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CONDITIONS OF ANALYTICITY FOR FUNCTIONS OF ONE COMPLEX VARIABLE

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ABSTRACT. In this paper certain necessary and sufficient conditions are considered for the analyticity of nonlinear functions of one complex variable in terms of the their monogenity set.

Let $f\colon D \to \mathbb{C}$ be a function, continuous over the domain $D \subset \mathbb{C}$, and let $z \in D$ be its any fixed point. Put

$$\varphi_z(h) = \frac{f(z+h) - f(z)}{h},$$

defined over the domain $Q_\varepsilon = \{h \in \mathbb{C} \mid 0 < |h| < \varepsilon\}$ with $\varepsilon = \varepsilon(z)$, where $\varepsilon(z)$ denotes the distance from z to the boundary of D.

Recall that for the set of monogenity (the set of differential numbers) $\mathfrak{M}_z(f)$ of f at the point z is given by Luzin's equality [1]

$$\mathfrak{M}_z = \cap_{\varepsilon>0} \overline{\mathfrak{M}_\varepsilon(z)},$$

where $\mathfrak{M}_\varepsilon(z) = \{\xi \in \mathbb{C} \mid \xi = \varphi_z(h),\, h \in Q_\varepsilon\}$.

The following assertion gives a sufficient condition for analyticity.

Theorem 1. *Let $f\colon D \to \mathbb{C}$ be a function which is continuous on the domain D and monogenic in each everywhere dense subset E of D. If f satisfies the condition*

(a) *at any point $\xi \in \mathbb{C}$ there are at most a countable family of sets $\mathfrak{M}_z$ containing ξ,*

then f is a nonlinear analytic function over D.

Proof. Assume the contrary. Then there exists a perfect subset $P \subset D$, at the points of P f is not analytic.

The condition (a) immediately implies that the set $\mathfrak{M}_z$ is not the complete plane for all $z \in D$, with the possible exception of countable set $H \subset D$.

Let $\{\xi_k\}$ be the set of points of the plane $\mathbb{C}$ with rational coordinates, and denote $P_{n,k}$ $(n, k \in \mathbb{N})$ the subset of the points $z \in P \setminus H$ with

$$|\varphi_z(h) - \xi_k| \geq \frac{1}{n} \tag{0.1}$$

for all h satisfying $0 < |h| < \frac{1}{n}$ and $z + h \in D$.

As $\{\xi_k\}$ is an everywhere dense subset of the $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and $\mathfrak{M}_z$ is closed in $\overline{\mathbb{C}}$, it is easy to see

$$P \setminus H = \bigcup_{n,k} P_{n,k}.$$

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Moreover, since f is continuous, all the sets $P_{n,k}$ are closed.

The perfect set P is of second category (in itself), hence there exist indices $n = n_0$ and $k = k_0$ such that P_{n_0,k_0} is everywhere dense in some subset P_0 of P. Since P_{n_0,k_0} is closed, we have $P_0 = P_{n_0,k_0}$ and it can be written in the form $P_0 = P \cap G_0$, where $G_0 \subset D$ is a domain. Consider the function $g(z) = f(z) - cz$, where $c = \xi_{k_0}$. By (1) for $z \in P_0$ and $0 < |h| < \frac{1}{n_0}$ the function $g(z)$ satisfies

$$\left| \frac{g(z+h) - g(z)}{h} \right| \geq \frac{1}{n_0}. \tag{0.2}$$

If we put $z = z_1$, $z + h = z_2$ into (2), we obtain

$$|g(z_2) - g(z_1)| \geq \frac{1}{n_0} |z_2 - z_1|, \tag{0.3}$$

for all $z_2 \in K_0$ and $z_1 \in K_0 \cap P_0 := P_1$, where $K_0 \subset G_0$ is an arbitrary circle of diameter $\frac{1}{n_0}$.

Therefore, the function $g(z)$ is single leafed on the perfect set P_1 and analytical on the open set $K_0 \setminus P_1$ (if it is nonempty). By Theorem 9 [2] there exists a domain $G_1 \subset K_0$ on which the function $g(z)$ single leafed if $G_1 \cap P_1 = P_2$ is nonempty. Let us consider the inverse $z = g^{-1}(w)$ of the function $w = g(z)$ on the domain $G_1^* = g(G_1)$.

