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A UNIFIED TREATMENT OF WELL–CHAINEDNESS AND CONNECTEDNESS PROPERTIES

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Abstract. A unified treatment of some old and new well-chainedness and connectedness properties of the most basic topological structures (such as closures, proximities and uniformities, for instance) is offered in the framework of relators (families of binary relations) and their fundamental refinements.

The results obtained show that the various connectedness properties are actually particular cases of Cantor's well-chainedness property neglected by several authors. Moreover, they show that the hyperconnectedness introduced by L.A. Steen and J.A. Seebach is a particular case of our paratopological connectedness.

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INTRODUCTION

A nonvoid family $\mathcal{R}$ of binary relations on a nonvoid set X is called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a relator space. Relator spaces are straightforward generalizations of ordered sets and uniform spaces [56].

If $\mathcal{R}$ is a relator on X, then the relator $\mathcal{R}^{-1} = \{ R^{-1} : \quad R \in \mathcal{R} \}$ is called the inverse of $\mathcal{R}$. And, the relator $\mathcal{R}^{\infty} = \{ R^{\infty} : \quad R \in \mathcal{R} \}$, where R^{∞} is the smallest preorder (reflexive and transitive) relation containing R, is called the preorder modification of $\mathcal{R}$.

Moreover, if $\mathcal{R}$ is a relator on X, then the relators

$$\begin{aligned}
\mathcal{R}^{*} &= \{ S \subset X^2 : \quad \exists \ R \in \mathcal{R} : \quad R \subset S \}, \\
\mathcal{R}^{\#} &= \{ S \subset X^2 : \quad \forall \ A \subset X : \quad \exists \ R \in \mathcal{R} : \quad R(A) \subset S(A) \}, \\
\mathcal{R}^{\wedge} &= \{ S \subset X^2 : \quad \forall \ x \in X : \quad \exists \ R \in \mathcal{R} : \quad R(x) \subset S(x) \}, \\
\mathcal{R}^{\triangle} &= \{ S \subset X^2 : \quad \forall \ x \in X : \quad \exists \ y \in X : \quad \exists \ R \in \mathcal{R} : \quad R(y) \subset S(x) \}
\end{aligned}$$

are called the uniform, the proximal, the topological, and the paratopological refinements of $\mathcal{R}$, respectively.

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And, the relators $\mathcal{R}^{\bullet} = \mathcal{R}^{\vee\vee}$, $\mathcal{R}^{\blacktriangle} = \mathcal{R}^{\vartriangle\bullet}$, and $\mathcal{R}^{\blacktriangle} = \mathcal{R}^{\triangledown\triangledown}$, where $\mathcal{R}^{\vee} = (\mathcal{R}^{\wedge})^{-1}$ and $\mathcal{R}^{\triangledown} = (\mathcal{R}^{\vartriangle})^{-1}$, are called the infinitesimal, the parainfinitesimal, and the ultimate refinements of $\mathcal{R}$, respectively.

Now, two relators $\mathcal{R}$ and $\mathcal{S}$ on X may, for instance, be called topologically (quasi-topologically) equivalent if $\mathcal{R}^{\wedge} = \mathcal{S}^{\wedge}$ $\big(\mathcal{R}^{\wedge\infty} = \mathcal{S}^{\wedge\infty} \big)$. Namely, it can be shown that the relators $\mathcal{R}$ and $\mathcal{S}$ give rise to the same interiors (open sets) if and only if they are topologically (quasi-topologically) equivalent.

Analogouosly to the problem of finding a powerful and flexible notion of a spatial structure, the problem of finding an appropriate notion of connectedness also has a long history. The three most important definitions were suggested by K. Weierstrass, G. Cantor and C. Jordan. (See [76] and [72, p. 29].)

According to Cantor, a metric space $X(d)$ may be called well-chained or chain-connected if for every $x, y \in X$ and every $\varepsilon > 0$ there exists a finite family $(x_i)_{i=0}^{n}$ of points of X such that $x_0 = x$, $x_n = y$ and $d(x_{i-1}, x_i) < \varepsilon$ for all $i = 1, \dots, n$.

That is, there exists a natural number n such that, for the ε-sized d-surrounding $B_{\varepsilon}^{d} = \big\{ (u, v) \in X^2 : \quad d(u, v) < \varepsilon \big\}$, which is only a tolerance (reflexive and symmetric) relation on X, we have $(x, y) \in \big(B_{\varepsilon}^{d} \big)^{n}$, where the nth power is taken with respect to composition.

Therefore, we may call a relator $\mathcal{R}$ on X well-chained, or more precisely properly well-chained, if for every $x, y \in X$ and every $R \in \mathcal{R}$ we have $(x, y) \in \bigcup_{n=0}^{\infty} R^n$, where $R^0 = \Delta_X$ and $R^n = R \circ R^{n-1}$. That is, if for every $R \in \mathcal{R}$ we have $R^{\infty} = X^2$, and thus $\mathcal{R}^{\infty} = \big\{ X^2 \big\}$.

Now, according to a general unifying principle of the theory of relators, we may naturally call a relator $\mathcal{R}$ on X uniformly, proximally, topologically, paratopologically, infinitesimally, parainfinitesimally, and ultimately well-chained if the relator $\mathcal{R}^{\square}$ is properly well-chained with $\square = *$, $\#$, $\wedge$, $\vartriangle$, $\bullet$, $\blacktriangle$, and $\blacklozenge$, respectively.

The well-chainedness of metric or uniform spaces is usually neglected by the authors of the standard textbooks on topology. The only exceptions seem to be Berge [3, pp. 96–99], Gaal [12, pp. 101 and 142] and Whyburn and Duda [75, pp. 34–37] .

Several interesting new characterizations of well-chained metric and uniform spaces were established by Mathews [36], Mrówka and Pervin [37] and Levine [24]. Moreover, the well-chainedness of nearness spaces has also been studied by Baboolal and Ori [2].

Some of the results of Levine were extended to reflexive relators by Kurdics and Száz in [22]. The latter authors also investigated the uniform, proximal and topological well-chainednesses of reflexive relators. But, the other well-chainedness properties have not been considered.

Now, according to the results of [19] and [23], a relator $\mathcal{R}$ on X may be naturally called properly connected if the relator $\mathcal{R} \triangledown \mathcal{R}^{-1} = \big\{ R \cup R^{-1} : \quad R \in \mathcal{R} \big\}$ is properly well-chained.

Moreover, analogously to the corresponding well-chainedness properties, the relator $\mathcal{R}$ may be naturally called uniformly, proximally, topologically, paratopologically, infinitesimally, parainfinitesimally, and ultimately connected if the relator $\mathcal{R}^{\square}$ is properly connected with $\square = *$, $\#$, $\wedge$, $\vartriangle$, $\bullet$, $\blacktriangle$, and $\blacklozenge$, respectively.

The appropriateness of the above definitions will already be quite obvious from the following four basic theorems.

Theorem 1. *A relator* $\mathcal{R}$ *on* X *is proximally (topologically) well-chained if and only if* $\emptyset$ *and* X *are the only proximally (topologically) open subsets of* $X(\mathcal{R})$.

Remark 1. A subset A of the relator space $X(\mathcal{R})$ is called proximally open if there exists an $R \in \mathcal{R}$ such that $R(A) \subset A$.

Theorem 2. *A relator* $\mathcal{R}$ *on* X *is proximally (topologically) connected if and only if* $\emptyset$ *and* X *are the only proximally (topologically) clopen subsets of* $X(\mathcal{R})$.

Remark 2. A subset A of the relator space $X(\mathcal{R})$ is called proximally clopen if both A and $X \setminus A$ are proximally open.

Theorem 3. *A relator* $\mathcal{R}$ *on* X, *with* $\operatorname{card}(X) > 1$, *is paratopologically well-chained if and only if* X *is the only fat subset of* $X(\mathcal{R})$, *or equivalently* $\mathcal{R} = \{X^2\}$.

Remark 3. A subset A of the relator space $X(\mathcal{R})$ is called fat if there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) \subset A$.

Theorem 4. *A relator* $\mathcal{R}$ *on* X, *with* $\operatorname{card}(X) > 1$, *is paratopologically connected if and only if each fat subset of* $X(\mathcal{R})$ *is dense, or equivalently* $\mathcal{R}^{-1} \circ \mathcal{R} = \{X^2\}$.

Remark 4. Therefore, the hyperconnectedness of Steen and Seebach [54, p. 29], studied also by Levine [25] and several other people, is a particular case of our paratopological connectedness.

1. A FEW BASIC FACTS ON RELATIONS AND RELATORS

A subset F of a product set $X \times Y$ is called a relation on X to Y. In particular, the relations $\Delta_X = \{(x, x) : \ x \in X\}$ and $X^2 = X \times X$ are called the identity and the universal relations on X, respectively.

Namely, if in particular $X = Y$, then we may simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X \cup Y$. Therefore, it is frequently not a severe restriction to assume that $X = Y$.

If F is a relation on X, and moreover $x \in X$ and $A \subset X$, then the sets $F(x) = \{y \in X : \ (x, y) \in F\}$ and $F[A] = \bigcup_{x \in A} F(x)$ are called the images of x and A under F, respectively. Whenever $A \in X$ is unlikely, we may write $F(A)$ in place of $F[A]$.

