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ADJOINTS OF COMPOSITION OPERATORS ON STANDARD BERGMAN AND DIRICHLET SPACES ON THE UNIT DISK

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Abstract. It is not known a satisfactory way to compute adjoints of composition operators, yet in classical functional Banach spaces (cf. [3]). If K is the reproducing kernel of a functional Hilbert space H, $g \in H$ and the composition operator C_φ is bounded then

$$C_\varphi^* g(z) = \langle g(t), K(z, \varphi(t))\rangle_H, \ z \in \mathbb{D}.$$

In general, although reproducing kernels might be described in series developments, it is not possible to determine C_φ^* in a closed form. In this article we establish some formulae in order to evaluate adjoints of composition operators on standard Bergman and Dirichlet spaces.

1. Introduction

Throughout this article we shall consider the standard Bergman and Dirichlet (Hilbert) spaces on the complex unit disk $\mathbb{D}$ defined by

$$\mathbb{A}_\alpha^2(\mathbb{D}) = \left\{ f \text{ analytic on } \mathbb{D} : \ \|f\|_{\mathbb{A}_\alpha^2(\mathbb{D})}^2 = \int_{\mathbb{D}} |f(z)|^2 \, \mathrm{d}A_\alpha(z) < \infty \right\}$$

and

$$\mathcal{D}_\alpha(\mathbb{D}) = \left\{ f \text{ analytic on } \mathbb{D} : \ \|f\|_{\mathcal{D}_\alpha(\mathbb{D})}^2 = |f(0)|^2 + \int_{\mathbb{D}} \left| \frac{df(z)}{dz} \right|^2 \mathrm{d}A_\alpha(z) < \infty \right\}$$

respectively, where $\alpha > -1$, $\mathrm{d}A_\alpha(z) = (1+\alpha)\left(1 - |z|^2\right)^\alpha \mathrm{d}A(z)/\pi$ and $\mathrm{d}A(z)$ is Lebesgue area measure. A composition operator on a Bergman space is bounded if its symbol is analytic. Indeed, if φ is an analytic authomorphism of the disk we can write $\varphi(z) = \lambda(z+a)/(1+\overline{a}\,z)$, where $|a| < 1$, $|\lambda| = 1$ and $z \in \mathbb{D}$ (cf. [5], Ch. 2, §6, Th. 6.1, page 63). Thus, if $f \in \mathbb{A}_\alpha^2(\mathbb{D})$ by the usual change of variables formula we obtain

$$\begin{aligned} \|\mathrm{C}_\varphi f\|_{\mathbb{A}^2(\mathbb{D})}^2 &= \int_{\mathbb{D}} |f(\varphi(z))|^2 \, \mathrm{d}A_\alpha(z) \\ &= \int_{\mathbb{D}} |f(w)|^2 \, (1+\alpha)\left(1 - \left|\varphi^{-1}(w)\right|^2\right)^\alpha \mathrm{d}A\left(\varphi^{-1}(w)\right)/\pi \\ &= \int_{\mathbb{D}} |f(w)|^2 \left(\frac{1 - \left|\varphi^{-1}(w)\right|^2}{1 - |w|^2}\right)^\alpha \left|\frac{d\varphi^{-1}}{dw}(w)\right|^2 \mathrm{d}A_\alpha(w) \\ &= \left(1 - |a|^2\right)^{2+\alpha} \int_{\mathbb{D}} \frac{|f(w)|^2}{\left|1 - w\,\overline{a\lambda}\right|^{2\alpha+4}} \mathrm{d}A_\alpha(w) \end{aligned}$$

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$$\leq \left(\frac{1+|a|}{1-|a|}\right)^{2+\alpha} \|f\|^2_{\mathbb{A}^2_\alpha(\mathbb{D})},$$

i.e. C_φ is bounded and $\|\mathrm{C}_\varphi\| \leq (1+|\varphi(0)|)/(1-|\varphi(0)|)$. Now, if φ_1 is an analytic self map of $\mathbb{D}$ and $\varphi_2 = \varphi \circ \varphi_1$ then $\mathrm{C}_{\varphi_2} = \mathrm{C}_{\varphi_1} \circ \mathrm{C}_\varphi$ and

$$\mathrm{C}_{\varphi_1} = \mathrm{C}_{\varphi_2} \circ \mathrm{C}_\varphi^{-1} = \mathrm{C}_{\varphi_2} \circ \mathrm{C}_{\varphi^{-1}}.$$