From (3) it follows that the function $g^{-1}(w)$ satisfies

$$\left| g^{-1}(w_1) - g^{-1}(w_2) \right| \leq n_0 |w_1 - w_2| \tag{0.4}$$

for any $w_1 \in P_2^* = g(P_2)$ and $w_2 \in G_1^*$. According to (4) the set $\mathfrak{M}_w(g^{-1})$ for $w \in P_2^*$ is bounded. By Theorem 2 [2] the function has a complete differential almost everywhere on P_2^*. Let us denote the corresponding subset of P_2^* by Q.

We have two cases to distinquish

Case 1. The set P_2^* is everywhere dense in a circle $K \subset G_1^*$. Since $P_2 \subset P_1$, we infer that the function $g(z)$ is single leafed in the domain $G_2 = g^{-1}(K)$.

We claim that the function $g^{-1}(w)$ is monogenic almost everywhere in K, i.e. in $Q \cap K = Q_1$.

Suppose the contrary, the function g^{-1} is not monogenic at a point $w_0 \in Q_1$. Then $\mathfrak{M}_{w_0}(g^{-1})$ is a complete circle ([2], p. 21). Put $E^* = g(E \cap G_2)$. Since the function $g(z)$ is continuous and single leafed in the domain G_2 the set E^* is everywhere dense in the circle $K = g(G_2)$.

Write

$$E_a^* = \left\{ w \in E^* \mid [g^{-1}(w)]' = a \right\},$$

where $a \in S$, S is a circle with boundary $\mathfrak{M}_{w_0}(g^{-1})$. It is easy to see that $E^* \supset \cup_a E_a^*$, therefore the sets E_a^* are disjoint. Since E^* is a countable set and S is not, there exists $a = a_0$ such that $E_{a_0}^* = \emptyset$, therefore $[g^{-1}(w)]' \neq a_0$ for any $w \in E^*$.

Let us consider the function $\psi(w) = g^{-1}(w) - a_0 w$ $(w \in K)$. By our assumption $0 \notin \mathfrak{M}_w(\psi)$ for $w \in K \setminus R$, with a countable set R. It is easy to see that the function $\psi(w)$ is single leafed in an open set Δ everywhere which is dense in K. To see this it suffices to take

$$M_n = \left\{ w \in K \mid \left| \frac{\varphi(w+t) - \varphi(w)}{t} \right| \geq \frac{1}{n}, 0 < |t| < \frac{1}{n} \right\},$$

and argue as in the proof of the inequality (3).

Since for any $w \in E^*$ there exists $\psi'(w) \neq 0$, the mapping $z = \psi(w)$ preserves the orientation of each component Δ_k $(k = 1, 2, \ldots)$ of the open set Δ.

Now we show that the function $z = \psi(w)$ realizes an inner mapping (in the sense of S. Stoylov) of the circle K.

Again, suppose the contrary, and let $\Delta \subset K$ ($\Delta \neq K$) be the maximal open subset of K on which the mapping $z = \psi(w)$ is inner.

Analogously, to our previous argument let us consider a subset L_0 of the perfect set $L = K \setminus \Delta$ on which the function $\psi(w)$ is single leafed. Clearly, we may assume $L_0 = K_1 \cap L$, where $K_1 \subset K$ is some circle. Since $\psi(w)$ is single leafed the set $\psi(L_0)$ is nowhere dense. Hence, by Theorem 8 [2], mapping $z = \psi(w)$ is inner in the circle K_1, which is a contradiction to the maximality of Δ.

Consequently, mapping $z = \psi(w)$ is inner in the circle K. It is well-known that an inner mapping either preserves or inverts the orientation at any point of the domain. As it is shown above, $z = \psi(w)$ preserves the orientation of the domains Δ_k ($k = 1, 2, \ldots$). On the other hand, at the point w_0 it inverts the orientation, therefore, the circle

$$\mathfrak{M}_{w_0}(\psi) = \left\{\omega \in \mathbb{C} \mid \omega = \psi_{w_0} + \psi_{\overline{w_0}} e^{-2i\beta}, \beta \in [0, 2\pi]\right\}$$

contains an inner point $\omega = 0$. Hence the Jacobian mapping

$$J(w_0) = |\psi_{w_0}|^2 - |\psi_{\overline{w_0}}|^2$$

is negative.

We have obtained a contradiction, so the function f is analytical over the domain D.

Case 2. Let P_2^* nowhere dense in the domain G_1^*. Then the function $\psi(w)$ is analytical on the open set $G_1^* \setminus P_2^*$ which is everywhere dense in the domain G_1^*, is single leafed on P_2^*. Hence, by Theorem 9 [2], function $z = \psi(w)$ realizes an inner mapping of the domain G_1^*. The remaining part of the statement can be proved analogously to Case 1.

