Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
19 (2003), 1–18
www.emis.de/journals

ON THE SHIFT-WINDOW PHENOMENON OF SUPER-FUNCTIONS

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Abstract. Using exploded numbers we consider the exploded Descartes-plane $\overset{\sqcap}{R^2} = \{(\overset{\sqcap}{x}, \overset{\sqcap}{y}) : x, y \in R\}$ in which have the graphs of super-functions. Of course certain parts of these graphs are visible in the traditional Descartes-plane R^2, while the other parts may be invisible. To show the invisible parts we introduce the super shift - and screw transformations which result the shifted - and screwed Descartes - coordinate systems. For the sake of understanding the paper contains some examples, too.

1. Computation with exploded numbers

In [1] we introduced the exploded numbers with the operations of super-addition and super-multiplication

$$\overset{\sqcap}{x} -\!\underset{\wedge}{\overset{\vee}{\oplus}}\!- \overset{\sqcap}{y} = \overset{\sqcap}{x+y}, \quad x, y \in R, \tag{1.1}$$

and

$$\overset{\sqcap}{x} -\!\underset{\wedge}{\overset{\vee}{\odot}}\!- \overset{\sqcap}{y} = \overset{\sqcap}{x \cdot y}, \quad x, y \in R, \tag{1.2}$$

respectively. Moreover, the rules of super-subtraction and super-division

$$\overset{\sqcap}{x} -\!\underset{\wedge}{\overset{\vee}{\ominus}}\!- \overset{\sqcap}{y} = \overset{\sqcap}{x-y}, \quad x, y \in R, \tag{1.3}$$

and

$$\overset{\sqcap}{x} -\!\underset{\wedge}{\overset{\vee}{(:)}}\!- \overset{\sqcap}{y} = \overset{\sqcap}{x : y}, \quad x, y \in R,\ y \neq 0, \tag{1.4}$$

were presented, too. The number $\overset{\sqcap}{x}$ was called the exploded of the real number $x \in R$ and the set of exploded numbers is denoted by $\overset{\sqcap}{R}$. Sometimes we say that $\overset{\sqcap}{R}$ is the exploded real axis. For any real number $x \in (-1, 1)$ explosion means

$$\overset{\sqcap}{x} = \operatorname{area}\operatorname{th} x. \tag{1.5}$$

Clearly, R is isomorphic with $\overset{\sqcap}{R}$ by the transformation

$$x \rightarrowtail \overset{\sqcap}{x},$$

$$+ \rightarrowtail -\!\underset{\wedge}{\overset{\vee}{\oplus}}\!-,$$

$$\cdot \rightarrowtail -\!\underset{\wedge}{\overset{\vee}{\odot}}\!-.$$

2000 *Mathematics Subject Classification.* 03C30.

Key words and phrases. Exploded numbers, exploded Descartes-plane, window-phenomenon, super-shift transformation, screw transformation.

Moreover,

$$0 = \overbracket{0} \tag{1.6}$$

is the unit element of super-addition and the unit element of super-multiplication is $\overbracket{1}$.

Considering that by (1.5) for any $x \in (-1, 1)$ we have

$$\overbracket{-x} = -\overbracket{x} \tag{1.7}$$

for any $x \in \overbracket{R}$ its additive inverse is denoted by $(-x)$.

For any pair $x, y \in \overbracket{R}$ the arrangement

$$x < y \quad \text{if and only if} \quad \underbracket{x} < \underbracket{y} \tag{1.8}$$

was also introduced in [1] (see [1], Definition 1.7), where $\underbracket{x}$ is called the compressed of x, defined by the identity

$$\overbracket{(\underbracket{x})} = x, \quad x \in \overbracket{R}. \tag{1.9}$$

Clearly, for any $x \in \overbracket{R}$ we have $\underbracket{x} \in R$. The inverse of explosion is called compression. Moreover,

$$\underbracket{x} = \operatorname{th} x \quad \text{if} \quad -\infty < x < \infty, \tag{1.10}$$

and we also use the identity

$$\underbracket{(\overbracket{x})} = x, \quad x \in R. \tag{1.11}$$

Clearly, for any $x \in R$ and $x \in \overbracket{R}$

$$\overbracket{-x} = -\overbracket{x} \quad \text{and} \quad \underbracket{-x} = -\underbracket{x}, \tag{1.12}$$

respectively. If $x \in \overbracket{R}$ and $x \leq -\overbracket{1}$ or $x \geq \overbracket{1}$ we say that x is invisible. The visible subset of the exploded real axis $\overbracket{R}$ is the real axis R itself. So, if $x \in R$ then $-\overbracket{1} < x < \overbracket{1}$.

2. Super-functions and super-curves

The concept of super-function was introduced in [1]. (See [1], (part 4).) Considering a basic function f with its definition-domain belonging to R we said that an $x \in \overbracket{R}$ belongs to the definition-domain of super-function $\operatorname{spr} f$ if $f(\underbracket{x})$ is defined. Moreover,

$$\operatorname{spr} f(x) = \overbracket{f(\underbracket{x})}. \tag{2.1}$$

Remark 2.2. If $S \subset \overbracket{R}$ and for any $x \in S$

$$y = F(x) \tag{2.3}$$

is unambiguous, then F is a super-function.

Really, considering

$$f_F(x) = \underbracket{F(\overbracket{x})}, \quad x \in \underbracket{S} \subset R$$

where $\underbracket{S}$ is the set of the compressed numbers of elements belonging to S as a basic function, we have by (2.1) and (1.9) that

$$\operatorname{spr} f_F(x) = \overbracket{f_F(\underbracket{x})} = \overbracket{(\underbracket{F(\overbracket{(\underbracket{x})})})} = F(x).$$

The graphs of super-functions are situated in the exploded Descartes-plane $\overbracket{R^2}$ which is a super-linear space with the operations

$$X \oplus Y = (x_1 \oplus y_1, x_2 \oplus y_2), \quad X = (x_1, x_2), \quad Y = (y_1, y_2)$$

and

$$c \odot X = (c \odot x_1, c \odot x_2), \quad x_1, x_2, y_1, y_2 \text{ and } c \in \overbracket{R}$$

as well as

$$X \ominus Y = (x_1 \ominus y_1, x_2 \ominus y_2).$$

Moreover, $\overbracket{R^2}$ is a super-eucledian space with the super-inner-product

$$X \odot Y = (x_1 \odot y_1) \oplus (x_2 \odot y_2)$$

and it is super-normed with

$$\|X\|_{\underbracket{R^2}} = \operatorname{spr} \sqrt{X \odot X}.$$

If $X, Y, Z \in \overbracket{R^2}$ and $X \neq Z$, $Y \neq Z$ then we say

$$\begin{aligned} \text{meas} \quad \operatorname{spr} \sphericalangle XZY = \operatorname{spr} \arccos\Big(&((X \ominus Z) \odot (Y \ominus Z)) \odot \\ &(\|X \ominus Z\|_{\underbracket{R^2}} \odot \|Y \ominus Z\|_{\underbracket{R^2}.75})\Big). \end{aligned} \tag{2.4}$$