Therefore, C_{φ_2} is bounded if and only if C_{φ_1} is. Choosing φ with $\varphi(\varphi_1(0)) = 0$ gives $\varphi_2(0) = 0$ and the claim follows by [2], Th. 3.1, p. 117. In the frame of Dirichlet spaces boundedness obeys to more restrictive conditions. Indeed, if φ is an analytic self map of $\mathbb{D}$ then $C_\varphi \in \mathcal{B}[\mathcal{D}_\alpha]$ if and only if for all $\zeta \in \mathbb{C}$ and all $0 < h < 1$ is $\mu_\alpha \varphi^{-1} S(\zeta, h) = O\left(h^{\alpha+2}\right)$, where $S(\zeta, h)$ is the unit disk intersected with the disk of radius h centered at ζ and μ_α is the measure on $\mathbb{D}$ defined as $d\mu_\alpha = |d\varphi/dz|^2 \, \mathrm{d}A_\alpha(z)$ (cf. [4]). Our goal in this article is to develop some formulae to evaluate adjoints of composition operators on Bergman and Dirichlet spaces. In general, with the exception of a few particular cases in the context of Hardy spaces, there is not known a satisfactory way to realize such computations (see [3] and [2], Ch. 9, §9.1, p. 321). This problem will be considered in Bergman and Dirichlet contexts in Sections 2 and 3 respectively.

2. Adjoints on Bergman spaces

Let $\alpha > -1$, $\Psi_n(z) = c_n\, z^n$, where $c_n = \binom{n+1+\alpha}{n}^{1/2}$, $n \in \mathbb{N}_0$. Then $\{\Psi_n\}_{n\in\mathbb{N}_0}$ is an orthonormal basis of $\mathbb{A}^2_\alpha(\mathbb{D})$. Let $f(z) = \sum_{n=0}^\infty a_n\, z^n$ be the Taylor series on $\mathbb{D}$ of a function $f \in \mathbb{A}^2_\alpha(\mathbb{D})$, $m \in \mathbb{N}_0$ and $0 < \varrho < 1$. Since the series converges uniformly on compact subsets of $\mathbb{D}$ and $\mathbb{D}$ has dA_α finite measure is

$$\int_{\varrho\overline{\mathbb{D}}} f(z)\ \overline{z}^m \,\mathrm{d}A_\alpha(z) = \sum_{n=0}^\infty a_n \int_{\varrho\overline{\mathbb{D}}} z^n\ \overline{z}^m \,\mathrm{d}A_\alpha(z) = a_m\ \varrho^{2(m+1+\alpha)}\ c_m^{-2}. \tag{1}$$

Since $|f(z)\ \overline{z}^m| \leq |f(z)|$ and by Hölder´s inequality $f \in \mathrm{L}^1(\mathbb{D}, dA_\alpha)$, on letting $\varrho \to 1^-$ in (1) we obtain

$$a_m = f^{(m)}(0)/m! = c_m\ \langle f, \Psi_m \rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}. \tag{2}$$

Theorem 1. *Let φ be an analytic self map of the complex unit disk $\mathbb{D}$ and C_φ be the composition operator $C_\varphi g = g \circ \varphi$ defined on the standard Bergman space $\mathbb{A}^2_\alpha(\mathbb{D})$. Then $C_\varphi \in \mathcal{B}\left[\mathbb{A}^2_\alpha(\mathbb{D})\right]$ and*

$$C^*_\varphi g(z) = \int_{\mathbb{D}} \frac{g(t)\ \mathrm{d}A_\alpha(t)}{\left(1 - z\ \overline{\varphi(t)}\right)^{\alpha+2}},\ g \in \mathbb{A}^2_\alpha(\mathbb{D}),\ z \in \mathbb{D}.$$