If F is a relation on X, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the inverse F^{-1} of F can be defined such that $F^{-1}(x) = \{y \in X : \ x \in F(y)\}$ for all $x \in X$.

Moreover, if F and G are relations on X then the composition $F \circ G$ and the box product $F \boxtimes G$ of F and G can be defined such that $(F \circ G)(x) = F\big(G(x)\big)$ for all $x \in X$ and $(F \boxtimes G)(x, y) = F(x) \times G(y)$ for all $x, y \in X$.

Thus, we have $(F \circ G)^{-1} = G^{-1} \circ F^{-1}$ and $(F \boxtimes G)^{-1} = F^{-1} \boxtimes G^{-1}$. Moreover, $(F \circ G)(A) = F\big(G(A)\big)$ for all $A \subset X$, and $(F \boxtimes G)(B) = G \circ B \circ F^{-1}$ for all $B \subset X^2$. And hence, in particular, $F \circ G = (G^{-1} \boxtimes F)(\Delta_X)$.

If F is a relation on X, then the sets $D_F = F^{-1}(X)$ and $R_F = F(X)$ are called the domain and range of F, respectively. Whenever, F is a relation on X to Y such that $X = D_F$ (and $Y = R_F$), then we say that F is a relation of X into (onto) Y.

A relation F is said to be a function if for each $x \in D_F$ there exists a $y \in R_F$ such that $F(x) = \{y\}$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$. Moreover, if f is a function, then sometimes we also write $(f_x)_{x \in D_f} = f$ and $\{f_x\}_{x \in D_f} = R_f$, where $f_x = f(x)$.

A relation R on X is called reflexive, symmetric, transitive, and directive if $\Delta_X \subset R$, $R = R^{-1}$, $R \circ R \subset R$, and $X^2 = R^{-1} \circ R$, respectively. Moreover, a reflexive relation is called a preorder (tolerance) if it is transitive (symmetric), and a directive preorder is called a direction. Note that $R = R \circ R$ if R is a preorder.

If R is a relation on X, then we write $R^n = R \circ R^{n-1}$ for all $n \in \mathbb{N}$ by agreeing that $R^0 = \Delta_X$. Moreover, we define $R^\infty = \bigcup_{n=0}^\infty R^n$. Thus, R^∞ is the smallest preorder on X such that $R \subset R^\infty$. Therefore, $R = R^\infty$ if and only if R is a preorder. Moreover, $R^\infty = R^{\infty\infty}$ and $\big(R^\infty\big)^{-1} = \big(R^{-1}\big)^\infty$.

A nonvoid family $\mathcal{R}$ of relations on a nonvoid set X is called a relator on X, and the ordered pair $X(\mathcal{R}) = (X, \mathcal{R})$ is called a relator space. Relator spaces are straightforward generalizations of ordered sets and uniform spaces [56]. They deserve to be widely investigated because of the following two facts.

If $\mathcal{D}$ is a nonvoid family of certain distance functions on X, then the relator $\mathcal{R}_\mathcal{D}$ consisting of all surroundings $B_r^d = \{(x, y) \in X^2 : \ d(x, y) < r\}$, where $d \in \mathcal{D}$ and $r > 0$, is a more convenient mean of defining the basic notions of analysis in the space $X(\mathcal{D})$ than the family of all open subsets of $X(\mathcal{D})$, or even the family $\mathcal{D}$ itself.

Moreover, all reasonable generalizations of the usual topological structures (such as proximities, closures, topologies, filters and convergences, for instance) can be easily derived from relators (according to the results of [62] and [55]), and thus they need not be studied separately.

For instance, if $\mathcal{A}$ is a certain generalized topology or a nonvoid stack (ascending system) in X, then $\mathcal{A}$ can be easily derived (according to the forthcoming definitions of the families $\tau_\mathcal{R}$, $\mathcal{T}_\mathcal{R}$ and $\mathcal{E}_\mathcal{R}$) from the Davis-Pervin relator $\mathcal{R}_\mathcal{A}$ consisting of all preorders $R_A = A^2 \cup (X \setminus A) \times X$, where $A \in \mathcal{A}$.

Note that, in contrast to the preorders R_A, the surroundings B_r^d are usually tolerances. Therefore, besides preorder relators, tolerance relators are also important particular cases of reflexive relators. Unfortunately, the class of all reflexive relators proved to be inadequate for several important purposes.

2. Set-valued functions and unary operations for relators

A function $\mathfrak{F}$ of the family of all relators on X into a family of sets is called a set-valued function for relators on X. And we write $\mathfrak{F}_\mathcal{R} = \mathfrak{F}(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

In particular, a function $\square$ of the family of all relators on X into itself is called a unary operation for relators on X. And we write $\mathcal{R}^\square = \square(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

The set-valued function $\mathfrak{F}$ is called increasing (decreasing) if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X with $\mathcal{S} \subset \mathcal{R}$ we have $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ $\left(\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}} \right)$. Note that, in this case, we have $\bigcup_{R \in \mathcal{R}} \mathfrak{F}_R \subset \mathfrak{F}_{\mathcal{R}}$ $\left(\mathfrak{F}_{\mathcal{R}} \subset \bigcap_{R \in \mathcal{R}} \mathfrak{F}_R \right)$.

Therefore, an increasing (decreasing) set-valued function $\mathfrak{F}$ for relators on X may be naturally called normal if for every relator $\mathcal{R}$ on X we have $\mathfrak{F}_{\mathcal{R}} = \bigcup_{R \in \mathcal{R}} \mathfrak{F}_R$ $\left(\mathfrak{F}_{\mathcal{R}} = \bigcap_{R \in \mathcal{R}} \mathfrak{F}_R \right)$.

If $\mathfrak{F}$ is an increasing (decreasing) set-valued function for relators on X, then the operation $\square_{\mathfrak{F}}$, defined by

$$\mathcal{R}^{\square_{\mathfrak{F}}} = \left\{ S \subset X^2 : \quad \mathfrak{F}_S \subset \mathfrak{F}_{\mathcal{R}} \right\} \qquad \left(\mathcal{R}^{\square_{\mathfrak{F}}} = \left\{ S \subset X^2 : \quad \mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_S \right\} \right)$$

for every relator $\mathcal{R}$ on X, is called the operation induced by the function $\mathfrak{F}$.

Note that if $\mathfrak{F}$ is an increasing (decreasing) set-valued function for relators on X, then for any relator $\mathcal{R}$ on X we have $\mathcal{R} \subset \mathcal{R}^{\square_{\mathfrak{F}}}$, and hence $\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$ $\left(\mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} \subset \mathfrak{F}_{\mathcal{R}} \right)$. Moreover, if $\mathfrak{F}$ is, in addition, normal, then we also have $\mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} = \bigcup_{S \in \mathcal{R}^{\square_{\mathfrak{F}}}} \mathfrak{F}_S \subset \mathfrak{F}_{\mathcal{R}}$ $\left(\mathfrak{F}_{\mathcal{R}} \subset \bigcap_{S \in \mathcal{R}^{\square_{\mathfrak{F}}}} \mathfrak{F}_S = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}} \right)$.

Therefore, a monotonic set-valued function $\mathfrak{F}$ for relators on X may be naturally called regular if for every relator $\mathcal{R}$ on X we have $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square_{\mathfrak{F}}}}$. Namely, thus every normal set-valued function for relators is, in particular, regular.

A set-valued function $\mathfrak{F}$ for relators on X is called $\square$–increasing ($\square$–decreasing), for some unary operation $\square$ for relators on X, if for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ $\left(\mathfrak{F}_{\mathcal{R}} \subset \mathfrak{F}_{\mathcal{S}} \right)$.

In particular, a unary operation $\square$ for relators on X is called self-increasing if it is $\square$–increasing. That is, for any two relators $\mathcal{R}$ and $\mathcal{S}$ on X we have $\mathcal{S} \subset \mathcal{R}^{\square} \iff \mathcal{S}^{\square} \subset \mathcal{R}^{\square}$.

A unary operation $\square$ for relators on X is called expansive and idempotent if for every relator $\mathcal{R}$ on X we have $\mathcal{R} \subset \mathcal{R}^{\square}$ and $\mathcal{R}^{\square} = \mathcal{R}^{\square\square}$, respectively. And the operation $\square$ is called stable if $\mathcal{R} = \mathcal{R}^{\square}$ whenever $\mathcal{R} = \{ X^2 \}$.

Moreover, an increasing and expansive (idempotent) operation for relators is called an extension (modification) operation. And an idempotent extension operation for relators is called a refinement or closure operation.

The appropriateness of the above definitions is apparent from the following results of [45].

Theorem 2.1. *If $\mathfrak{F}$ is an increasing (decreasing) set-valued function for relators on X, then*

(1) $\square_{\mathfrak{F}}$ is an extension operation;

(2) $\mathfrak{F}_{\mathcal{S}} \subset \mathfrak{F}_{\mathcal{R}}$ implies $\mathcal{S} \subset \mathcal{R}^{\square_{\mathfrak{F}}}$ $\left(\mathcal{R} \subset \mathcal{S}^{\square_{\mathfrak{F}}} \right)$.