The super-curve G is defined as follows:

$$G = \big\{X = (x, y) \in \overbracket{R^2} : x = x(t), y = y(t), \quad \alpha \leq t \leq \beta, \quad (\alpha, \beta) \subset \overbracket{R}\big\}$$

such that $u = \underbracket{x(t)}$ and $v = \underbracket{y(t)}$ are continuously differentiable functions of the parameter $\varphi = \underbracket{t}$ on the interval $[\underbracket{\alpha}, \underbracket{\beta}]$. Moreover, we require for any $\varphi \in [\underbracket{\alpha}, \underbracket{\beta}]$ that

$$(u'(\varphi))^2 + (v'(\varphi))^2 \neq 0.$$

An important special case is the super-line $L_C(V)$ defined by the equation

$$X = C \oplus (t \odot V), \quad t \in \overbracket{R}, \tag{2.5}$$

where $C = (c_1, c_2)$, $V = (v_1, v_2) \in \overbracket{R^2}$ are given such that

$$(v_1 \odot v_1) \oplus (v_2 \odot v_2) = \overbracket{1}. \tag{2.6}$$

The super measure of super-angle of super-lines $L_C(V)$ and $L_C(U)$ with

$$\|U\|_{\underbracket{R^2}} = \overbracket{1}$$

is defined as follows:

$$\begin{aligned} \text{meas} \quad \operatorname{spr} \sphericalangle(L_C(V), L_C(U)) = \min(&\operatorname{spr} \arccos(V \odot U), \\ &\operatorname{spr} \arccos(-V \odot U)). \end{aligned} \tag{2.7}$$

3. Window phenomenon and shift-window phenomenon

Considering a super-curve G (which may be a graph of a super-function) the intersection

$$G \cap R^2$$

is called the window phenomenon of G (or a super-function of f). For example, the window phenomenon of the super-unit-circle with the centre origo $\|x\|_{\underbracket{R^2}} = \overbracket{1}$ is itself the super-unit-circle except for points $(-\overbracket{1}, 0)$, $(0, \overbracket{1})$, $(\overbracket{1}, 0)$ and $(0, -\overbracket{1})$ which are invisible points in the window R^2.

Considering a fixed point $(x_0, y_0) \in \overbracket{R^2}$ we introduce the super-shift transformation

$$
\begin{aligned}
x &= \xi \mathbin{-\!\oplus\!-} x_0 \\
y &= \eta \mathbin{-\!\oplus\!-} y_0
\end{aligned}
\tag{3.1}
$$

which moves the exploded Descartes-coordinate system "x, y" into the system "ξ, η" having the new coordinate axes "ξ" and "η" with the new origo (x_0, y_0). While the Descartes-plane

$$
\begin{aligned}
-\overbracket{1} < x < \overbracket{1} \\
-\overbracket{1} < y < \overbracket{1}
\end{aligned}
$$

is moving on the exploded Descartes-plane $\overbracket{R^2}$ into the Descartes-plane

$$
\begin{aligned}
x_0 \mathbin{-\!\ominus\!-} \overbracket{1} < x < x_0 \mathbin{-\!\oplus\!-} \overbracket{1} \\
y_0 \mathbin{-\!\ominus\!-} \overbracket{1} < y < y_0 \mathbin{-\!\oplus\!-} \overbracket{1}
\end{aligned}
$$

we can see the another subset of $\overbracket{R^2}$. This transformation gives a new window on the wall $\overbracket{R^2}$.

We can observe this transformation on the compressed model where the compressed wall $\overbracket{R^2}$ is R^2 and the compressed window R^2 is $\underbracket{R^2}$, that is, the set of points $(\underbracket{x}, \underbracket{y})$ for which

$$
\begin{aligned}
&-1 < \underbracket{x} < 1 \\
&-1 < \underbracket{y} < 1 \quad (x, y) \in R^2.
\end{aligned}
$$

By the compression the super-shift transformation becomes the familiar shift-transformation

$$
\begin{aligned}
\underbracket{x} &= \underbracket{\xi} + \underbracket{x_0} \\
\underbracket{y} &= \underbracket{\eta} + \underbracket{y_0}
\end{aligned}
$$

and the new sub-window is the set of points $(\underbracket{x}, \underbracket{y})$ for which

$$
\begin{aligned}
\underbracket{x_0} - 1 < \underbracket{x} < \underbracket{x_0} + 1 \\
\underbracket{y_0} - 1 < \underbracket{y} < \underbracket{y_0} + 1.
\end{aligned}
$$

Example 3.2. In the window we can see the visible parts of graphs of super-parabola

$$
y = x \mathbin{-\!\odot\!-} x, \quad x \in \overbracket{R}
$$

and of super-square-root function $\operatorname{spr}\sqrt{x} = \overbracket{\sqrt{\underbracket{x}}}, x \geq 0$. Their super-curves have two common points $(0, 0)$ and $(\overbracket{1}, \overbracket{1})$ such that the latter is invisible. Using the super-shift transformation (3.1) with $(x_0, y_0) = (\overbracket{1}, \overbracket{1})$, the new equation of super-parabola is

$$
\eta = (\xi \mathbin{-\!\oplus\!-} \overbracket{2}) \mathbin{-\!\odot\!-} \xi, \quad \xi \in \overbracket{R}
$$

while the super-square-root has the equation

$$
\eta = \overbracket{\left(\sqrt{\underbracket{\xi \mathbin{-\!\oplus\!-} \overbracket{1}}}\right)} \mathbin{-\!\ominus\!-} \overbracket{1}, \quad -\overbracket{1} \leq \xi \in \overbracket{R}
$$

having the equations of window-curves (window phenomenons)

$$
\eta = \operatorname{area\,th}((\operatorname{th}\xi)(\operatorname{th}\xi + 2)), \quad -\infty < \xi < \operatorname{area\,th}(\sqrt{2} - 1)
$$

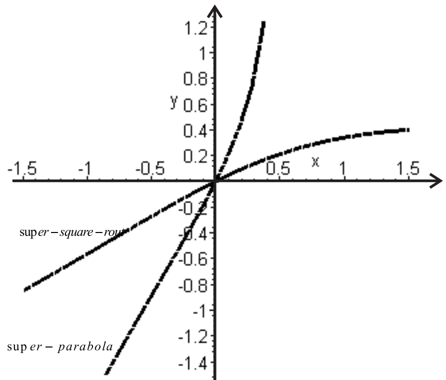

FIGURE 3.3.

