures on $\mathcal{P}(X^2)$.

In the sequel, concerning the composition of relations, we shall also frequently need the fact that, for any two families $(R_i)_{i \in I}$ and $(S_j)_{j \in J}$ of relations on $X$, we have $\Big(\bigcup_{j \in J} S_j\Big) \circ \Big(\bigcup_{i \in I} R_i\Big) = \bigcup_{j \in J} \bigcup_{i \in I} S_j \circ R_i$.

## 2. Composition powers of commuting relations

**Theorem 2.1.** *If $R$ and $S$ are relations on $X$ such that $R \circ S \subset S \circ R$, then for all $n, m \in \mathbb{N}$ we have*

$$R^n \circ S^m \subset S^m \circ R^n.$$

2000 *Mathematics Subject Classification.* 08A02.

*Key words and phrases.* Preorders and equivalences, commutativity of composition.

*Proof.* If $n \in N$ such that $R^n \circ S \subset S \circ R^n$, then we can see that

$$R^{n+1} \circ S = (R \circ R^n) \circ S = R \circ (R^n \circ S) \subset R \circ (S \circ R^n) = (R \circ S) \circ R^n \subset (S \circ R) \circ R^n = S \circ (R \circ R^n) = S \circ R^{n+1}.$$

Hence, by induction, it is clear that $R^n \circ S \subset S \circ R^n$ holds true for all $n \in \mathbb{N}$.

Moreover, if $n, m \in \mathbb{N}$ such that $R^n \circ S^m \subset S^m \circ R^n$, then we can see that

$$R^n \circ S^{m+1} = R^n \circ (S \circ S^m) = (R^n \circ S) \circ S^m \subset (S \circ R^n) \circ S^m = S \circ (R^n \circ S^m) \subset S \circ (S^m \circ R^n) = (S \circ S^m) \circ R^n = S^{m+1} \circ R^n.$$

Hence, by induction on $m$, it is clear that $R^n \circ S^m \subset S^m \circ R^n$ for all $n, m \in \mathbb{N}$.

Now, as an immediate consequence of the above theorem, we can also state

**Corollary 2.2.** If $R$ and $S$ are relations on $X$ such that $R \circ S = S \circ R$, then for all $n, m \in \mathbb{N}$ we have

$$R^n \circ S^m = S^m \circ R^n.$$

Moreover, by using Theorem 2.1, we can also prove the following

**Theorem 2.3.** *If $R$ and $S$ are relations on $X$ such that $R \circ S \subset S \circ R$, then for all $n \in \mathbb{N}$ we have*

$$\big(R \cup S\big)^n \subset \bigcup_{k=0}^{n} S^{n-k} \circ R^k.$$

*Proof.* If $n \in \mathbb{N}$ such that $\big(R \cup S\big)^n \subset \bigcup_{k=0}^{n} S^{n-k} \circ R^k$, then by using Theorem 2.1, we can see that

$$\big(R \cup S\big)^{n+1} = \big(R \cup S\big) \circ \big(R \cup S\big)^n \subset (R \cup S) \circ \bigcup_{k=0}^{n} S^{n-k} \circ R^k =$$

$$\bigcup_{k=0}^{n} (R \cup S) \circ \big(S^{n-k} \circ R^k\big) = \bigcup_{k=0}^{n} \Big(R \circ \big(S^{n-k} \circ R^k\big) \cup \ S \circ \big(S^{n-k} \circ R^k\big)\Big) =$$

$$\bigcup_{k=0}^{n} \Big(\big(R \circ S^{n-k}\big) \circ R^k \cup \ S^{n+1-k} \circ R^k\Big) \subset \bigcup_{k=0}^{n} \Big(\big(S^{n-k} \circ R\big) \circ R^k \cup \ S^{n+1-k} \circ R^k\Big) =$$

$$\bigcup_{k=0}^{n} \Big(S^{n-k} \circ R^{k+1} \cup \ S^{n+1-k} \circ R^k\Big) = \left(\bigcup_{k=0}^{n} S^{n-k} \circ R^{k+1}\right) \cup \left(\bigcup_{k=0}^{n} S^{n+1-k} \circ R^k\right) =$$

$$\left(\bigcup_{k=0}^{n} S^{n+1-k} \circ R^k\right) \cup \left(\bigcup_{k=1}^{n+1} S^{n+1-k} \circ R^k\right) = \bigcup_{k=0}^{n+1} S^{n+1-k} \circ R^k.$$

Hence, by induction, it is clear that $\big(R \cup S\big)^n \subset \bigcup_{k=0}^{n} S^{n-k} \circ R^k$ for all $n \in \mathbb{N}$.