Theorem 2.2. *If $\square$ is a unary operation for relators on X, then the following assertions are equivalent:*

(1) $\square$ is a refinement; *(2) $\square$ is self–increasing;*

(3) there exists a $\square$–monotonic set-valued function $\mathfrak{F}$ for relators on X.

Theorem 2.3. *If* $\square$ *is a unary operation and* $\mathfrak{F}$ *is a* $\square$*–increasing (* $\square$*–decreasing) set-valued function for relators on* X *, then*

(1) $\square$ *is a refinement and* $\square = \square_{\mathfrak{F}}$ *;*

(2) $\mathfrak{F}$ *is increasing (decreasing) and regular;*

(3) for every relator $\mathcal{R}$ *on* X *,* $\mathcal{R}^{\square}$ *is the largest relator on* X *such that* $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$ *.*

Corollary 2.4. *If* $\mathfrak{F}$ *is a set-valued function for relators on* X *, then there exists at most one unary operation* $\square$ *for relators on* X *such that* $\mathfrak{F}$ *is* $\square$*–monotonic.*

Theorem 2.5. *If* $\mathfrak{F}$ *is a set-valued function for relators on* X *, then the following assertions are equivalent:*

(1) $\mathfrak{F}$ *is* $\square_{\mathfrak{F}}$*–increasing (* $\square_{\mathfrak{F}}$*–decreasing);*

(2) $\mathfrak{F}$ *is increasing (decreasing) and regular;*

(3) $\mathfrak{F}$ *is* $\square$*–increasing (*$\square$*–decreasing) for some unary oparation* $\square$ *for relators on* X *.*

Corollary 2.6. *If* $\diamondsuit$ *is a unary operation for relators on* X *, then* $\diamondsuit$ *is a refinement if and only if it is increasing and regular. Moreover, in this case, we have* $\diamondsuit = \square_{\diamondsuit}$ *.*

To briefly express some further useful properties of extension and modification operations for relators, we must also have the following

Definition 2.7. If $\diamondsuit$ and $\square$ are unary operations for relators on X, then we say that the operation $\square$ is $\diamondsuit$–dominating, $\diamondsuit$–invariant, $\diamondsuit$–absorbing, and $\diamondsuit$–compatible if for every relator $\mathcal{R}$ on X we have $\mathcal{R}^{\diamondsuit} \subset \mathcal{R}^{\square}$, $\mathcal{R}^{\square} = \mathcal{R}^{\square\diamondsuit}$, $\mathcal{R}^{\square} = \mathcal{R}^{\diamondsuit\square}$, and $\mathcal{R}^{\square\diamondsuit} = \mathcal{R}^{\diamondsuit\square}$, respectively.

Concerning the latter definitions, we shall only quote here the following result of [45].

Theorem 2.8. *If* $\diamondsuit$ *is an expansive and* $\square$ *is a* $\diamondsuit$*–dominating idempotent operation for relators on* X *, then* $\square$ *is* $\diamondsuit$*–invariant. Moreover, if in addition* $\square$ *is increasing, then* $\square$ *is also* $\diamondsuit$*–absorbing.*

Remark 2.9. In this respect, it is also worth noticing that if $\diamondsuit$ is an expansive and $\square$ is a $\diamondsuit$–dominating operation for relators on X, then $\square$ is also expansive.

Moreover, if $\diamondsuit$ is an arbitrary (increasing) and $\square$ is an expansive operation for relators on X such that $\mathcal{R}^{\diamondsuit\square} \subset \mathcal{R}^{\square}$ $\big(\mathcal{R}^{\square\diamondsuit} \subset \mathcal{R}^{\square} \big)$ for every relator $\mathcal{R}$ on X, then $\square$ is $\diamondsuit$–dominating.

3. Some important set-valued functions for relators

If $\mathcal{R}$ is a relator on X, then for any $A, B \subset X$ and $x, y \in X$ we write:

(1) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ $\big(B \in \operatorname{Cl}_{\mathcal{R}}(A) \big)$ if $R(B) \subset A$ $\big(R(B) \cap A \neq \emptyset \big)$ for some (all) $R \in \mathcal{R}$;

(2) $x \in \operatorname{int}_{\mathcal{R}}(A) \quad (\, x \in \operatorname{cl}_{\mathcal{R}}(A)\,)$ if $\{x\} \in \operatorname{Int}_{\mathcal{R}}(A) \quad (\, \{x\} \in \operatorname{Cl}_{\mathcal{R}}(A)\,)$;

(3) $y \in \sigma_{\mathcal{R}}(x) \quad (\, y \in \rho_{\mathcal{R}}(x)\,)$ if $y \in \operatorname{int}_{\mathcal{R}}(\{x\}) \quad (\, y \in \operatorname{cl}_{\mathcal{R}}(\{x\})\,)$;

and moreover

(4) $A \in \tau_{\mathcal{R}} \quad (\, A \in \tau\!\!\!\!-_{\mathcal{R}}\,)$ if $A \in \operatorname{Int}_{\mathcal{R}}(A) \quad (\, X \setminus A \notin \operatorname{Cl}_{\mathcal{R}}(A)\,)$;

(5) $A \in \mathcal{T}_{\mathcal{R}} \quad (\, A \in \mathcal{F}_{\mathcal{R}}\,)$ if $A \subset \operatorname{int}_{\mathcal{R}}(A) \quad (\, \operatorname{cl}_{\mathcal{R}}(A) \subset A\,)$;

(6) $A \in \mathcal{E}_{\mathcal{R}} \quad (\, A \in \mathcal{D}_{\mathcal{R}}\,)$ if $\operatorname{int}_{\mathcal{R}}(A) \neq \emptyset \quad (\, \operatorname{cl}_{\mathcal{R}}(A) = X\,)$.

The relations $\operatorname{Int}_{\mathcal{R}}$, $\operatorname{int}_{\mathcal{R}}$ and $\sigma_{\mathcal{R}}$ are called the proximal, the topological and the infinitesimal interiors induced on X by $\mathcal{R}$, respectively. While, the members of the families $\tau_{\mathcal{R}}$, $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{E}_{\mathcal{R}}$ are called the proximally open, the topologically open and the fat subsets of $X(\mathcal{R})$, respectively.

The fat sets are frequently more important tools than the open sets. For instance, if $\prec$ is a preorder on X, then $\mathcal{T}_{\prec}$ and $\mathcal{E}_{\prec}$ are just the families of all ascending and residual subsets of the preordered set $X(\prec)$, respectively. And the latter sets are certainly more important than the former ones.

In this respect, it is also worth mentioning that if for instance R is a relation on $\mathbb{R}$ such that $R(x) = \,]-\infty, x] \cup \{x+1\}$ for all $x \in \mathbb{R}$, then $\mathcal{T}_R = \{\emptyset, \mathbb{R}\}$, but $\mathcal{E}_R \neq \{\mathbb{R}\}$. Therefore, in contrast to the open sets, the fat sets may be useful tools even in a topologically indiscrete relator space.

Hence, it is not surprising that if $\mathcal{R}$ is a relator on X, then sometimes we shall also need the sets

$$E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} \qquad \text{and} \qquad D_{\mathcal{R}} = \bigcup \left(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}} \right).$$

A function x of a preordered set Γ into a set X is called a Γ-net in X. The Γ-net x is said to be fatly (densely) in a subset A of X if $x^{-1}(A)$ is a fat (dense) subset of Γ. Therefore, Γ could here be an arbitrary relator space, however preordered nets are usually sufficient.

Now, if $\mathcal{R}$ is a relator on X, then for any Γ-nets x and y in X and $a \in X$ we write:

(7) $y \in \operatorname{Lim}_{\mathcal{R}}(x) \quad (\, y \in \operatorname{Adh}_{\mathcal{R}}(x)\,)$ if the net (y, x) is fatly (densely) in each $R \in \mathcal{R}$;

(8) $a \in \operatorname{lim}_{\mathcal{R}}(x) \quad (\, a \in \operatorname{adh}_{\mathcal{R}}(x)\,)$ if $a_{\Gamma} \in \operatorname{Lim}_{\mathcal{R}}(x) \quad (\, a_{\Gamma} \in \operatorname{Adh}_{\mathcal{R}}(x)\,)$, where $a_{\Gamma} = \Gamma \times \{a\}$.

Concerning the above basic tools we shall only quote here the following theorems which have been mostly proved in [62] and [45].

Theorem 3.1. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then*

$$\operatorname{Cl}_{\mathcal{R}}(A) = \mathcal{P}(X) \setminus \operatorname{Int}_{\mathcal{R}}(X \setminus A) \qquad \textit{and} \qquad \operatorname{cl}_{\mathcal{R}}(A) = X \setminus \operatorname{int}_{\mathcal{R}}(X \setminus A).$$

Theorem 3.2. *If* $\mathcal{R}$ *is a relator on* X *and* $\mathcal{R}^{-1} = \{ R^{-1} : \quad R \in \mathcal{R} \}$, *then*

$$\mathrm{Cl}_{\mathcal{R}}^{-1} = \mathrm{Cl}_{\mathcal{R}^{-1}} \qquad \textit{and} \qquad \mathrm{Int}_{\mathcal{R}}^{-1} = \mathcal{C}_X \circ \mathrm{Int}_{\mathcal{R}^{-1}} \circ \mathcal{C}_X \, ,$$

where $\mathcal{C}_X(A) = X \setminus A$ *for all* $A \subset X$.