Proof. If φ is constant, say $\varphi(z) \equiv z_0$, then C_φ is the evaluation functional at z_0. If $f,\ g \in \mathbb{A}^2_\alpha(\mathbb{D})$, the reproducing formula

$$f(z_0) = \int_{\mathbb{D}} \frac{f(z)}{(1 - z_0\ \overline{z})^{2+\alpha}} \,\mathrm{d}A_\alpha(z) \tag{3}$$

holds (cf. [2], p. 27, Ex. 2.1.5 (a)) and

$$\begin{aligned} \langle C_\varphi f, g \rangle_{\mathbb{A}^2_\alpha(\mathbb{D})} &= f(z_0) \int_{\mathbb{D}} \overline{g(w)} \,\mathrm{d}A_\alpha(w) \\ &= \int_{\mathbb{D}} \frac{f(z)}{(1 - z_0\ \overline{z})^{2+\alpha}} \,\mathrm{d}A_\alpha(z) \int_{\mathbb{D}} \overline{g(w)} \,\mathrm{d}A_\alpha(w) \\ &= \int_{\mathbb{D}} f(z)\, \overline{(1 - \overline{z_0}\ z)^{-2-\alpha} \int_{\mathbb{D}} g(w) \,\mathrm{d}A_\alpha(w)} \,\mathrm{d}A_\alpha(z). \end{aligned}$$

Since f is arbitrary we obtain $C_\varphi^* g(z) = (1-\overline{z_0}\ z)^{-2-\alpha}\int_{\mathbb{D}} g(w)\,\mathrm{d}A_\alpha(w)$ and our claim follows. Let φ be non constant, $\varrho\in(0,1)$, $m,\ n\in\mathbb{N}_0$. Then

$$\varphi(z)^n=\sum_{k=0}^{\infty}a_{n,k}\ z^k,\ \textit{with}\ a_{n,k}=\frac{1}{2\pi i}\int_{|t|=\varrho}\varphi(t)^n\ \frac{dt}{t^{k+1}},\ z\in\mathbb{D}, \tag{4}$$

the series in (4) being uniformly convergent on compact subsets of $\mathbb{D}$. By (2) and (4) we write

$$\begin{aligned}\langle C_\varphi f, z^m\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})} &= \left\langle \sum_{n=0}^{\infty}\langle f,\Psi_n\rangle\ C_\varphi\Psi_n,\ z^m\right\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}\\ &=\sum_{n=0}^{\infty}c_n\ \langle f,\Psi_n\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}\ \langle\varphi(z)^n, z^m\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}\\ &=c_m^{-2}\sum_{n=0}^{\infty}c_n^2\int_{\mathbb{D}}f(z)\ \overline{z}^n\,\mathrm{d}A_\alpha(z)\ \frac{1}{2\pi i}\int_{|t|=\varrho}\varphi(t)^n\ \frac{dt}{t^{m+1}}.\end{aligned} \tag{5}$$

Now, for each non negative integer n we have

$$\left|\frac{(\varphi^n)^{(m)}(0)}{m!}\right|=\left|\frac{1}{2\pi i}\int_{|t|=\varrho}\varphi(t)^n\ \frac{dt}{t^{m+1}}\right|\le\int_0^{2\pi}|\varphi(\varrho\exp(it))|^n\ \frac{dt}{2\pi\varrho^m}\le\frac{\mathrm{M}_\varphi(\varrho)^n}{\varrho^m},$$

where $\mathrm{M}_\varphi(\varrho)=\max\{|\varphi(z)|:\ |z|\le\varrho\}$. If φ is non constant by the maximum modulus principle $\mathrm{M}_\varphi(\varrho)<1$. So, for $p\in\mathbb{N}$ and $z\in\mathbb{D}$ we have

$$\begin{aligned}\left|f(z)\ \sum_{n=0}^{p}\binom{n+1+\alpha}{n}\ \frac{(\varphi^n)^{(m)}(0)}{m!}\ \overline{z}^n\right| &\le \frac{|f(z)|}{\varrho^m}\sum_{n=0}^{p}\binom{n+1+\alpha}{n}\ \mathrm{M}_\varphi(\varrho)^n\\ &\le\varrho^{-m}\ (1-\mathrm{M}_\varphi(\varrho))^{-2-\alpha}\ |f(z)|.\end{aligned}$$

Since $f\in\mathrm{L}^1(\mathbb{D},dA_\alpha)$ we apply Lebesgue dominated convergence theorem in (5) obtaining

$$\langle C_\varphi f, z^m\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}=c_m^{-2}\int_{\mathbb{D}}f(z)\left[\sum_{n=0}^{\infty}c_n^2\ \overline{z}^n\ \frac{1}{2\pi i}\int_{|t|=\varrho}\varphi(t)^n\ \frac{dt}{t^{m+1}}\right]\mathrm{d}A_\alpha(z) \tag{6}$$