The nonlinearity of f easily follows from the condition (a), because for a linear function $f(z) = cz + d$ we have $\mathfrak{M}_z(f) = \{c\}$ for any $z \in \mathbb{C}$. $\square$

Remark 1. *We show that the condition (a) is also necessary for the analyticity of a nonlinear function $f\colon D \to \mathbb{C}$ defined on a domain $D \subset \mathbb{C}$.*

Indeed, for an analytic function f and for $z \in D$ we have

$$\mathfrak{M}_z(f) = \{f'(z)\}\,. \tag{0.5}$$

Assume that our assertion is not true. Then, by (5), there exist a $c \in \mathbb{C}$ and an uncountable set $M \subset D$ such that $f'(z) = c$ for $z \in M$. It is easy to see that there exists a subdomain $\overline{D_1} \subset D$ such that $M_1 = M \cap \overline{D_1}$ is an un-countable set. By the Theorem of uniqueness for analytic functions we get $f'(z) \equiv c$ ($z \in D$). It follows that $f(z) = cz + c_0$ ($z \in D$), i.e., f is a linear function, which contradicts our assumption.

We point out another property of analytic functions in the next statement.

Theorem 2. *Let $w = f(z)$ be a nonlinear function which is analytic in the domain D and let $S_0 \subset D$ be a set of points $z \in D$ with $f''(z) = 0$. Then for any closed domain $\overline{D_0} \subset D \setminus S_0$ there exists $\varepsilon > 0$ such that*

$$f'(z) \notin \mathfrak{M}_\varepsilon(z), \tag{0.6}$$

where $z_0 \in \overline{D_0}$.

Proof. First note that each subdomain $\overline{D_0} \subset D$ contains at most a finite set of points of S_0. In the opposite case by the Theorem of uniqueness for analytic functions we have $f''(z) \equiv 0$ for any $z \in D$, i.e., f is linear.

Suppose that the assertion of theorem 2 is false. Then either in some subdomain $\overline{D_0} \subset D \setminus S_0$ for any $n \in \mathbb{N}$ ($n \geq n_0$) there exists a point $z_n \in \overline{D_0}$ such that

$f'(z_n) \in \mathfrak{M}_{\frac{1}{n}}(z_n)$, or there exists $\xi_n \in \mathbb{C}$ such that

$$f'(z_n) = \frac{f(\xi_n) - f(z_n)}{\xi_n - z_n}, \tag{0.7}$$

and $0 < |\xi_n - z_n| < \frac{1}{n}$.

We shall assume that the sequence $\{z_n\}$ converges to a point $z_0 \in \overline{D_0}$. (Otherwise a convergent subsequence of $\{z_n\}$ can be considered).

Clearly, $\xi_n \to z_0$ ($n \to \infty$).

By decomposing the function f into its Taylor series in the neighbourhoods of the points z_n, (7) can be rewritten as

$$f'(z_n) = f'(z_n) + \frac{f''(z_n)}{2!}(\xi_n - z_n) + \frac{f'''(z_n)}{3!}(\xi_n - z_n)^2 + \cdots$$

From this we obtain

$$\frac{f''(z_n)}{2!} + \frac{f'''(z_n)}{3!}(\xi_n - z_n) + \cdots = 0.$$

Taking limits we conclude $f''(z_0) = 0$. But this contradicts that $f''(z) \neq 0$ for any $z \in \overline{D_0}$. □

Since for an analytic function f monogenic at the point z we have (5), the condition in Theorem 2 can be reformulated as follows:

(b) *for any closed domain* $\overline{D_0} \subset D \setminus S_0$ *there exists* $\varepsilon > 0$ *such that*

$$\mathfrak{M}_z(f) \cap \mathfrak{M}_\varepsilon(z) = \emptyset, \tag{0.8}$$

where $z_0 \in \overline{D_0}$.

Note that the condition (b) (if $\overline{D_0}$ is a circle) with certain additional restrictions is also sufficient for the analyticity of a nonlinear function.

Theorem 3. *Let* $f\colon D \to \mathbb{C}$ *be a continuous function in the domain* $D \subset \mathbb{C}$, *monogenic almost everywhere in* D, *and let* $H \subset D$ *be an countable set.*

If for any closed circle $\overline{K} \subset D$ *there exists* $\varepsilon > 0$ *such that every* $z \in \overline{K} \setminus H$ *satisfies (8), then* f *is a nonlinear function which is analytic in the domain* D.

Proof. Assume the contrary; then there exists a perfect set $P \subset D$, at the points where f is not analytic.