Theorem 3.3. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then*

$$\mathrm{cl}_{\mathcal{R}}(A) = \bigcap_{R \in \mathcal{R}} R^{-1}(A) \qquad \textit{and} \qquad \rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = \Big(\bigcap \mathcal{R} \Big)^{-1}.$$

Theorem 3.4. *If* $\mathcal{R}$ *is a relator on* X *and* $x \in X$, *then*

$$\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = \rho_{\mathcal{R}^{-1}} \qquad \textit{and} \qquad \rho_{\mathcal{R}}^{-1}(x) = \bigcap \{ A \subset X : \quad x \in \mathrm{int}_{\mathcal{R}}(A) \} .$$

Theorem 3.5. *If* $\mathcal{R}$ *is a relator on* X *and* $x \in X$, *then*

$$\sigma_{\mathcal{R}}(x) = X \setminus \mathrm{cl}_{\mathcal{R}}\big(X \setminus \{x\} \big) = \bigcup_{R \in \mathcal{R}} \big(X \setminus \rho_{R}\big(X \setminus \{x\} \big) \big) \, .$$

Theorem 3.6. *If* $\mathcal{R}$ *is a relator on* X, *then* $\tau\!\!\!\!-_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$,

$$\tau\!\!\!\!-_{\mathcal{R}} = \big\{ A \subset X : \quad X \setminus A \in \tau_{\mathcal{R}} \big\} \qquad \textit{and} \qquad \mathcal{F}_{\mathcal{R}} = \big\{ A \subset X : \quad X \setminus A \in \mathcal{T}_{\mathcal{R}} \big\} .$$

Theorem 3.7. *If* $\mathcal{R}$ *is a relator on* X, *then*

$$\mathcal{D}_{\mathcal{R}} = \big\{ A \subset X : \quad X \setminus A \notin \mathcal{E}_{\mathcal{R}} \big\} = \big\{ A \subset X : \quad \forall \; B \in \mathcal{E}_{\mathcal{R}} : \quad A \cap B \neq \emptyset \big\} .$$

Theorem 3.8. *If* $\mathcal{R}$ *is a relator on* X, *then*

$$E_{\mathcal{R}} = \bigcap_{x \in X} \rho_{\mathcal{R}}^{-1}(x) \qquad \textit{and} \qquad D_{\mathcal{R}} = X \setminus E_{\mathcal{R}}.$$

Theorem 3.9. *If* $\mathcal{R}$ *is a relator on* X, *then for any* $A, B \subset X$ *we have* $B \in \mathrm{Cl}_{\mathcal{R}}(A)$ *if and only if there exist nets* x *and* y *in* A *and* B, *respectively, such that* $y \in \mathrm{Lim}_{\mathcal{R}}(x)$ $\big(y \in \mathrm{Adh}_{\mathcal{R}}(x) \big)$.

Remark 3.10. The nets x and y in the above theorem can, in general, be required to be only partially ordered (directed).

However, if the relator $\mathcal{R}$ is uniformly filtered in the sense that for any $R, S \in \mathcal{R}$ there exists a $T \in \mathcal{R}$ such that $T \subset R \cap S$, then the above nets can already be required to be both partially ordered and directed.

Theorem 3.11. *If* $\mathcal{R}$ *is a relator on* X, *then we have* $\mathrm{Lim}_{\mathcal{R}}^{-1} = \mathrm{Lim}_{\mathcal{R}^{-1}}$ *and* $\mathrm{Adh}_{\mathcal{R}}^{-1} = \mathrm{Adh}_{\mathcal{R}^{-1}}$.

Theorem 3.12. *If* $\mathcal{R}$ *is a relator on* X, *then for any* $a \in X$ *and* $A \subset X$ *we have* $a \in \mathrm{cl}_{\mathcal{R}}(A)$ *if and only if there exists a net* x *in* A *such that* $a \in \mathrm{lim}_{\mathcal{R}}(x)$ $\big(a \in \mathrm{adh}_{\mathcal{R}}(x) \big)$.

Remark 3.13. The net x in the above theorem can, in general, be required to be only partially ordered (directed).

However, if the relator $\mathcal{R}$ is topologically filtered in the sense that for any $x \in X$ and $R, S \in \mathcal{R}$ there exists a $T \in \mathcal{R}$ such that $T(x) \subset R(x) \cap S(x)$, then the net x can already be required to be both partially ordered and directed.

Theorem 3.14. *If $\mathcal{R}$ is a relator on X, then for any Γ-net x in X we have*

$$\lim_{\mathcal{R}}(x) = \bigcap_{A \in \mathcal{D}_\Gamma} \operatorname{cl}_{\mathcal{R}}\big(x(A)\big) \qquad \textit{and} \qquad \operatorname{adh}_{\mathcal{R}}(x) = \bigcap_{A \in \mathcal{E}_\Gamma} \operatorname{cl}_{\mathcal{R}}\big(x(A)\big).$$

Remark 3.15. Unfortunately, the relationships between the induced limits and adherences are not so straightforward even if $\mathcal{R}$ is uniformly filtered.

However, it is now more important to note that we also have the following

Theorem 3.16. *(1)* Int, int, σ, τ, $\tau\!\!\!-$, $\mathcal{E}$, *and* D *are normal increasing set-valued functions for relators on* X*;*

(2) Lim, Adh, $\lim$, adh, Cl, cl, ρ, $\mathcal{D}$, *and* E *are normal decreasing set-valued functions for relators on* X.

Remark 3.17. Unfortunately, the increasing set-valued functions $\mathcal{T}$ and $\mathcal{F}$ are not, in general, even regular.

Namely, if $\mathcal{R}$ is a relator on X, then by [30, Example 5.3] there does not, in general, exist a largest relator $\mathcal{R}^{\square}$ on X such that $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\square}}$.

4. SOME IMPORTANT UNARY OPERATIONS FOR RELATORS

If $\mathcal{R}$ is a relator on X, then the relators

$$\begin{aligned}
\mathcal{R}^{*} &= \{ S \subset X^2 : \quad \exists\ R \in \mathcal{R} : \quad R \subset S \}, \\
\mathcal{R}^{\#} &= \{ S \subset X^2 : \quad \forall\ A \subset X : \quad A \in \operatorname{Int}_{\mathcal{R}}\big(S(A)\big) \}, \\
\mathcal{R}^{\wedge} &= \{ S \subset X^2 : \quad \forall\ x \in X : \quad x \in \operatorname{int}_{\mathcal{R}}\big(S(x)\big) \}, \\
\mathcal{R}^{\triangle} &= \{ S \subset X^2 : \quad \forall\ x \in X : \quad S(x) \in \mathcal{E}_{\mathcal{R}} \}
\end{aligned}$$

are called the uniform, the proximal, the topological, and the paratopological refinements of $\mathcal{R}$, respectively.

Moreover, the relators

$$\mathcal{R}^{\bullet} = \{ \rho_{\mathcal{R}}^{-1} \}^{*}, \qquad \mathcal{R}^{\bigstar} = \{ \rho_{\mathcal{R}}^{-1} \}^{\triangle} \qquad \text{and} \qquad \mathcal{R}^{\blacktriangle} = \{ X \times E_{\mathcal{R}} \}^{*}$$

are called the infinitesimal refinement, the ultrainfinitesimal extension and the parainfinitesimal refinement of $\mathcal{R}$, respectively.

And, the relator $\mathcal{R}^{\blacklozenge}$, defined by

$$\mathcal{R}^{\blacklozenge} = \{ X^2 \} \quad \text{if} \quad \mathcal{R} = \{ X^2 \} \qquad \text{and} \qquad \mathcal{R}^{\blacklozenge} = \mathcal{P}(X^2) \quad \text{if} \quad \mathcal{R} \neq \{ X^2 \},$$

is called the ultimate stable refinement of $\mathcal{R}$.

Unfortunately, thus we only have

$$\mathcal{R} \subset \mathcal{R}^{*} \subset \mathcal{R}^{\#} \subset \mathcal{R}^{\wedge} \subset \mathcal{R}^{\vartriangle} \cap \mathcal{R}^{\bullet} \qquad \text{and} \qquad \mathcal{R}^{\vartriangle} \cup \mathcal{R}^{\bullet} \subset \mathcal{R}^{\bigstar} \subset \mathcal{R}^{\blacktriangle} \subset \mathcal{R}^{\blacklozenge}.$$

Namely, by [45, Example 1], the relators $\mathcal{R}^{\vartriangle}$ and $\mathcal{R}^{\bullet}$ are, in general, incomparable.

On the other hand, if $\mathcal{R}$ is a relator on X, then the relators

$$\mathcal{R}^{\infty} = \{ R^{\infty} : \quad R \in \mathcal{R} \} \qquad \text{and} \qquad \mathcal{R}^{\partial} = \{ S \subset X^{2} : \quad S^{\infty} \in \mathcal{R} \}$$

are called the direct and the inverse preorder modifications of $\mathcal{R}$, respectively. Thus, we have $\mathcal{R}^{\partial\infty} \subset \mathcal{R} \subset \mathcal{R}^{\infty\partial}$, and hence $\mathcal{R}^{\infty} = \mathcal{R}^{\infty\partial\infty}$ and $\mathcal{R}^{\partial} = \mathcal{R}^{\partial\infty\partial}$.