**Remark 2.4.** By Theorem 2.1, we can also state that

$$\bigcup_{k=0}^{n} R^{n-k} \circ S^k \subset \bigcup_{k=0}^{n} S^k \circ R^{n-k} = \bigcup_{k=0}^{n} S^{n-k} \circ R^k.$$

The following example shows that Theorem 2.3 cannot be improved by writing $\bigcup_{k=0}^{n} R^{n-k} \circ S^k$ in place of $\bigcup_{k=0}^{n} S^{n-k} \circ R^k$.

**Example 2.5.** If $X = \{1, 2\}$ and

$$R = \{(1, 2), (2, 2)\} \qquad \text{and} \qquad S = \{(2, 1), (2, 2)\},$$

then we can easily see that

$$R \circ S \subset S \circ R, \qquad \text{but} \qquad \big(R \cup S\big)^2 \not\subset \bigcup_{k=0}^{2} R^{n-k} \circ S^k.$$

By using Corollary 2.2, we can similarly prove the following counterpart of Newton's binomial theorem.

**Theorem 2.6.** *If $R$ and $S$ are relations on $X$ such that $R \circ S = S \circ R$, then for all $n \in \mathbb{N}$ we have*

$$\big(R \cup S\big)^n = \bigcup_{k=0}^{n} S^{n-k} \circ R^k.$$

**Remark 2.7.** Note that, in the latter theorem, we may write $\bigcup_{k=0}^{n} R^{n-k} \circ S^k$ in place of $\bigcup_{k=0}^{n} S^{n-k} \circ R^k$.

## 3. Preorders and equivalences generated by commuting relations

**Theorem 3.1.** *If $R$ and $S$ are relations on $X$ such that $R \circ S \subset S \circ R$, then*

*(1) $R^\star \circ S \subset S \circ R^\star$;* *(2) $R \circ S^\star \subset S^\star \circ R$;* *(3) $R^\star \circ S^\star \subset S^\star \circ R^\star$.*

*Proof.* By Theorem 2.1, we have

$$R^\star \circ S = \left(\bigcup_{k=0}^{\infty} R^n\right) \circ S = \bigcup_{k=0}^{\infty} R^n \circ S \subset \bigcup_{k=0}^{\infty} S \circ R^n = S \circ \left(\bigcup_{k=0}^{\infty} R^n\right) = S \circ R^\star$$

and

$$R \circ S^\star = R \circ \left(\bigcup_{k=0}^{\infty} S^n\right) = \bigcup_{k=0}^{\infty} R \circ S^n \subset \bigcup_{k=0}^{\infty} S^n \circ R = \left(\bigcup_{k=0}^{\infty} S^n\right) \circ R = S^\star \circ R.$$

Therefore, the assertions (1) and (2) are true. The assertion (3) follows from the assertion (1) by the the assertion (2).

Now, as an immediate consequence of Theorem 3.1, we can also state

**Corollary 3.2.** If $R$ and $S$ are relations on $X$ such that $R \circ S = S \circ R$, then

(1) $R^\star \circ S = S \circ R^\star$; (2) $R \circ S^\star = S^\star \circ R$; (3) $R^\star \circ S^\star = S^\star \circ R^\star$.

*Proof.* Since $R \circ S \subset S \circ R$ and $S \circ R \subset R \circ S$, by Theorem 3.1 we have

$$R^\star \circ S \subset S \circ R^\star, \quad R \circ S^\star \subset S^\star \circ R \quad \text{and} \quad S^\star \circ R \subset R \circ S^\star, \quad S \circ R^\star \subset R^\star \circ S.$$

Therefore, the assertions (1) and (2) are true. The assertion (3) again follows from the assertion (1) by the assertion (2).