Let $z_0 \in P$ be an arbitrary point, $K \subset D$ the circle with centre z_0 of radius $r \leq \frac{1}{2}\rho(z, \partial D)$, K_{ε_0} the concentric circle of radius $\varepsilon_0 \leq \min\{\varepsilon, \frac{r}{2}\}$, where ε is the number in (8). (Note that (8) also holds for any ε_0 with $0 < \varepsilon_0 < \varepsilon$).

For any fixed $z \in \overline{K_{\varepsilon_0}}$ consider the function $\varphi_z(h)$, $h \in Q_{\varepsilon_0}$. We have

$$\frac{\varphi_z(h+t) - \varphi_z(h)}{t} = \frac{1}{t}\left[\frac{f(z+h+t) - f(z)}{h+t} - \frac{f(z+h) - f(z)}{h}\right] =$$

$$= \frac{1}{th(h+t)}\{[f(z+h+t) - f(z+h)]h - [f(z+h) - f(z)]t\} =$$

$$= \frac{f(z+h+t) - f(z+h)}{t(h+t)} - \frac{f(z+h) - f(z)}{h(h+t)}.$$

This implies that each differential number $\omega(\varphi_z; h)$ of the function $\varphi_z(h)$ at the point h is determined by the equality

$$\omega(\varphi_z; h) = \frac{1}{h}\omega(f; z+h) - \frac{1}{h}\varphi_z(h).$$

We show that $0 \notin \mathfrak{M}_h(\varphi_z)$ for any $h \in Q_{\varepsilon_0} \setminus H_0$, where H_0 is an countable set. Indeed, $0 \in \mathfrak{M}_h(\varphi_z)$ implies that there exists h such that

$$\varphi_z(h) \in \mathfrak{M}_{z+h}(f),$$

where $z+h \in K$, or

$$\frac{f(z+h)-f(z)}{h} \in \mathfrak{M}_{z+h}(f).$$

Putting $z+h=z'$ we have

$$\frac{f(z)-f(z')}{z-z'} \in \mathfrak{M}_{z'}(f),$$

where $z, z' \in K$. However, this contradicts the condition of theorem 2.

If the function $\varphi_z(h)$ has nonzero differential in a set everywhere dense in Q_{ε_0}, and $0 \notin \mathfrak{M}_h(\varphi_z)$ for $h \in Q_{\varepsilon_0} \setminus H_0$, analogously as in the proof of Theorem 1 we claim that $\varphi_z(h)$ (for any fixed $z \in \overline{K_{\varepsilon_0}}$) realizes an inner mapping of the domain Q_{ε_0}.

Let M be the set of points for which, in accordance with the conditions of theorem 2, there exists the differential $f'(z) = \varphi_z(0)$, and $R = \max_{|h|=\varepsilon_0} |\varphi_z(h)|$ for any $z \in \overline{D_0}$.

Since the mapping $\xi = \varphi_z(h)$ is inner in the circle $K_0 = \{h \mid |h| < \varepsilon_0\}$, we assume $|\varphi_z(h)| \leq R$ for any h, $|h| < \varepsilon_0$ and $z \in M$.

Choose an arbitrary point $z_0 \in K_{\varepsilon_0} \setminus M$. By the continuity of the function $\varphi_z(h)$ of the variable z (for any fixed h) we get

$$\lim_{z \to z_0 z \in M} \varphi_z(h) = \varphi_{z_0}(h).$$

It follows that $|\varphi_z(h)| \leq R$ for any $h \in Q_{\varepsilon_0}$ and $z \in K_{\varepsilon_0}$, i.e. the sets of monogenity $\mathfrak{M}_z(f)$ are bounded in the circle K_{ε_0}. By Lemma 11 [2] we obtain that f is analytic in the circle K_{ε_0}, which contradicts $P_0 = P \cap K_{\varepsilon_0} \neq \emptyset$.

The nonlinearity of f follows from (8), since for a linear function $f(z) = cz + c_0$ we have

$$\mathfrak{M}_z(f) = \mathfrak{M}_\varepsilon(z) = \{c\}$$

for any $z \in \mathbb{C}$. $\square$

References

[1] V.S. Fedorov. The works of N.N. Luzin on the theory of functions of complex variable. *Uspehi-Matem.-Nauk (N.S.)*, 2(48):7–16, 1952.

[2] Yu.Yu. Trokhimchuk. Continuous mappings and conditions of monogeneity. In *Israel Program for Scientific Translations.* Daniel Davey & Co., Inc., New York, Jerusalem, 1964.

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