Moreover, for instance, the relators $\mathcal{R}^{*\infty}$ and $\mathcal{R}^{\infty *}$ are called the quasi–uniform and the almost uniform modifications of $\mathcal{R}$, respectively. Thus, we have $\mathcal{R}^{\infty} \subset \mathcal{R}^{*\infty} \subset \mathcal{R}^{\infty *} \subset \mathcal{R}^{*}$, and hence $\mathcal{R}^{*\infty} = \mathcal{R}^{\infty * \infty}$ and $\mathcal{R}^{\infty *} = \mathcal{R}^{*\infty *}$.

While, for instance, the relators

$$\mathcal{R}^{\sharp} = \mathcal{R}^{\#\partial} \qquad \text{and} \qquad \mathcal{R}^{\curlywedge} = \mathcal{R}^{\wedge\partial}$$

are called the superproximal refinement and the supertopological extension of $\mathcal{R}$, respectively. Thus, we have $\mathcal{R}^{\#} \subset \mathcal{R}^{\sharp}$ and $\mathcal{R}^{\wedge} \subset \mathcal{R}^{\curlywedge}$ such that $\mathcal{R}^{\#\infty} = \mathcal{R}^{\sharp\infty}$ and $\mathcal{R}^{\wedge\infty} = \mathcal{R}^{\curlywedge\infty}$.

And, the relator

$$\mathcal{R}^{\star} = \{ S \subset X^{2} : \quad \sigma_{S} \subset \sigma_{\mathcal{R}} \}$$

is called the σ–infinitesimal refinement of $\mathcal{R}$. Thus, we have

$$\mathcal{R}^{\star} = \{ S \subset X^{2} : \quad \forall\ x \in X : \quad \mathrm{cl}_{\mathcal{R}}(X \setminus \{x\}) \subset S^{-1}(X \setminus \{x\}) \} .$$

The appropriateness of most of the above definitions is apparent from the following theorem of [45].

Theorem 4.1. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{R}^{*} = \mathcal{R}^{\square_{\mathrm{Lim}}} = \mathcal{R}^{\square_{\mathrm{Adh}}}$;

(2) $\mathcal{R}^{\#} = \mathcal{R}^{\square_{\mathrm{Int}}} = \mathcal{R}^{\square_{\mathrm{Cl}}}$;

(3) $\mathcal{R}^{\wedge} = \mathcal{R}^{\square_{\mathrm{lim}}} = \mathcal{R}^{\square_{\mathrm{adh}}}$;

(4) $\mathcal{R}^{\wedge} = \mathcal{R}^{\square_{\mathrm{int}}} = \mathcal{R}^{\square_{\mathrm{cl}}}$;

(5) $\mathcal{R}^{\vartriangle} = \mathcal{R}^{\square_{\mathcal{E}}} = \mathcal{R}^{\square_{\mathcal{D}}}$;

(6) $\mathcal{R}^{\blacktriangle} = \mathcal{R}^{\square_{E}} = \mathcal{R}^{\square_{D}}$;

(7) $\mathcal{R}^{\sharp} = \mathcal{R}^{\square_{\tau}} = \mathcal{R}^{\square_{\mathcal{F}}}$;

(8) $\mathcal{R}^{\curlywedge} = \mathcal{R}^{\square_{\mathcal{T}}} = \mathcal{R}^{\square_{\mathcal{F}}}$;

(9) $\mathcal{R}^{\star} = \mathcal{R}^{\square_{\sigma}}$;

(10) $\mathcal{R}^{\bullet} = \mathcal{R}^{\square_{\rho}}$.

Hence, by Theorems 3.16, 2.5 and 2.3, it is clear that we have the following

Theorem 4.2. $*$, $\#$, $\wedge$, $\vartriangle$, $\blacktriangle$, $\sharp$, $\star$ *and* $\bullet$ *are refinement operations for relators on* X *such that for every relator* $\mathcal{R}$ *on* X

(1) $\mathcal{R}^{*}$ *is the largest relator on* X *such that* $\operatorname{Lim}_{\mathcal{R}} = \operatorname{Lim}_{\mathcal{R}^{*}}$*, or equivalently* $\operatorname{Adh}_{\mathcal{R}} = \operatorname{Adh}_{\mathcal{R}^{*}}$*;*

(2) $\mathcal{R}^{\#}$ *is the largest relator on* X *such that* $\operatorname{Int}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R}^{\#}}$*, or equivalently* $\operatorname{Cl}_{\mathcal{R}} = \operatorname{Cl}_{\mathcal{R}^{\#}}$*;*

(3) $\mathcal{R}^{\wedge}$ *is the largest relator on* X *such that* $\operatorname{lim}_{\mathcal{R}} = \operatorname{lim}_{\mathcal{R}^{\wedge}}$*, or equivalently* $\operatorname{adh}_{\mathcal{R}} = \operatorname{adh}_{\mathcal{R}^{\wedge}}$*;*

(4) $\mathcal{R}^{\wedge}$ *is the largest relator on* X *such that* $\operatorname{int}_{\mathcal{R}} = \operatorname{int}_{\mathcal{R}^{\wedge}}$*, or equivalently* $\operatorname{cl}_{\mathcal{R}} = \operatorname{cl}_{\mathcal{R}^{\wedge}}$*;*

(5) $\mathcal{R}^{\vartriangle}$ *is the largest relator on* X *such that* $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{\vartriangle}}$*, or equivalently* $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{\vartriangle}}$*;*

(6) $\mathcal{R}^{\blacktriangle}$ *is the largest relator on* X *such that* $E_{\mathcal{R}} = E_{\mathcal{R}^{\blacktriangle}}$*, or equivalently* $D_{\mathcal{R}} = D_{\mathcal{R}^{\blacktriangle}}$*;*

(7) $\mathcal{R}^{\sharp}$ *is the largest relator on* X *such that* $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\sharp}}$*, or equivalently* $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\sharp}}$*;*

(8) $\mathcal{R}^{\star}$ *and* $\mathcal{R}^{\bullet}$ *are the largest relators on* X *such that* $\sigma_{\mathcal{R}} = \sigma_{\mathcal{R}^{\star}}$ *and* $\rho_{\mathcal{R}} = \rho_{\mathcal{R}^{\bullet}}$*, respectively.*

Remark 4.3. Unfortunately, by [45, Examples 7.1 and 7.2], the operations $\bigstar$ and $\curlywedge$ are not, in general, idempotent.

Therefore, if $\square = \bigstar$ or $\curlywedge$, then by [45, Theorem 1.5] there does not, in general, exist a set-valued function $\mathfrak{F}$ for relators on X such that, for every relator $\mathcal{R}$ on X, $\mathcal{R}^{\square}$ is the largest relator on X such that $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\square}}$.

However, in addition to Theorem 4.2, we can also at once state

Theorem 4.4. $\blacklozenge$ *is a refinement operation for relators on* X *such that, for every relator* $\mathcal{R}$ *on* X*,* $\mathcal{R}^{\blacklozenge}$ *is the largest relator on* X *such that* $\mathcal{R} = \mathcal{R}^{\blacklozenge}$ *whenever* $\mathcal{R} = \{X^2\}$.

Moreover, by the results of [45, Section 5], we also have the following

Theorem 4.5. *If* $\mathcal{R}$ *is a relator on* X*, then*

$$\mathcal{R}^{\bullet} = \mathcal{R}^{\vee\vee}, \qquad \mathcal{R}^{\bigstar} = \mathcal{R}^{\bullet\vartriangle}, \qquad \mathcal{R}^{\blacktriangle} = \mathcal{R}^{\vartriangle\bullet} \qquad \text{and} \qquad \mathcal{R}^{\blacklozenge} = \mathcal{R}^{\triangledown\triangledown},$$

where $\mathcal{R}^{\vee} = \mathcal{R}^{\wedge -1}$ *and* $\mathcal{R}^{\triangledown} = \mathcal{R}^{\vartriangle -1}$.

Remark 4.6. Unfortunately, the operations $\sharp$, $\curlywedge$ and $\star$ are not, in general, stable. That is, they are not, in general, dominated by the operation $\blacklozenge$.

Therefore, even instead of the operation $\sharp$, it seems more convenient to use the operation $\#\infty$. Namely, according to the results of [35], we have the following

Theorem 4.7. $\#\infty$, $\wedge\infty$ *and* $\vartriangle\infty$ *are modification operations for relators on* X *such that for every relator* $\mathcal{R}$ *on* X

(1) $\mathcal{R}^{\#\infty}$ *is the largest preorder relator on* X *such that* $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\#\infty}}$*, or equivalently* $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\#\infty}}$*;*

(2) $\mathcal{R}^{\wedge\infty}$ *is the largest preorder relator on* X *such that* $\mathcal{T}_{\mathcal{R}} = \mathcal{T}_{\mathcal{R}^{\wedge\infty}}$, *or equivalently* $\mathcal{F}_{\mathcal{R}} = \mathcal{F}_{\mathcal{R}^{\wedge\infty}}$;

(3) $\mathcal{R}^{\vartriangle\infty}$ *is the largest preorder relator on* X *such that* $\mathcal{E}_{\mathcal{R}} = \mathcal{E}_{\mathcal{R}^{\vartriangle\infty}}$, *or equivalently* $\mathcal{D}_{\mathcal{R}} = \mathcal{D}_{\mathcal{R}^{\vartriangle\infty}}$, *whenever* $\mathcal{R}$ *is total.*

Remark 4.8. A relator $\mathcal{R}$ on X will be called total if X is the domain of each member of $\mathcal{R}$.