and

$$\eta = \operatorname{area\,th}(\sqrt{\operatorname{th}\xi + 1} - 1), \quad -\infty < \xi < \infty,$$

respectively. Their intersection is visible in the new window:

Example 3.3. Considering the super-function

$$y = (\overset{\sqcup}{2} \mathbin{\circledast\!\!\odot} x) \mathbin{\circledast\!\!(:)} (\overset{\sqcup}{1} \mathbin{\circledast\!\!\oplus} (x \mathbin{\circledast\!\!\odot} x)), \quad x \in \overset{\sqcup}{R} \tag{3.4}$$

by (1.6), (1.9), (1.2) (1.1), (1.4), (1.10) and (1.5) we can see that for any $x \in R$

$$(\overset{\sqcup}{2} \mathbin{\circledast\!\!\odot} x) \mathbin{\circledast\!\!(:)} (\overset{\sqcup}{1} \mathbin{\circledast\!\!\oplus} (x \mathbin{\circledast\!\!\odot} x)) = \overline{\left(\frac{2\underset{\sqcup}{x}}{1 + (\underset{\sqcup}{x})^2}\right)} =$$

$$= \overline{\left(\frac{2\operatorname{th} x}{1 + \operatorname{th}^2 x}\right)} = \overline{\operatorname{th} 2x} = \operatorname{area\,th}(\operatorname{th} 2x) = 2x$$

is obtained. This means that the (familiar) line, having the equation

$$y = 2x \quad (x \in R), \tag{3.5}$$

is the window-curve of the super-function defined under (3.4). In the window the graph of super-function appears to be a line. Applying the super-shift transformation (3.1) with $(x_0, y_0) = (\overset{\sqcup}{1}, \overset{\sqcup}{1})$ we have the new equation of the original super-function (3.4):

$$\eta = -((\xi \mathbin{\circledast\!\!\odot} \xi) \mathbin{\circledast\!\!(:)} (\overset{\sqcup}{2} \mathbin{\circledast\!\!\oplus} (\xi \mathbin{\circledast\!\!(:)} (\xi \mathbin{\circledast\!\!\oplus} \overset{\sqcup}{2})))), \quad \xi \in \overset{\sqcup}{R}. \tag{3.6}$$

The window phenomenon of (3.6) has the equation

$$\eta = \frac{1}{2} \ln \frac{2(e^{4\xi} + e^{2\xi})}{3e^{4\xi} + 1}, \quad \xi \in R \tag{3.7}$$

and we can see it in the shift-window which is the following subset of $\overset{\sqcup}{R}$:

$$\begin{aligned} 0 &< x < \overset{\sqcup}{2} \\ 0 &< y < \overset{\sqcup}{2}. \end{aligned} \tag{3.8}$$

Let us consider the points $O = (0, 0), P_1 = (\frac{1}{2}, 1)$, $P_2 = (1, 2)$ and $\Omega = (\overset{\sqcup}{1}, \overset{\sqcup}{1})$ of the graph of super-function (3.4). Points O, P_1, and P_2 are situated on the line

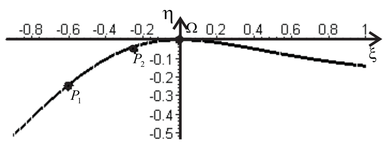

FIGURE 3.9.

(3.5) but Ω is invisible. On the other hand P_1, P_2 and Ω has new coordinates in the Descartes-system "ξ, η":

$$\left(\frac{1}{2} \overset{\vee}{\underset{\wedge}{\ominus}} \overset{\sqcup}{1}, 1 \overset{\vee}{\underset{\wedge}{\ominus}} \overset{\sqcup}{1}\right), \left(1 \overset{\vee}{\underset{\wedge}{\ominus}} \overset{\sqcup}{1}, 2 \overset{\vee}{\underset{\wedge}{\ominus}} \overset{\sqcup}{1}\right) \quad \text{and} \quad (0,0)$$

respectively, so we can see them in the shift-window (3.8) on the window-phenomenon (3.7):

In Fig. 3.9 the point O is invisible. However, another super-shift transformation, namely (3.1) with

$$(x_0, y_0) = \left(\overset{\sqcup}{(\tfrac{1}{2})},\ \overset{\sqcup}{(\tfrac{1}{2})}\right),$$

results that O, P_1, P_2 and Ω are visible in the new window. In this case, the new equation of the super-function defined under (3.4) is

$$\eta = \left(\left(\overset{\sqcup}{(-\tfrac{1}{2})} \overset{\vee}{\underset{\wedge}{\odot}} \xi \overset{\vee}{\underset{\wedge}{\odot}} \xi\right) \overset{\vee}{\underset{\wedge}{\oplus}} \left(\overset{\sqcup}{(\tfrac{3}{2})} \overset{\vee}{\underset{\wedge}{\odot}} \xi\right) \overset{\vee}{\underset{\wedge}{\oplus}} \overset{\sqcup}{(\tfrac{3}{8})}\right) \overset{\vee}{\underset{\wedge}{\oslash}} \left((\xi \overset{\vee}{\underset{\wedge}{\odot}} \xi) \overset{\vee}{\underset{\wedge}{\oplus}} \xi \overset{\vee}{\underset{\wedge}{\oplus}} \overset{\sqcup}{(\tfrac{5}{4})}\right), \quad \xi \in \overset{\sqcup}{R} \tag{3.10}$$

The window phenomenon of (3.10) has the equation

$$\eta = \operatorname{area\,th} \frac{-\frac{1}{2}(\operatorname{th}\xi)^2 + \frac{3}{2}\operatorname{th}\xi + \frac{3}{8}}{(\operatorname{th}\xi)^2 + \operatorname{th}\xi + \frac{5}{4}}, \quad \operatorname{area\,th}(\sqrt{3} - \frac{5}{2}) < \xi < \infty \tag{3.11}$$

and we have:

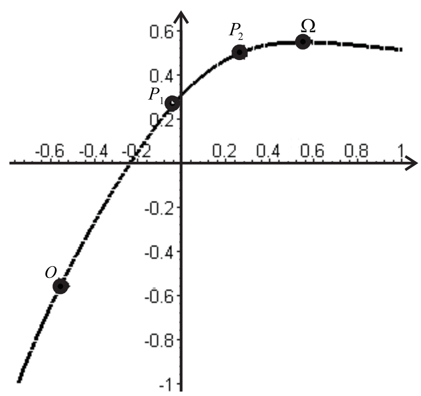

FIGURE 3.12.