Moreover, by an analogous argument applied in (6) we have

$$\begin{aligned}&\langle C_\varphi f, z^m\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}\\ &=\frac{1}{c_m^2}\int_{\mathbb{D}}f(z)\left[\frac{1}{2\pi i}\int_{|t|=\varrho}\sum_{n=0}^{\infty}\binom{n+1+\alpha}{n}\ (\overline{z}\varphi(t))^n\ \frac{dt}{t^{m+1}}\right]\mathrm{d}A_\alpha(z)\\ &=\frac{1}{c_m^2}\int_{\mathbb{D}}f(z)\left[\frac{1}{2\pi i}\int_{|t|=\varrho}(1-\overline{z}\ \varphi(t)\ )^{-2-\alpha}\ \frac{dt}{t^{m+1}}\right]\mathrm{d}A_\alpha(z)\\ &=\int_{\mathbb{D}}f(z)\ \frac{1}{m!c_m^2}\frac{\partial^m}{\partial t^m}(1-\overline{z}\ \varphi(t)\ )^{-2-\alpha}\ |_{t=0}\ dA_\alpha(z)\\ &=\left\langle f(z),\ \frac{1}{m!c_m^2}\frac{\partial^m}{\partial t^m}\left(1-z\ \overline{\varphi(t)}\ \right)^{-2-\alpha}\ |_{t=0}\right\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}.\end{aligned}$$

Since f is arbitrary we have

$$C_\varphi^*\Psi_m(z)=\frac{1}{m!\ c_m}\frac{\partial^m}{\partial t^m}\left(1-z\ \overline{\varphi(t)}\ \right)^{-2-\alpha}\ |_{t=0}\ . \tag{7}$$

On the other hand, if $p\in\mathbb{N}$ and $z,\ t\in\mathbb{D}$ we have

$$\left|t^m\sum_{n=0}^{p}\binom{n+1+\alpha}{n}(\overline{z}\ \varphi(t))^n\right|\le(1-|z|)^{-2-\alpha}$$

and by Lebesgue dominated convergence theorem, (2) and (4)

$$\begin{aligned}\int_{\mathbb{D}} \frac{\Psi_m(t)\,\mathrm{d}A_\alpha(t)}{\left(1 - z\,\overline{\varphi(t)}\right)^{\alpha+2}} &= \sum_{n=0}^{\infty} \binom{n+1+\alpha}{n} z^n\, \overline{\langle \varphi(t)^n, \Psi_m(t)\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}} \\ &= c_m^{-1} \sum_{n=0}^{\infty} \binom{n+1+\alpha}{n} z^n\, \overline{a_{n,m}} \\ &= c_m^{-1} \sum_{n=0}^{\infty} \binom{n+1+\alpha}{n} z^n\, \frac{\left(\overline{\varphi}^n\right)^{(m)}(0)}{m!}.\end{aligned}$$

As before, if $0 < \varrho < 1$ a new application of Lebesgue dominated convergence theorem and (7) give

$$\begin{aligned}\int_{\mathbb{D}} \frac{\Psi_m(t)\,dA_\alpha(t)}{\left(1 - z\,\overline{\varphi(t)}\right)^{\alpha+2}} &= c_m^{-1} \sum_{n=0}^{\infty} \binom{n+1+\alpha}{n} \frac{1}{2\pi i} \int_{|t|=\varrho} z^n\, \overline{\varphi(t)}^n\, \frac{dt}{t^{m+1}} \\ &= \frac{1}{2\pi i c_m} \int_{|t|=\varrho} \left(1 - z\,\overline{\varphi(t)}\right)^{-2-\alpha} \frac{dt}{t^{m+1}} = C_\varphi^* \Psi_m(z).\end{aligned} \tag{8}$$