Finally, we note that the following theorem is also true.

Theorem 4.9. *The operations* ∞, ∂, $*$, $\#$ *and* $\bullet$ *are inversion compatible. While, the operation* $\blacklozenge$ *is both inversion invariant and inversion absorbing.*

Remark 4.10. The map $\mathcal{R} \mapsto \mathcal{R}'$, where $\mathcal{R}'$ is the family of all finite intersections of the members of $\mathcal{R}$, is also an important inversion compatible refinement operation for relators. Moreover, by Theorem 3.3, we have $\mathcal{R}' \subset \mathcal{R}^{\bullet}$.

However, unfortunately, the operations $\wedge$, $\vartriangle$ and $\blacktriangle$ are not, in general, inversion compatible. Namely, for instance, by using Theorems 4.5 and 3.3, we can show that $\left(\mathcal{R}^{-1}\right)^{\wedge} \subset \left(\mathcal{R}^{\wedge}\right)^{-1}$ if and only if $\mathcal{R}^{\wedge} = \mathcal{R}^{\bullet}$. (See also [34]).

5. SOME FURTHER RESULTS ON THE BASIC SET-VALUED FUNCTIONS AND UNARY OPERATIONS FOR RELATORS

Definition 5.1. If $\mathcal{R}$ is a relator on X, then we say that:

(1) $\mathcal{R}$ is total if $R(x) \neq \emptyset$ for all $x \in X$ and $R \in \mathcal{R}$;

(2) $\mathcal{R}$ is reflexive if $x \in R(x)$ for all $x \in X$ and $R \in \mathcal{R}$;

(3) $\mathcal{R}$ is quasi-topological if $x \in \operatorname{int}_{\mathcal{R}}\left(\operatorname{int}_{\mathcal{R}}\left(R(x)\right)\right)$ for all $x \in X$ and $R \in \mathcal{R}$;

(4) $\mathcal{R}$ is topological if for all $x \in X$ and $R \in \mathcal{R}$ there exists $V \in \mathcal{T}_{\mathcal{R}}$ such that $x \in V \subset R(x)$.

Remark 5.2. Quite similarly, a relator $\mathcal{R}$ on X may be called proximal if for all $A \subset X$ and $R \in \mathcal{R}$ there exists $V \in \tau_{\mathcal{R}}$ such that $A \subset V \subset R(A)$.

The appropriateness of Definition 5.1 is already quite obvious from the following four basic theorems which have been mostly proved in [60].

Theorem 5.3. *If* $\mathcal{R}$ *is a relator on* X, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is total;* *(2)* $\emptyset \notin \mathcal{E}_{\mathcal{R}}$ $\left(X \in \mathcal{D}_{\mathcal{R}}\right)$; *(3)* $\mathcal{E}_{\mathcal{R}} \neq \mathcal{P}(X)$ $\left(\mathcal{D}_{\mathcal{R}} \neq \emptyset\right)$.

Theorem 5.4. *If* $\mathcal{R}$ *is a relator on* X, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is reflexive;* *(2)* $\rho_{\mathcal{R}}$ *is reflexive;*

(3) $\operatorname{int}_{\mathcal{R}}(A) \subset A$ $\left(A \subset \operatorname{cl}_{\mathcal{R}}(A)\right)$ *for all* $A \subset X$;

(4) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ $\left(B \cap A \neq \emptyset\right)$ *implies* $B \subset A$ $\left(B \in \operatorname{Cl}_{\mathcal{R}}(A)\right)$ *for all* $A, B \subset X$.

Theorem 5.5. *If* $\mathcal{R}$ *is a relator on* X*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi-topological;*

(2) $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{T}_{\mathcal{R}}$ *for all* $x \in X$ *and* $R \in \mathcal{R}$*;*

(3) $\operatorname{int}_{\mathcal{R}}(A) \in \mathcal{T}_{\mathcal{R}}$ $\big(\operatorname{cl}_{\mathcal{R}}(A) \in \mathcal{F}_{\mathcal{R}}\big)$ *for all* $A \subset X$.

Theorem 5.6. *If* $\mathcal{R}$ *is a relator on* X*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological;*

(2) $\mathcal{R}$ *is reflexive and quasi-topological;*

(3) $\operatorname{int}_{\mathcal{R}}(A) = \bigcup\{V \in \mathcal{T}_{\mathcal{R}} : \ V \subset A\}$ $\big(\operatorname{cl}_{\mathcal{R}}(A) = \bigcap\{W \in \mathcal{F}_{\mathcal{R}} : \ A \subset W\}\big)$ *for all* $A \subset X$.

Remark 5.7. By Theorem 5.5, a relator $\mathcal{R}$ on X may be called weakly (strongly) quasi-topological if $\rho_{\mathcal{R}}(x) \in \mathcal{F}_{\mathcal{R}}$ for all $x \in X$ $\big(R(x) \in \mathcal{T}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}\big)$.

Moreover, by Theorem 5.6, the relator $\mathcal{R}$ may be called weakly (strongly) topological if it is reflexive and weakly (strongly) quasi-topological.

Also by Theorems 5.5 and 5.6, it is clear that in particular we have the following

Theorem 5.8. *If* $\mathcal{R}$ *is a topological relator on* X*, then*

$$\mathcal{E}_{\mathcal{R}} = \big\{A \subset X : \quad \exists\ V \in \mathcal{T}_{\mathcal{R}} : \quad \emptyset \neq V \subset A\big\}.$$

Remark 5.9. Unfortunately, if the above equality holds, then we can only state that $\operatorname{int}_{\mathcal{R}}\big(\operatorname{int}_{\mathcal{R}}\big(R(x)\big)\big) \neq \emptyset$, and hence $\operatorname{int}_{\mathcal{R}}\big(R(x)\big) \in \mathcal{E}_{\mathcal{R}}$ for all $x \in X$ and $R \in \mathcal{R}$.

Definition 5.10. If $\square$ is a unary operation on relators on X, then two relators $\mathcal{R}$ and $\mathcal{S}$ on X are called $\square$–equivalent if $\mathcal{R}^{\square} = \mathcal{S}^{\square}$.

In particular, the relator $\mathcal{R}$ is called $\square$–simple if it is $\square$–equivalent to a singleton relator. Moreover, the relator $\mathcal{R}$ is called $\square$–fine if $\mathcal{R} = \mathcal{R}^{\square}$.

Remark 5.11. In this respect, it is worth noticing that the relator $\mathcal{R}^{\vee}$ is always proximally fine and topologically simple. Moreover, every relator is already infinitesimally simple [44].

Now, in addition to Theorem 5.6, we can also state the following theorem of [60].

Theorem 5.12. *If* $\mathcal{R}$ *is a relator on* X*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topological;*

(2) $\mathcal{R}$ *is topologically equivalent to* $\mathcal{R}_{\mathcal{T}_{\mathcal{R}}}$ $\big(\mathcal{R}^{\wedge\infty}\big)$*;*

(3) $\mathcal{R}$ *is topologically equivalent to a preorder (topological) relator on* X.

Remark 5.13. Moreover, it is also worth mentioning that a unary operation $\square$ for relators on X is a refinement (modification) if and only if there exists a topological (quasi-topological) relator $\mathfrak{R}$ on $\mathcal{P}(X^2)$ such that $\mathcal{R}^{\square} = \operatorname{cl}_{\mathfrak{R}}(\mathcal{R})$ for every relator $\mathcal{R}$ on X.

In addition to the results of Section 4, we shall also need the following theorems.

Theorem 5.14. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then*

$$\operatorname{Int}_{\mathcal{R}^\wedge}(A) = \mathcal{P}\big(\operatorname{int}_{\mathcal{R}}(A)\big) \qquad \textit{and} \qquad \operatorname{Cl}_{\mathcal{R}^\wedge}(A) = \mathcal{P}(X) \setminus \mathcal{P}\big(X \setminus \operatorname{cl}_{\mathcal{R}}(A)\big).$$

Corollary 5.15. *If* $\mathcal{R}$ *is a relator on* X, *then* $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$ *and* $\bar{\tau}_{\mathcal{R}^\wedge} = \mathcal{F}_{\mathcal{R}}$.

Theorem 5.16. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then*

(1) $\operatorname{Int}_{\mathcal{R}^\vartriangle}(A) = \{\emptyset\}$ *if* $A \notin \mathcal{E}_{\mathcal{R}}$ *and* $\operatorname{Int}_{\mathcal{R}^\vartriangle}(A) = \mathcal{P}(X)$ *if* $A \in \mathcal{E}_{\mathcal{R}}$;

(2) $\operatorname{Cl}_{\mathcal{R}^\vartriangle}(A) = \emptyset$ *if* $A \notin \mathcal{D}_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^\vartriangle}(A) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $A \in \mathcal{D}_{\mathcal{R}}$.