The case of super-shift transformation (3.1) with $(x_0, y_0) = (0, \overbracket{1})$:

$$x = \xi \quad \text{and} \quad y = \eta \mathbin{-\!\!\oplus\!\!-} \overbracket{1}.$$

The new equation of super-function defined under (3.4) is the following:

$$\eta = -\Big(((\overbracket{1} \mathbin{-\!\!\ominus\!\!-} \xi) \mathbin{-\!\!\odot\!\!-} (\overbracket{1} \mathbin{-\!\!\ominus\!\!-} \xi)) \mathbin{-\!\!\odot\!\!-} (\overbracket{1} \mathbin{-\!\!\odot\!\!-} (\xi \mathbin{-\!\!\odot\!\!-} \xi))\Big), \quad \xi \in \overbracket{R} \tag{3.13}$$

The window phenomenon of (3.13) is

$$\eta = \frac{1}{2} \ln \frac{e^{4\xi} - 1}{e^{4\xi} + 3}, \quad 0 < \xi < \infty, \tag{3.14}$$

which shows that both O and Ω are invisible:

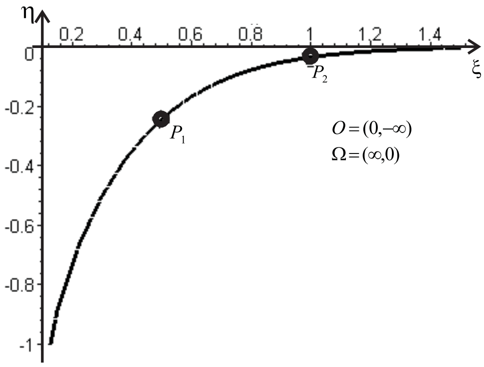

FIGURE 3.15.

The super-shift transformation (3.1) with $\underbracket{(x_0, y_0)} = (\overbracket{1}, 0)$ is the most interesting. Now, the shift-window is the following subset of $\overbracket{R^2}$:

$$\begin{aligned} 0 < x < \overbracket{2} \\ -\infty < y < \infty. \end{aligned} \tag{3.16}$$

Similarly to the case mentined above, O and Ω are invisible again. The new equation of super-function defined under (3.4) is the following:

$$\begin{aligned} \eta = \Big((\overbracket{2} \mathbin{-\!\!\odot\!\!-} \xi) \mathbin{-\!\!\oplus\!\!-} \overbracket{2}\Big) \mathbin{-\!\!\odot\!\!-} \\ \Big((\xi \mathbin{-\!\!\odot\!\!-} \xi) \mathbin{-\!\!\oplus\!\!-} (\overbracket{2} \mathbin{-\!\!\odot\!\!-} \xi) \mathbin{-\!\!\odot\!\!-} \overbracket{2}\Big), \quad \xi \in \overbracket{R}. \end{aligned} \tag{3.17}$$

The window phenomenon of (3.17) has the equation

$$\eta = \operatorname{area\,th} \frac{2(\operatorname{th}\xi + 1)}{1 + (\operatorname{th}\xi + 1)^2}, \quad -\infty < \xi < \infty, \quad \xi \neq 0 \tag{3.18}$$

By Fig. 3.9, 3.12, 3.15 and 3.19 it seems that Ω is the maximum point of the graph of the super-function (3.4). Really, we have

Remark 3.20. Considering that for any $\underbracket{x} \in R$ the inequalities

$$-1 \leq \frac{2\underbracket{x}}{1 + (\underbracket{x})^2} \leq 1$$

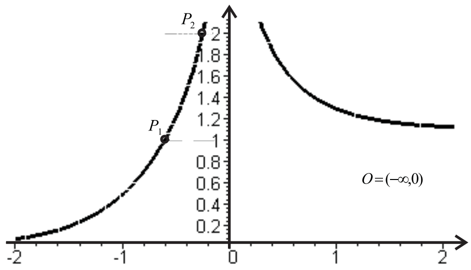

FIGURE 3.19.

are valid such that the equalities are valid if and only if the left-hand side, $\underset{\sqcup}{x} = -1$ and, on the right-hand side, $\underset{\sqcup}{x} = 1$. Hence (1.6), (1.7) and (1.8) yield

$$-\overset{\sqcup}{1} \le (\overset{\sqcup}{2} \mathbin{-\!\!\odot\!\!-} x) \mathbin{-\!\!\odot\!\!-} (\infty \mathbin{-\!\!\oplus\!\!-} (x \mathbin{-\!\!\odot\!\!-} x)) \le \overset{\sqcup}{1}, \quad x \in \overset{\sqcup}{R} \tag{3.21}$$

with the equalities on the left-hand or right-hand sides in the cases $x = -\overset{\sqcup}{1}$ or $x = \overset{\sqcup}{1}$, respectively.

Remark 3.22. By (3.21) we can see that the super-function defined under (3.4) is bounded but not constant. So, we have that (3.4) is not a super-line though its window phenomenon is line (3.5).

Example 3.23. Let us consider the super-square determined by the super-lines

$$\begin{aligned} L_{\underset{1,0}{\sqcup}} &: y = x \\ L_{-\infty,\infty} &: y = \overset{\sqcup}{1} \mathbin{-\!\!\ominus\!\!-} x \\ L_{\infty,-\infty} &: y = x \mathbin{-\!\!\ominus\!\!-} \overset{\sqcup}{1} \\ L_{-\infty,0} &: y = -x \end{aligned} \tag{3.23}$$

Using the super-shift transformation (3.1) with $(x_0, y_0) = (\overset{\sqcup}{(\frac{1}{2})}, 0)$ we can see the symmetric form of the super-square in the shift window:

$$\overset{\sqcup}{(\tfrac{1}{2})} < x < \overset{\sqcup}{(\tfrac{3}{2})}$$
$$-\infty < y < \infty$$

where the new equations of super-lines are:

$$\begin{aligned} L_{\infty,0} &: \eta = \xi \mathbin{-\!\!\oplus\!\!-} \overset{\sqcup}{(\tfrac{1}{2})} \\ L_{-\infty,\infty} &: \eta = \overset{\sqcup}{(\tfrac{1}{2})} \mathbin{-\!\!\ominus\!\!-} \xi \\ L_{\infty,-\infty} &: \eta = \xi \mathbin{-\!\!\ominus\!\!-} \overset{\sqcup}{(\tfrac{1}{2})} \\ L_{-\infty,0} &: \eta = -(\xi \mathbin{-\!\!\oplus\!\!-} \overset{\sqcup}{(\tfrac{1}{2})}) \end{aligned}$$

with the equations of window phenomena $y = \operatorname{area\,th}(\operatorname{th}\xi + \frac{1}{2})$, $y = \operatorname{area\,th}(\frac{1}{2} - \operatorname{th}\xi)$, $y = \operatorname{area\,th}(\operatorname{th}\xi - \frac{1}{2})$, and $y = -\operatorname{area\,th}(\operatorname{th}\xi + \frac{1}{2})$, respectively.