Finally, we observe that $C_\varphi^* g = \sum_{m=0}^{\infty} \langle g, \Psi_m\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}\, C_\varphi^*\Psi_m$ for all $g \in \mathbb{A}^2_\alpha(\mathbb{D})$ and evaluations are bounded linear functionals on $\mathbb{A}^2_\alpha(\mathbb{D})$.[1] Thus if $z \in \mathbb{D}$ by (7) and (8) is

$$\begin{aligned}C_\varphi^* g(z) &= \sum_{m=0}^{\infty} \langle g, \Psi_m\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}\, \frac{1}{m!\, c_m} \frac{\partial^m}{\partial t^m} \left(1 - z\,\overline{\varphi(t)}\right)^{-2-\alpha} |_{t=0} \\ &= \sum_{m=0}^{\infty} \int_{\mathbb{D}} g(s)\, \overline{s}^m \frac{1}{m!} \frac{\partial^m}{\partial t^m} \left(1 - z\,\overline{\varphi(t)}\right)^{-2-\alpha} |_{t=0}\, dA_\alpha(s)\end{aligned}$$

and the result follows by a last application of Lebesgue dominated convergence theorem. □

Corollary 1. *If $a \in \mathbb{D}$ and φ is an analytic self map on $\mathbb{D}$ then*

$$C_\varphi^*\left[(1 - t\,\overline{a})^{-\alpha-2}\right](z) = \left(1 - z\,\overline{\varphi(a)}\right)^{-\alpha-2},\ z \in \mathbb{D}.$$

Remark 1. Observe that if φ is an analytic self map of the disk, $\alpha > -1$ and we define

$$(H_\varphi g)(z) = \int_{\mathbb{D}} \frac{g(t)\,\mathrm{d}A_\alpha(t)}{\left(1 - z\,\overline{\varphi(t)}\right)^{\alpha+2}},\ g \in \mathbb{A}^2_\alpha(\mathbb{D}),\ z \in \mathbb{D},$$

then $H_\varphi \in \mathcal{B}\left[\mathbb{A}^2_\alpha(\mathbb{D})\right]$ and $\langle C_\varphi f, g\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})} = \langle f, H_\varphi g\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})}$. Of course, this is consequence of Th. 1 and the existence and uniqueness of adjoints. None of the two assertions are trivial and a direct and perhaps natural point of view is not appropriate nor conducent. For instance, if φ is an automorphism on $\mathbb{D}$ the usual change of variable formula gives

$$\langle C_\varphi f, g\rangle_{\mathbb{A}^2_\alpha(\mathbb{D})} = \int_{\mathbb{D}} f(z)\, \overline{g\left(\varphi^{-1}(z)\right)} \left(\frac{1 - \left|\varphi^{-1}(z)\right|^2}{1 - |z|^2}\right)^\alpha \left|\varphi^{(1)}\left(\varphi^{-1}(z)\right)\right|^2 \mathrm{d}A_\alpha(z),$$

but it is immediate that in general the function

$$z \to g\left(\varphi^{-1}(z)\right) \left(\frac{1 - \left|\varphi^{-1}(z)\right|^2}{1 - |z|^2}\right)^\alpha \left|\varphi^{(1)}\left(\varphi^{-1}(z)\right)\right|^2$$

[1] If $z_0 \in \mathbb{D}$, $f \in \mathbb{A}^2_\alpha(\mathbb{D})$ and $\delta(z - z_0)$ is the linear evaluation at z_0 by (3) is $|\langle f(z), \delta(z - z_0)\rangle| \leq \|f\|_{\mathbb{A}^2_\alpha(\mathbb{D})} (1 - |z_0|)^{-2-\alpha}$, i.e. $\delta(z - z_0)$ is bounded.

does not belong to $\mathbb{A}^2_\alpha(\mathbb{D})$.

Example 1. If $a,\ b \in \mathbb{C}$ are such that $0 < |a| \le |a| + |b| \le 1$ the function $\varphi(z) = az + b$ is an analytic self map of $\mathbb{D}$ and

$$C^*_\varphi g(z) = g\left(\frac{\overline{a}\ z}{1-\overline{b}\ z}\right)\ \frac{1}{\left(1-\overline{b}\ z\right)^{\alpha+2}},\ g \in \mathbb{A}^2_\alpha(\mathbb{D}),\ z \in \mathbb{D}.$$

3. Adjoints on Dirichlet spaces

Results of Section 3 are outlined, their proofs follow similarly as in Section 2. Let $\Lambda_0(z) = 1$ and $\Lambda_n(z) = c_{n-1}\ z^n/n,\ n \in \mathbb{N}$. Now $\{\Lambda_n\}_{n\in\mathbb{N}_0}$ is an orthonormal basis of $\mathcal{D}_\alpha(\mathbb{D})$. Let $f(z) = \sum_{n=0}^\infty a_n\ z^n$ be the Taylor series on $\mathbb{D}$ of a function $f \in \mathcal{D}_\alpha(\mathbb{D})$ and $m \in \mathbb{N}_0$.