Corollary 5.17. *If* $\mathcal{R}$ *is a relator on* X, *then* $\tau_{\mathcal{R}^\vartriangle} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$ *and* $\bar{\tau}_{\mathcal{R}^\vartriangle} = \big(\mathcal{P}(X) \setminus \mathcal{D}_{\mathcal{R}}\big) \cup \{X\}$.

Theorem 5.18. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then*

$$\operatorname{Int}_{\mathcal{R}^\bullet}(A) = \mathcal{P}\big(X \setminus \rho_{\mathcal{R}}(X \setminus A)\big) \qquad \textit{and} \qquad \operatorname{Cl}_{\mathcal{R}^\bullet}(A) = \mathcal{P}(X) \setminus \mathcal{P}\big(X \setminus \rho_{\mathcal{R}}(A)\big).$$

Corollary 5.19. *If* $\mathcal{R}$ *is a relator on* X, *then*

$$\tau_{\mathcal{R}^\bullet} = \big\{ A \subset X : \quad A \cap \rho_{\mathcal{R}}(X \setminus A) = \emptyset \big\}$$

and $\bar{\tau}_{\mathcal{R}^\bullet} = \{ A \subset X : \quad (X \setminus A) \cap \rho_{\mathcal{R}}(A) = \emptyset \}$.

Theorem 5.20. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then*

(1) $\operatorname{Int}_{\mathcal{R}^\star}(A) = \emptyset$ *if* $X = \rho_{\mathcal{R}}(X \setminus A)$ *and* $\operatorname{Int}_{\mathcal{R}^\star}(A) = \mathcal{P}(X)$ *if* $X \neq \rho_{\mathcal{R}}(X \setminus A)$;

(2) $\operatorname{Cl}_{\mathcal{R}^\star}(A) = \emptyset$ *if* $X \neq \rho_{\mathcal{R}}(A)$ *and* $\operatorname{Cl}_{\mathcal{R}^\blacktriangle}(A) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $X = \rho_{\mathcal{R}}(A)$.

Corollary 5.21. *If* $\mathcal{R}$ *is a relator on* X, *then*

$$\tau_{\mathcal{R}^\star} = \big\{ A \subset X : \quad X \neq \rho_{\mathcal{R}}(X \setminus A) \big\} \cup \{\emptyset\}$$

and $\bar{\tau}_{\mathcal{R}^\star} = \{ A \subset X : \quad X \neq \rho_{\mathcal{R}}(A) \} \cup \{X\}$.

Theorem 5.22. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then*

(1) $\operatorname{Int}_{\mathcal{R}^\blacktriangle}(A) = \{\emptyset\}$ *if* $E_{\mathcal{R}} \not\subset A$ *and* $\operatorname{Int}_{\mathcal{R}^\blacktriangle}(A) = \mathcal{P}(X)$ *if* $E_{\mathcal{R}} \subset A$;

(2) $\operatorname{Cl}_{\mathcal{R}^\blacktriangle}(A) = \emptyset$ *if* $A \subset D_{\mathcal{R}}$ *and* $\operatorname{Cl}_{\mathcal{R}^\blacktriangle}(A) = \mathcal{P}(X) \setminus \{\emptyset\}$ *if* $A \not\subset D_{\mathcal{R}}$.

Corollary 5.23. *If* $\mathcal{R}$ *is a relator on* X *, then*

$$\tau_{\mathcal{R}^{\blacktriangle}} = \{ A \subset X : \quad E_{\mathcal{R}} \subset A \} \cup \{ \emptyset \} \qquad \text{and} \qquad \bar{\tau}_{\mathcal{R}^{\blacktriangle}} = \mathcal{P}(D_{\mathcal{R}}) \cup \{ X \} .$$

Theorem 5.24. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *, then*

(1) $\operatorname{Int}_{\mathcal{R}^{\blacklozenge}}(A) = \{ \emptyset \}$ *if* $A \neq X$ *and* $\mathcal{R} = \{ X^2 \}$ *and* $\operatorname{Int}_{\mathcal{R}^{\blacklozenge}}(A) = \mathcal{P}(X)$ *if* $A = X$ *or* $\mathcal{R} \neq \{ X^2 \}$*;*

(2) $\operatorname{Cl}_{\mathcal{R}^{\blacklozenge}}(A) = \emptyset$ *if* $A = \emptyset$ *or* $\mathcal{R} \neq \{ X^2 \}$ *and* $\operatorname{Cl}_{\mathcal{R}^{\blacklozenge}}(A) = \mathcal{P}(X) \setminus \{ \emptyset \}$ *if* $A \neq \emptyset$ *and* $\mathcal{R} = \{ X^2 \}$*.*

Corollary 5.25. *If* $\mathcal{R}$ *is a relator on* X*, then* $\tau_{\mathcal{R}^{\blacklozenge}} = \{ \emptyset, X \}$ *if* $\mathcal{R} = \{ X^2 \}$ *and* $\tau_{\mathcal{R}^{\blacklozenge}} = \mathcal{P}(X)$ *if* $\mathcal{R} \neq \{ X^2 \}$*, and moreover* $\bar{\tau}_{\mathcal{R}^{\blacklozenge}} = \tau_{\mathcal{R}^{\blacklozenge}}$*.*

6. Some important binary operations for relators

Definition 6.1. If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then we define

$$\mathcal{R} \circ \mathcal{S} = \{ R \circ S : \quad R \in \mathcal{R}, \; S \in \mathcal{S} \}, \qquad \mathcal{R} \,\square\, \mathcal{S} = \{ R \,\square\, S : \quad R \in \mathcal{R}, \; S \in \mathcal{S} \},$$

$$\mathcal{R} \wedge \mathcal{S} = \{ R \cap S : \quad R \in \mathcal{R}, \; S \in \mathcal{S} \}, \qquad \mathcal{R} \vee \mathcal{S} = \{ R \cup S : \quad R \in \mathcal{R}, \; S \in \mathcal{S} \}.$$

Remark 6.2. By the corresponding definitions, it is clear that $\mathcal{R} \cap \mathcal{S} \subset \mathcal{R} \wedge \mathcal{S}$ and $\mathcal{R} \cap \mathcal{S} \subset \mathcal{R} \vee \mathcal{S}$.

Moreover, concerning the binary operations $\circ$ and $\vee$, we can easily prove the following

Theorem 6.3. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are reflexive relators on* X *, then*

$$\left(\mathcal{R} \circ \mathcal{S} \right)^{*} \subset \left(\mathcal{R} \vee \mathcal{S} \right)^{*}.$$

Proof. In this case, for any $R \in \mathcal{R}$ and $S \in \mathcal{S}$, we have $\Delta_X \subset R$ and $\Delta_X \subset S$. Hence, it follows that $R = R \circ \Delta_X \subset R \circ S$ and $S = \Delta_X \circ S \subset R \circ S$, and thus $R \cup S \subset R \circ S$. Therefore, $\mathcal{R} \circ \mathcal{S} \subset \left(\mathcal{R} \vee \mathcal{S} \right)^{*}$, and hence by the monotonicity and the idempotency of $*$ it is clear that the required inclusion is also true.

Remark 6.4. Note that in the above theorem we may write any increasing $*$-absorbing operation $\square$ in place of $*$.

Therefore, it is also of some interest to prove the following

Theorem 6.5. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are relators on* X *and* $\square \in \{ *, \# \}$*, then*

$$\left(\mathcal{R} \circ \mathcal{S} \right)^{\square} = \left(\mathcal{R}^{\square} \circ \mathcal{S}^{\square} \right)^{\square}.$$

Hint. Since $\mathcal{R} \subset \mathcal{R}^{\#}$ and $\mathcal{S} \subset \mathcal{S}^{\#}$, we evidently have $\mathcal{R} \circ \mathcal{S} \subset \mathcal{R}^{\#} \circ \mathcal{S}^{\#}$. And hence, by the monotonicity of $\#$, it is clear that $\left(\mathcal{R} \circ \mathcal{S} \right)^{\#} \subset \left(\mathcal{R}^{\#} \circ \mathcal{S}^{\#} \right)^{\#}$.

On the other hand, if $W \in \left(\mathcal{R}^{\#} \circ \mathcal{S}^{\#} \right)^{\#}$, then for each $A \subset X$ there exist $U \in \mathcal{R}^{\#}$ and $V \in \mathcal{S}^{\#}$ such that $(U \circ V)(A) \subset W(A)$. Moreover, there exists $S \in \mathcal{S}$ such that $S(A) \subset V(A)$, and there exists $R \in \mathcal{R}$ such that $R\big(S(A)\big) \subset U\big(S(A)\big)$. Hence, it is clear that

$$(R \circ S)(A) = R\big(S(A)\big) \subset U\big(S(A)\big) \subset U\big(V(A)\big) = (U \circ V)(A) \subset W(A).$$

Therefore, $W \in \big(\mathcal{R} \circ \mathcal{S}\big)^{\#}$, and thus $\big(\mathcal{R}^{\#} \circ \mathcal{S}^{\#}\big)^{\#} \subset \big(\mathcal{R} \circ \mathcal{S}\big)^{\#}$ also holds.