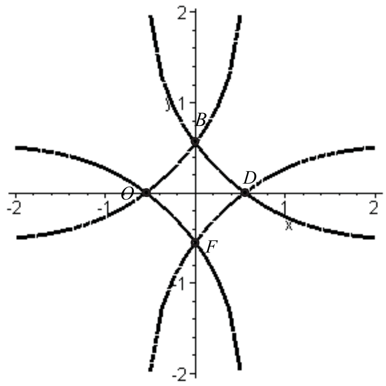

FIGURE 3.24.

Example 3.25. That the super-lines

$$L_{\infty,\infty} : y = x \mathbin{-\!\oplus\!-} \overset{\sqcup}{1}$$

$$L_{\underset{2,0}{\sqcup}} : y = \overset{\sqcup}{2} \mathbin{-\!\odot\!-} x$$

have the common (invisible) point $M = (\overset{\sqcup}{1}, \overset{\sqcup}{2})$. It is easy too see that point $Q = (-\overset{\sqcup}{1}, 0)$ lies on $L_{\infty,\infty}$ and $O = (0,0)$ lies on $L_{\underset{2,0}{\sqcup}}$. Using (2.4), by (1.1) - (1.4), (1.6), (1.7) and (2.1) we compute

$$\text{meas} \quad \text{spr} \sphericalangle QMO = \text{area th}(\arccos \frac{3}{\sqrt{10}}).$$

By the super-shift transformation (3.1) with $(x_0, y_0) = M$, we can see the intersection in the shift-window:

$$0 < x < \overset{\sqcup}{2}$$
$$\overset{\sqcup}{1} < y < \overset{\sqcup}{3}.$$

Now, the new equations of super-lines are

$$L_{\infty,\infty} : \eta = \xi$$

$$L_{\underset{2,0}{\sqcup}} : \eta = \overset{\sqcup}{2} \mathbin{-\!\odot\!-} \xi$$

with the equations of shift-window phenomena $\eta = \xi$, $(\xi \in R)$ and $\eta = \text{area th}(2 \,\text{th}\, \xi)$, $(|\xi| < \text{area th} \frac{1}{2})$ respectively.

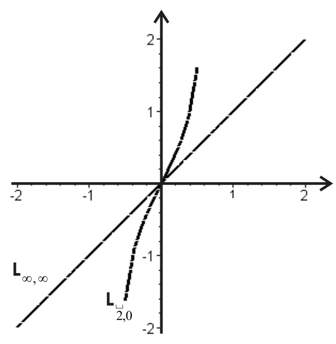

FIGURE 3.26.

Example 3.27. Let us consider super-lines

$$L_C(V) : X = C \oplus (t \odot V) \tag{3.28}$$

and

$$L_C(U) : X = C \oplus (t \odot U), \tag{3.29}$$

where

$$C = \Big(\overline{(\frac{2+\sqrt{3}}{2})} \;,\; \overline{(\frac{\sqrt{3}}{2})}\Big), \quad V = \Big(\overline{\sqrt{\frac{2+\sqrt{3}}{4}}} \;,\; \overline{\sqrt{\frac{2-\sqrt{3}}{4}}}\Big)$$

and

$$U = \Big(\overline{\sqrt{\frac{1}{2}}} \,,\, \overline{\sqrt{\frac{1}{2}}}\Big).$$

Computing their super-angle by (1.1), (1.2), (1.8), (1.9), (1.12), (2.1) and (2.7)

$$\text{meas} \quad \text{spr} \sphericalangle(L_C(V), L_C(U)) = \overline{(\frac{\pi}{6})}$$

is obtained, but the intersection of super-lines cannot be seen in the window.

Now, using the super-shift transformation (3.1) with $(x_0, y_0) = C$, we can see it in the shift-window:

$$\overline{(\frac{\sqrt{3}}{2})} < x < \overline{(2 + \frac{\sqrt{3}}{2})}$$

$$\overline{(\frac{\sqrt{3}}{2} - 1)} < y < \overline{(1 + \frac{\sqrt{3}}{2})}.$$

Considering that the equations under (3.28) and (3.29) are equivalent with the equations:

$$y = \overline{(\frac{\sqrt{3}}{2})} \oplus (\overline{2-\sqrt{3}} \odot (x \ominus \overline{(\frac{2+\sqrt{3}}{2})}))$$

and

$$y = x \ominus \overline{1},$$

respectively. Hence, the new equations of super-lines

$$L_C(V) : \eta = \overline{2 - \sqrt{3}} \odot \xi, \quad \xi \in \overline{R}$$

and

$$L_C(U) : \eta = \xi, \quad \xi \in \overline{R}.$$

Moreover, their shift-window phenomena have equations

$$\eta = \operatorname{area\,th}((2 - \sqrt{3}) \operatorname{th} \xi), \quad \xi \in R$$

and

$$\eta = \xi, \quad \xi \in R,$$

respectively, and have the graphs below:

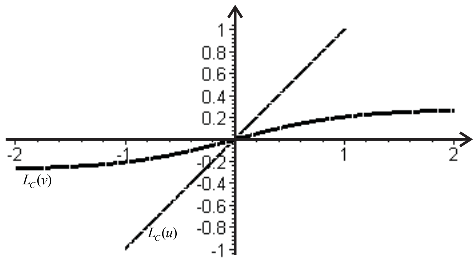

FIGURE 3.30.

4. SCREW TRANSFORMATION

Having the exploded Descartes-coordinate system "x, y", for any point $(x, y) \in \overbracket{R^2}$ we introduce the screw coordinates ω and ϑ by the equation system

$$\begin{aligned} x &= (\omega \odot \operatorname{spr}\cos\varphi_0) \ominus (\vartheta \odot \operatorname{spr}\sin\varphi_0) \\ y &= (\omega \odot \operatorname{spr}\sin\varphi_0) \oplus (\vartheta \odot \operatorname{spr}\cos\varphi_0) \end{aligned} \tag{4.1}$$

where $\varphi_0 \in \overbracket{R}$ is a given (exploded) number. The transformation given by the equation system (4.1) is called screw transformation. The name of coordinate system "ω, ϑ" is the screwed system.

By (1.1), (1.2), (1.6), (1.9) and (2.1) for any $\varphi \in \overbracket{R}$ we easily prove the identity

$$(\operatorname{spr}\sin\varphi \odot \operatorname{spr}\sin\varphi) \oplus (\operatorname{spr}\cos\varphi \odot \operatorname{spr}\cos\varphi) = \overbracket{1}$$

and using it, we can solve the equation system (4.1) for ω and ϑ, so

$$\begin{aligned} \omega &= (x \odot \operatorname{spr}\cos\varphi_0) \oplus (y \odot \operatorname{spr}\sin\varphi_0) \\ \vartheta &= (y \odot \operatorname{spr}\cos\varphi_0) \ominus (x \odot \operatorname{spr}\sin\varphi_0). \end{aligned} \tag{4.2}$$

The equation systems (4.1) and (4.2) show the mutual and unambigous connection beetween coordinates (x, y) and (ω, ϑ) which characterise the same point of $\overbracket{R^2}$. The systems "x, y" and "ω, ϑ" have a common origo.