The following example shows that the converse of the above corollary need not be true.

**Example 3.3.** If $X = \{1, 2\}$ and

$$R = \{(1, 2)\} \qquad \text{and} \qquad S = \{(1, 2), (2, 2)\},$$

then we can easily see that $R \circ S \subset S \circ R$, and moreover

$$R^\star \circ S = S \circ R^\star, \quad R \circ S^\star = S^\star \circ R, \quad R^\star \circ S^\star = S^\star \circ R^\star, \quad \text{but} \quad S \circ R \not\subset R \circ S.$$

As a partial analogue of Corollary 3.2, we can also prove the following

**Theorem 3.4.** *If $R$ and $S$ are relations on $X$ such that*

$$R \circ S = S \circ R \qquad \text{and} \qquad R \circ S^{-1} = S^{-1} \circ R,$$

*then*

*(1)* $R^\bigstar \circ S = S \circ R^\bigstar$; *(2)* $R \circ S^\bigstar = S^\bigstar \circ R$; *(3)* $R^\bigstar \circ S^\bigstar = S^\bigstar \circ R^\bigstar$.

*Proof.* Note that, in addition to the conditions of the theorem, we also have

$$R^{-1} \circ S = \left(S^{-1} \circ R\right)^{-1} = \left(R \circ S^{-1}\right)^{-1} = S \circ R^{-1}.$$

Therefore,

$$\left(R \cup R^{-1}\right) \circ S = R \circ S \ \cup\ R^{-1} \circ S = S \circ R \ \cup\ S \circ R^{-1} = S \circ \left(R \cup R^{-1}\right)$$

and

$$R \circ \left(S \cup S^{-1}\right) = R \circ S \ \cup\ R \circ S^{-1} = S \circ R \ \cup\ S^{-1} \circ R = \left(S \cup S^{-1}\right) \circ R.$$

Hence, by Corollary 3.2, it is clear that

$$R^\bigstar \circ S = \left(R \cup R^{-1}\right)^\star \circ S = S \circ \left(R \cup R^{-1}\right)^\star = S \circ R^\bigstar$$

and

$$R \circ S^\bigstar = R \circ \left(S \cup S^{-1}\right)^\star = \left(S \cup S^{-1}\right)^\star \circ R = S^\bigstar \circ R.$$

Therefore, the assertions (1) and (2) are true.

The assertion (3) again follows from the assertion (1) by the assertion (2). Namely, we also have

$$R^\bigstar \circ S^{-1} = \left(R^\bigstar\right)^{-1} \circ S^{-1} = \left(S \circ R^\bigstar\right)^{-1} = \left(R^\bigstar \circ S\right)^{-1} = S^{-1} \circ \left(R^\bigstar\right)^{-1} = S^{-1} \circ R^\bigstar.$$

The following example shows that the extra condition $R \circ S^{-1} = S^{-1} \circ R$ cannot be omitted from the above theorem.

**Example 3.5.** If $X = \{1, 2, 3\}$ and

$$R = \{(1, 2)\} \qquad \text{and} \qquad S = \{(3, 2)\},$$

then we can easily see that $R \circ S = S \circ R$, and moreover $S \circ R^\bigstar \subset R^\bigstar \circ S$ and $R \circ S^\bigstar \subset S^\bigstar \circ R$, but

$$R^\bigstar \circ S \not\subset S \circ R^\bigstar, \quad S^\bigstar \circ R \not\subset R \circ S^\bigstar, \quad R^\bigstar \circ S^\bigstar \not\subset S^\bigstar \circ R^\bigstar, \quad S^\bigstar \circ R^\bigstar \not\subset R^\bigstar \circ S^\bigstar.$$

**Acknowledgment.** The author is indebted to Árpád Száz for many helpful comments and suggestions.

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*Received March 15, 2002.*

Institute of Mathematics and Informatics,
University of Debrecen,
H-4010 Debrecen, Pf. 12, Hungary
*E-mail address*: glavosit@dragon.klte.hu