$$a_m = \frac{f^{(m)}(0)}{m!} = \begin{cases} \langle f, \Lambda_0\rangle_{\mathcal{D}_\alpha(\mathbb{D})} & \text{if } m = 0, \\ c_{m-1}/m\ \ \langle f, \Lambda_m\rangle_{\mathcal{D}_\alpha(\mathbb{D})} & \text{if } m > 0. \end{cases} \tag{9}$$

Moreover, let us consider the function $K_{\mathcal{D}_\alpha}$ on $\mathbb{D}\times\mathbb{D}$ defined as

$$K_{\mathcal{D}_\alpha}(z,w) = 1 + \int_0^w \frac{\left(1-t\ \overline{z}\right)^{-\alpha-1} - 1}{(\alpha+1)\ t}\, \mathrm{d}t.$$

So, for $z \in \mathbb{D}$ fixed the function $w \to K_{\mathcal{D}_\alpha}(z,w)\ (= \overline{K_{\mathcal{D}_\alpha}(w,z)}\ )$ belongs to $\mathcal{D}_\alpha$, and $f(z) = \langle f(w), K_{\mathcal{D}_\alpha}(z,w)\rangle_{\mathcal{D}_\alpha}$ if $f \in \mathcal{D}_\alpha$ (cf. [1]). Let φ be an analytic self map of $\mathbb{D}$ such that $C_\varphi \in \mathcal{B}[\mathcal{D}_\alpha(\mathbb{D})]$. Then

$$\langle C_\varphi f, \Lambda_0\rangle = f(\varphi(0)) = \langle f(z), K_{\mathcal{D}_\alpha}(\varphi(0), z)\rangle_{\mathcal{D}_\alpha},\ \ f \in \mathcal{D}_\alpha,$$

i.e. $C^*_\varphi \Lambda_0(z) = K_{\mathcal{D}_\alpha}(\varphi(0), z)$ if $z \in \mathbb{D}$. Moreover, for $m \in \mathbb{N}$ is

$$C^*_\varphi \Lambda_m(z) = \frac{1}{(m-1)!\ c_{m-1}}\ \frac{\partial^m}{\partial t^m}\left[K_{\mathcal{D}_\alpha}(\varphi(t), z)\right]|_{t=0}\ .$$

Therefore, if $f(z) = \sum_{m=0}^\infty \langle f, \Lambda_m\rangle\ \ \Lambda_m(z)$ in $\mathcal{D}_\alpha$ we write

$$\begin{aligned} &C^*_\varphi f(z) \\ &= \sum_{m=0}^\infty \langle f, \Lambda_m\rangle\ \ C^*_\varphi \Lambda_m(z) \\ &= f(0)\ \ K_{\mathcal{D}_\alpha}(\varphi(0), z) + \sum_{m=0}^\infty \int_{\mathbb{D}} \frac{df}{ds}\ \ \frac{\overline{s}^{m-1}}{(m-1)!}\, \mathrm{d}A_\alpha(s)\ \ \frac{\partial^m}{\partial t^m}\left[K_{\mathcal{D}_\alpha}(\varphi(t), z)\right]|_{t=0} \\ &= f(0)\ \ \overline{K_{\mathcal{D}_\alpha}(z, \varphi(0))} + \int_{\mathbb{D}} \frac{df}{ds}\ \overline{\frac{\partial}{\partial s} K_{\mathcal{D}_\alpha}(z, \varphi(s))}\, \mathrm{d}A_\alpha(s)\,. \end{aligned}$$

Theorem 2. *Let φ be an analytic self map of the complex unit disk $\mathbb{D}$ such that $C_\varphi \in \mathcal{B}[\mathcal{D}_\alpha(\mathbb{D})]$. Then*

$$C^*_\varphi g(z) = \left\langle g(t), 1 + \int_0^{\varphi(t)} \frac{\left(1-\overline{z}\ s\right)^{-1-\alpha} - 1}{(1+\alpha)\ \ s}\, \mathrm{d}s \right\rangle_{\mathcal{D}_\alpha},\ \ g \in \mathcal{D}_\alpha,\ z \in \mathbb{D}.$$

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