Considering the points characterised by

$$\vartheta = 0 \tag{4.3}$$

the equation system (4.1) yields

$$\begin{aligned} x &= \omega \odot \operatorname{spr}\cos\varphi_0 \\ y &= \omega \odot \operatorname{spr}\sin\varphi_0. \end{aligned} \tag{4.4}$$

If

$$\varphi_0 = \overbracket{k} \odot \overbracket{2\pi}, \quad k = 0, \pm 1, \pm 2, \ldots$$

then by (4.4)

$$\vartheta = 0 \quad \text{if and only if} \quad y = 0,$$

and the screw transformation is identical.

If

$$\varphi_0 = \overline{\pi} \oplus (\overline{k} \odot \overline{2\pi}), \quad k = 0, \pm 1, \pm 2, \dots$$

then by (4.4)

$$\vartheta = 0 \quad \text{if and only if} \quad y = 0,$$

but the point $(x, y) = (\overline{1}, 0)$ has the new coordinates $(\omega, \vartheta) = (-\overline{1}, 0)$.

If

$$\varphi_0 = \overline{(\frac{\pi}{2})} \oplus (\overline{k} \odot \overline{2\pi}), \quad k = 0, \pm 1, \pm 2, \dots$$

then

$$\vartheta = 0 \quad \text{if and only if} \quad x = 0, \tag{4.5}$$

and the point $(x, y) = (\overline{1}, 0)$ has the new coordinates $(\omega, \vartheta) = (0, \overline{1})$.

If

$$\vartheta_0 = \overline{(-\frac{\pi}{2})} \oplus (\overline{k} \odot \overline{2\pi}), \quad k = 0, \pm 1, \pm 2, \dots$$

then

$$\vartheta = 0 \quad \text{if and only if} \quad x = 0, \tag{4.6}$$

but the point $(x, y) = (\overline{1}, 0)$ has the new coordinates $(\omega, \vartheta) = (0, -\overline{1})$.

In general, except for the cases (4.5) and (4.6), the equation system (4.4) yields

$$\vartheta = 0 \quad \text{if and only if} \quad y = (\operatorname{spr} \operatorname{tg} \varphi_0) \odot x, \tag{4.7}$$

that is, the "ω-axis" of the screwed system is a super-line in the exploded Descartes-coordinate system "x, y".

Similarly, considering the points characterised by

$$\omega = 0, \tag{4.8}$$

the equation system (4.1) yields

$$\begin{aligned} x &= -(\vartheta \odot \operatorname{spr} \sin \varphi_0) \\ y &= \vartheta \odot \operatorname{spr} \cos \varphi_0. \end{aligned} \tag{4.9}$$

Hence, except for cases

$$\varphi_0 = \overline{k} \odot \overline{\pi}, \quad k = 0, \pm 1, \pm 2, \dots$$

we have that

$$\omega = 0 \quad \text{if and only if} \quad y = (-\operatorname{spr} \operatorname{ctg} \varphi_0) \odot x, \tag{4.10}$$

that is, the "ϑ-axis" of the screwed system is a super-line in the exploded Descartes system "x, y".

Moreover, for any $\varphi_0 \in \overline{R}$ we have

$$\text{meas } \operatorname{spr} \sphericalangle(\text{“}\omega - \text{axis}'', \text{ “}\vartheta - \text{axis}'') = \overline{(\frac{\pi}{2})}. \tag{4.11}$$

The window phenomena of screwed axes in the cases

$$\varphi_0 = \overline{k} \odot \overline{(\frac{\pi}{4})}, \quad k = 0, \pm 1, \pm 2, \dots$$

are lines.

In the special case $\varphi_0 = \overbracket{(\frac{\pi}{3})}$, the window phenomena of screwed axes having the equations

$$\text{“}\omega-\text{axis''}: \quad y = \operatorname{area\,th}(\sqrt{3}\operatorname{th} x), \quad |x| < \operatorname{area\,th}\frac{1}{\sqrt{3}}$$

and

$$\text{“}\vartheta-\text{axis''}: \quad y = -\operatorname{area\,th}(\frac{\operatorname{th} x}{\sqrt{3}}), \quad x \in R$$

are situated in the following figure:

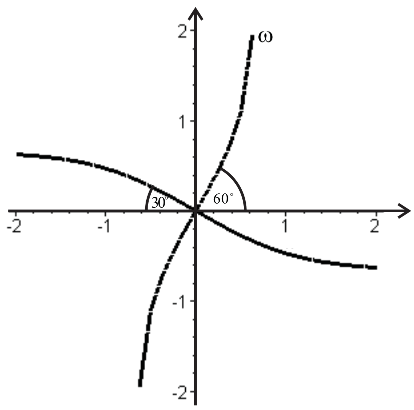

FIGURE 4.12.

5. CRITERION FOR SUPER-LINES

Now turning to the lines with equations

$$y = -x, \quad x \in R \tag{5.1}$$

and

$$y = 1 - x, \quad x \in R \tag{5.2}$$

we can consider as the window phenomena of some kind of super-functions. Clearly, the equation

$$y = -x, \quad x \in \overbracket{R} \tag{5.3}$$

represents a super-line and its window phenomenon has the equation under (5.1). The case of (5.2) is more complicated. Using (1.1) - (1.4) we can write for any $x \in R$

$$1 - x = (1 \ominus x) \odot (\overbracket{1} \ominus (1 \odot x))$$

which means that the super-function represented by the equation

$$y = (1 \ominus x) \odot (\overbracket{1} \ominus (1 \odot x)), \quad x \in \overbracket{R}, \;\; x \neq \overbracket{(\frac{1}{\underbracket{1}})}(> \infty), \tag{5.4}$$

has the window phenomenon with equation (5.2). The graph of this super-function is not a super-line because starting from

$$f(x) = \frac{\operatorname{th} 1 - x}{1 - (\operatorname{th} 1)x}, \quad x \in R, \; x \neq \frac{1}{\operatorname{th} 1} \tag{5.5}$$

by (1.2)-(1.4), (1.6), (1.10) and (2.1) we have

$$\operatorname{spr} f(x) = (1 \ominus x) \odot (\overbracket{1} \ominus (1 \odot x)), \quad \underbracket{x} \neq \frac{1}{\operatorname{th} 1}.$$

Having that (5.5) gives a hyperbola we can say that the super-function (5.4) represents a super-hyperbola.

To see this super-hyperbola in a better position, we use the super-shift transformation (3.1) with

$$(x_0, y_0) = \Big(\overbrace{(\frac{1}{\operatorname{th} 1})}, \ \overbrace{(\frac{1}{\operatorname{th} 1})}\Big).$$

Now the equation (instead of (5.4)) of super-hyperbola is:

$$\eta = \overbrace{(\frac{1}{\operatorname{th} 1})^2 - 1} \ominus \xi, \quad \xi \in \overbrace{R}, \quad \xi \neq 0 \tag{5.6}$$

and we can see its window phenomenon

$$\eta = \operatorname{area th} \frac{(\frac{1}{\operatorname{th} 1})^2 - 1}{\operatorname{th} \xi}, \quad |\xi| > \operatorname{area th}((\frac{1}{\operatorname{th} 1})^2 - 1)$$

in the shift-window:

$$\operatorname{area th}(\frac{1}{\operatorname{th} 1} - 1) < x < \overbrace{1 + \frac{1}{\operatorname{th} 1}}$$
$$\operatorname{area th}(\frac{1}{\operatorname{th} 1} - 1) < y < \overbrace{1 + \frac{1}{\operatorname{th} 1}},$$

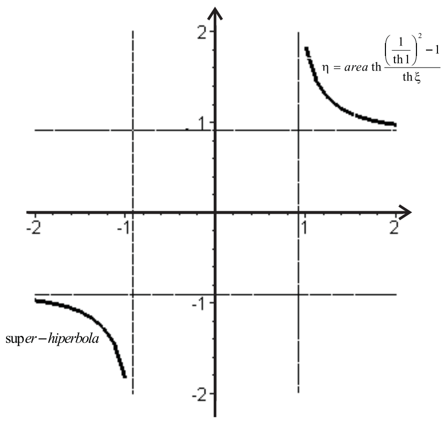

FIGURE 5.7.

Considering the sets

$$S_{x_0} = \{(x, y) \in \overbrace{R^2} : x = x_0, \quad y \in \overbrace{R} \quad \text{is arbitrary}\}$$

and

$$S_{y_0} = \{(x, y) \in \overbrace{R^2} : x \in \overbrace{R} \quad \text{is arbitrary}, \ y = y_0\}.$$

We see at a glance, that their points form super-lines. (The cases $x_0 = 0$ and $y_0 = 0$ are the exploded coordinate-axes.)

If x is running over $\overbrace{R}$, and $y = F(x)$ represents a mutual and unambiguous connection between x and $y \in \overbrace{R}$, it is not trivial whether the set

$$S = \{(x, y) \in \overbrace{R^2} : x \in \overbrace{R} \quad \text{is arbitrary and} \quad y = F(x)\} \tag{5.8}$$

forms a super-line or not. (Exceptions are $F(x) = x$ and $F(x) = -x$.) Now we give a criterion in

Theorem 5.9. *Let us assume that x is running over $\overline{R}$ and $y = F(x)$ is a mutual and unabiguous connection between x and $y \in \overline{R}$. The set (5.8) forms a super-line if and only if there exists one and only one $x_0 \in \overline{R}$ so that*

$$F(x_0) = 0, \tag{5.10}$$

moreover, there exists a $\varphi_0 \in \overline{R}$, $\varphi_0 \neq \overline{k} -\!\odot\!- \overline{(\tfrac{\pi}{2})}$, $k = 0, \pm 1, \dots$ such that the transformations first, the super-shift transformation

$$\begin{aligned} x &= \xi -\!\oplus\!- x_0 \\ y &= \eta \end{aligned} \tag{5.11}$$

second, the screw transformation

$$\begin{aligned} \xi &= (\omega -\!\odot\!- \operatorname{spr}\cos\varphi_0) -\!\ominus\!- (\vartheta -\!\odot\!- \operatorname{spr}\sin\varphi_0) \\ \eta &= (\omega -\!\odot\!- \operatorname{spr}\sin\varphi_0) -\!\oplus\!- (\vartheta -\!\odot\!- \operatorname{spr}\cos\varphi_0) \end{aligned} \tag{5.12}$$

result that the graph of super-function F becomes the "ω-axis" of the coordinate-system "ω, ϑ" with the origo $(x_0, 0)$.

Example 5.13. Let us consider the set (5.8) with

$$F(x) = \overline{2} -\!\odot\!- x, \quad x \in \overline{R}.$$

In this case $x_0 = 0$ (see (5.10)), so the super-shift transformation is identical. (See (3.1) and (5.11).) Moreover, the screw transformation (5.12) with the necessary condition $\vartheta = 0$ has the form

$$x = \omega -\!\odot\!- \operatorname{spr}\cos\varphi_0$$

$$\overline{2} -\!\odot\!- x = \omega -\!\odot\!- \operatorname{spr}\sin\varphi_0.$$

Hence $\varphi_0 = \operatorname{spr}\operatorname{arc}\operatorname{tg}\overline{2}$. So, the set (5.8) forms a super-line.

Example 5.14. Let us a consider the set (5.8) with

$$F(x) = \operatorname{spr}\operatorname{arc}\operatorname{tg} x, \quad x \in \overline{R}. \ \text{(See (2.1).)}$$

In this case $x_0 = 0$ (see (5.10)), so the super-shift transformation in identical. (See (3.1) and (5.11).) Moreover, the screw transformation (5.12) with the necessaary condition $\vartheta = 0$ yields

$$x = \omega -\!\odot\!- \operatorname{spr}\cos\varphi_0$$

$$\operatorname{spr}\operatorname{arc}\operatorname{tg} x = \omega -\!\odot\!- \operatorname{spr}\sin\varphi_0$$

Hence,

$$\operatorname{spr}\operatorname{arc}\operatorname{tg}\varphi_0 = (\operatorname{spr}\operatorname{arc}\operatorname{tg} x) -\!\oslash\!- x =$$

$$= \overline{\operatorname{arc}\operatorname{tg}\underline{x}} -\!\oslash\!- \overline{(\underline{x})} = \overline{\left(\frac{\operatorname{arc}\operatorname{tg}\underline{x}}{\underline{x}}\right)}$$

depends on x, so φ_0 does not exists.

6. Proof of Theorem 5.9.

We need the following

Lemma 6.1. *If* $v_1 \in \overline{R^+}$, v_1 *and* v_2 *satisfy condition (2.6) then we have*

$$\operatorname{spr}\sin(\operatorname{spr}\operatorname{arc}\operatorname{tg}(v_2 \mathbin{-\!\odot\!-} v_1)) = v_2 \tag{6.2}$$

and

$$\operatorname{spr}\cos(\operatorname{spr}\operatorname{arc}\operatorname{tg}(v_2 \mathbin{-\!\odot\!-} v_1)) = v_1 . \tag{6.3}$$

Proof of the Lemma. Using (2.1), (1.9), (1.4) and (1.11) we get

$$\operatorname{spr}\operatorname{arc}\operatorname{tg}(v_2 \mathbin{-\!\odot\!-} v_1) = \overline{\mathit{arc}\operatorname{tg}\underline{v_2 \mathbin{-\!\odot\!-} v_1}} =$$

$$= \overline{\operatorname{arc}\operatorname{tg}\underline{(\overline{\underline{v_2}}) \mathbin{-\!\odot\!-} (\overline{\underline{v_1}})}} = \overline{\operatorname{arc}\operatorname{tg}\underline{(\overline{\underline{v_2} : \underline{v_1}})}} =$$

$$= \overline{\operatorname{arc}\operatorname{tg}(\underline{v_2} : \underline{v_1})}.$$

Moreover, (2.6) implies

$$(\underline{v_1})^2 + (\underline{v_2})^2 = 1.$$

Hence, using that $\underline{v_1} > 0$ (see (1.8)) we get

$$\sin\left(\operatorname{arc}\operatorname{tg}\frac{\underline{v_2}}{\underline{v_1}}\right) = \underline{v_2}.$$

Moreover, (2.1), (1.11) and (1.9) yield

$$\operatorname{spr}\sin(\operatorname{spr}\operatorname{arc}\operatorname{tg}(v_2 \mathbin{-\!\odot\!-} v_1)) = \overline{\sin\underline{\operatorname{spr}\operatorname{arc}\operatorname{tg}(v_2 \mathbin{-\!\odot\!-} v_1)}}$$

$$= \overline{\sin(\operatorname{arc}\operatorname{tg}(\underline{v_2} : \underline{v_1}))} = \overline{(\underline{v_2})} = v_2,$$

that is we have (6.2).

Similarly,

$$\cos(\operatorname{arc}\operatorname{tg}\frac{\underline{v_2}}{\underline{v_1}}) = \underline{v_1}$$

implies

$$\operatorname{spr}\cos(\operatorname{spr}\operatorname{arc}\operatorname{tg}(v_2 \mathbin{-\!\odot\!-} v_1)) = v_1$$

that is we have (6.3). □

Proof of Theorem. Necessity. Let us assume that set S (see (5.8)) forms a super-line which has the equation system

$$x = c_1 \mathbin{-\!\oplus\!-} (t \mathbin{-\!\odot\!-} v_1)$$

$$y = c_2 \mathbin{-\!\oplus\!-} (t \mathbin{-\!\odot\!-} v_2)$$

where $C = (c_1, c_2)$ and $V = (v_1, v_2)$ with (2.6) belong to $\overline{R^2}$. As the connection between x and y is mutual and unambiguous, v_1 and v_2 are different from 0. Moreover, we can assume that $v_1 > 0$. Hence, we have that

$$y = (m \mathbin{-\!\odot\!-} x) \mathbin{-\!\oplus\!-} b \tag{6.4}$$

where

$$m = v_2 \mathbin{-\!\odot\!-} v_1 , (m \neq 0) \tag{6.5}$$

and

$$b = c_2 \ominus (m \odot c_1),$$

that is

$$F(x) = (m \odot x) \oplus b \tag{6.6}$$

is a super-linear function. Now, by (6.6) we have

$$\underset{(0)}{x} = (-b) \odot m \tag{6.7}$$

as the unique point such that

$$F(\underset{(0)}{x}) = 0,$$

so (5.10) is fulfilled. Let now

$$\varphi_0 = \operatorname{spr} \operatorname{arc} \operatorname{tg} m. \tag{6.8}$$

Clearly, $\varphi_0 \neq \overset{\sqcup}{k} \odot \overset{\sqcup}{(\frac{\pi}{2})}$, $k = 0, \pm 1, \pm 2, \ldots$. Using transformation (5.11) with $x_0 = \underset{(0)}{x}$, by (6.4) and (6.7) we have the equation super-line (5.8) with (6.6) in the system "ξ, η" with the origo $(\underset{(0)}{x}, 0)$:

$$\eta = m \odot \xi. \tag{6.9}$$

Considering transformation (5.12) with φ_0 under (6.8), by (6.5) and Lemma 6.1 we have

$$\xi = (\omega \odot v_1) \ominus (\vartheta \odot v_2)$$

$$\eta = (\omega \odot v_2) \oplus (\vartheta \odot v_1)$$

Using (6.9) the equation

$$(\omega \odot v_1) \oplus (\vartheta \odot v_1) = m \odot ((\omega \odot v_1) \ominus (\vartheta \odot v_2)) \tag{6.10}$$

is the equation of super-line (5.8) in the coordinate system "ω, ϑ" with the origo $(\underset{(0)}{x}, 0)$. Spermultipliying both sides of the equation (6.10) by v_1 and using (6.5) we can write

$$((\omega \odot v_2) \odot v_1) \oplus ((\vartheta \odot v_1) \odot v_1) =$$

$$= ((v_2 \odot \omega) \odot v_1) \ominus (v_2 \odot (\vartheta \odot v_2)).$$

Hence

$$\vartheta \odot ((v_1 \odot v_1) \oplus (v_2 \odot v_2)) = 0$$

and (2.6) gives that $\vartheta = 0$. This shows that super-line (5.8) lies on the "ω-axis" of the coordinate-system "ω, ϑ".

Sufficiency. Now we assume that there exists a (unique) x_0 with (5.10) and $\varphi_0 \neq \overset{\sqcup}{k} \odot \overset{\sqcup}{(\frac{\pi}{2})}$ such that after the transformations (5.11) and (5.12) the points of the set S lie on the "ω-axis" of the coordinate system "ω, ϑ" with the origo $(x_0, 0)$. These mean that if $(x, y) \in S$ having the coordinates ω and ϑ such that

$$x \ominus x_0 = (\omega \odot \operatorname{spr} \cos \varphi_0) \ominus (\vartheta \odot \operatorname{spr} \sin \varphi_0)$$

$$y = (\omega \odot \operatorname{spr} \sin \varphi_0) \oplus (\vartheta \odot \operatorname{spr} \cos \varphi_0)$$

is valid with $\vartheta = 0$. Hence, for any $x \in \overline{R}$

$$x \ominus x_0 = \omega \odot \operatorname{spr}\cos\varphi_0$$

$$F(x) = \omega \odot \operatorname{spr}\sin\varphi_0$$

Having that $\operatorname{spr}\cos\varphi_0 \neq 0$ an assuming that $x \neq x_0$ (that is $\omega \neq 0$) by (2.1) we can write

$$F(x) \oslash (x \ominus x_0) = \operatorname{spr}\operatorname{tg}\varphi_0 (\neq 0).$$

Moreover,

$$F(x) = (\operatorname{spr}\operatorname{tg}\varphi_0) \odot (x \ominus x_0)$$

which, by (5.10), remains valid for $x = x_0$, too. As F is a super-linear function, the points of S form a super-line. $\square$

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Received November 1, 2001.

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