By using a similar argument, concerning the unary operation $\wedge$, we can only prove the following

Theorem 6.6. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then*

$$\big(\mathcal{R} \circ \mathcal{S}\big)^{\wedge} = \big(\mathcal{R}^{\#} \circ \mathcal{S}^{\wedge}\big)^{\wedge}$$

Hence, by writing $\mathcal{R}^{\wedge}$ in place of $\mathcal{R}$, we can immediately get

Corollary 6.7. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then*

$$\big(\mathcal{R}^{\wedge} \circ \mathcal{S}\big)^{\wedge} = \big(\mathcal{R}^{\wedge} \circ \mathcal{S}^{\wedge}\big)^{\wedge}.$$

Moreover, analogously to Theorem 6.5, we can also easily prove the following

Theorem 6.8. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X and $\Box \in \{*, \wedge\}$, then*

$$\big(\mathcal{R} \wedge \mathcal{S}\big)^{\Box} = \big(\mathcal{R}^{\Box} \wedge \mathcal{S}^{\Box}\big)^{\Box}.$$

The binary operation $\vee$ has some more satisfactory properties than $\circ$ and $\wedge$ since we have the following

Theorem 6.9. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X and $\Box \in \{*, \#, \wedge\}$, then*

$$\big(\mathcal{R} \vee \mathcal{S}\big)^{\Box} = \mathcal{R}^{\Box} \cap \mathcal{S}^{\Box}.$$

Hint. If $V \in \big(\mathcal{R} \vee \mathcal{S}\big)^{\#}$, then for each $A \subset X$ there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that $(R \cup S)(A) \subset V(A)$. Hence, since $(R \cup S)(A) = R(A) \cup S(A)$, it follows that $R(A) \subset V(A)$ and $S(A) \subset V(A)$. Therefore, $V \in \mathcal{R}^{\#}$ and $V \in \mathcal{S}^{\#}$, and hence $V \in \mathcal{R}^{\#} \cap \mathcal{S}^{\#}$.

On the other hand, if $V \in \mathcal{R}^{\#} \cap \mathcal{S}^{\#}$, then $V \in \mathcal{R}^{\#}$ and $V \in \mathcal{S}^{\#}$. Therefore, for each $A \subset X$, there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that $R(A) \subset V(A)$ and $S(A) \subset V(A)$. Hence, it follows that $(R \cup S)(A) = R(A) \cup S(A) \subset V(A)$, and thus $V \in (\mathcal{R} \vee \mathcal{S})^{\#}$.

Now, as a close analogue of Theorem 6.5, we can also easily establish

Corollary 6.10. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X and $\Box \in \{*, \#, \wedge\}$, then*

$$\big(\mathcal{R} \vee \mathcal{S}\big)^{\Box} = \big(\mathcal{R}^{\Box} \vee \mathcal{S}^{\Box}\big)^{\Box} \qquad \textit{and} \qquad \mathcal{R}^{\Box} \cap \mathcal{S}^{\Box} = \big(\mathcal{R}^{\Box} \cap \mathcal{S}^{\Box}\big)^{\Box}.$$

Proof. By using Theorem 6.9, we can see that

$$\big(\mathcal{R} \vee \mathcal{S}\big)^{\Box} = \mathcal{R}^{\Box} \cap \mathcal{S}^{\Box} = \mathcal{R}^{\Box\Box} \cap \mathcal{S}^{\Box\Box} = \big(\mathcal{R}^{\Box} \vee \mathcal{S}^{\Box}\big)^{\Box}$$

and $\mathcal{R}^{\Box} \cap \mathcal{S}^{\Box} = \big(\mathcal{R} \vee \mathcal{S}\big)^{\Box} = \big(\mathcal{R} \vee \mathcal{S}\big)^{\Box\Box} = \big(\mathcal{R}^{\Box} \cap \mathcal{S}^{\Box}\big)^{\Box}$.

In this respect, it is also worth proving the following

Theorem 6.11. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X and $\square \in \{*, \#, \wedge\}$, then the following assertions are equivalent:*

(1) $\mathcal{R} \vee \mathcal{S} \subset (\mathcal{R} \cap \mathcal{S})^{\square}$; *(2)* $(\mathcal{R} \vee \mathcal{S})^{\square} = (\mathcal{R} \cap \mathcal{S})^{\square}$;

(3) $\mathcal{R}^{\square} \cap \mathcal{S}^{\square} \subset (\mathcal{R} \cap \mathcal{S})^{\square}$; *(4)* $\mathcal{R}^{\square} \cap \mathcal{S}^{\square} = (\mathcal{R} \cap \mathcal{S})^{\square}$.

Proof. By the self-increasingness of $\square$, it is clear the assertion (1) is equivalent to the inclusion $(\mathcal{R} \vee \mathcal{S})^{\square} \subset (\mathcal{R} \cap \mathcal{S})^{\square}$. Moreover, from Remark 6.2 we can at once see that the converse inclusion is always true. Therefore, the assertions (1) and (2) are equivalent. On the other hand, by Theorem 6.9, it is clear that the equivalences (1) $\Longleftrightarrow$ (3) and (2) $\Longleftrightarrow$ (4) are also true.

The importance of the binary operation $\vee$ lies mainly in the following

Theorem 6.12. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then*

$$\operatorname{Int}_{\mathcal{R} \vee \mathcal{S}} = \operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}} \qquad \textit{and} \qquad \operatorname{Cl}_{\mathcal{R} \vee \mathcal{S}} = \operatorname{Cl}_{\mathcal{R}} \cup \operatorname{Cl}_{\mathcal{S}}.$$

Proof. If $A \subset X$ and $B \in \operatorname{Int}_{\mathcal{R} \vee \mathcal{S}}(A)$, then there exist $R \in \mathcal{R}$ and $S \in \mathcal{S}$ such that $(R \cup S)(B) \subset A$. Hence, since $(R \cup S)(B) = R(B) \cup S(B)$, it follows that $R(B) \subset A$ and $S(B) \subset A$. Therefore, $B \in \operatorname{Int}_{\mathcal{R}}(A)$ and $B \in \operatorname{Int}_{\mathcal{S}}(A)$, and hence $B \in \operatorname{Int}_{\mathcal{R}}(A) \cap \operatorname{Int}_{\mathcal{S}}(A) = \big(\operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}}\big)(A)$. Therefore, $\operatorname{Int}_{\mathcal{R} \vee \mathcal{S}} \subset \operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}}$.

The converse inclusion can be proved quite similarly. Moreover, the second assertion of the theorem can be derived from the first one by using Theorem 3.1.

Now, as an immediate consequence of Theorem 6.12, we can also state

Corollary 6.13. *If $\mathcal{R}$ is a relator on X, then $\tau_{\mathcal{R} \vee \mathcal{S}} = \tau_{\mathcal{R}} \cap \tau_{\mathcal{S}}$, and thus $\mathcal{T}_{\mathcal{R} \vee \mathcal{S}} = \mathcal{T}_{\mathcal{R}} \cap \mathcal{T}_{\mathcal{S}}$.*

Moreover, combining Theorems 6.11 and 6.12, we can also easily establish the following

Theorem 6.14. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the following assertions are equivalent:*

(1) $\mathcal{R} \vee \mathcal{S} \subset (\mathcal{R} \cap \mathcal{S})^{\#}$; *(2)* $\mathcal{R}^{\#} \cap \mathcal{S}^{\#} \subset (\mathcal{R} \cap \mathcal{S})^{\#}$

(3) $\operatorname{Int}_{\mathcal{R} \cap \mathcal{S}} = \operatorname{Int}_{\mathcal{R}} \cap \operatorname{Int}_{\mathcal{S}}$; *(4)* $\operatorname{Cl}_{\mathcal{R} \cap \mathcal{S}} = \operatorname{Cl}_{\mathcal{R}} \cup \operatorname{Cl}_{\mathcal{S}}$.

Proof. Note that if the assertion (1) holds, then by Theorem 6.11 we also have $(\mathcal{R} \cap \mathcal{S})^{\#} = (\mathcal{R} \vee \mathcal{S})^{\#}$. Hence, by Theorem 4.3 (2), it follows that $\operatorname{Int}_{\mathcal{R} \cap \mathcal{S}} = \operatorname{Int}_{\mathcal{R} \vee \mathcal{S}}$. Therefore, by Theorem 6.12, the assertion (3) also holds.

On the other hand, if the assertion (3) holds, then by Theorem 6.12 we also have $\operatorname{Int}_{\mathcal{R} \cap \mathcal{S}} = \operatorname{Int}_{\mathcal{R} \vee \mathcal{S}}$. Hence, again by Theorem 4.3 (2), it follows that $(\mathcal{R} \cap \mathcal{S})^{\#} = (\mathcal{R} \vee \mathcal{S})^{\#}$. Therefore, in particular, the assertion (1) also holds.

Finally, to complete the proof, we note that the equivalences (1) $\Longleftrightarrow$ (2) and (3) $\Longleftrightarrow$ (4) are immediate from Theorems 6.11 and 3.1, respectively.

7. Some further important binary operations for relators

Definition 7.1. If $\mathcal{R}=\{R_i\}_{i\in I}$ and $\mathcal{S}=\{S_i\}_{i\in I}$ are relators on X, then by trusting to the reader's good sense to avoid confusions we also define

$$\mathcal{R}\circledcirc\mathcal{S}=\{R_i\circ S_i:\quad i\in I\},\qquad \mathcal{R}\boxdot\mathcal{S}=\{R_i\Box S_i:\quad i\in I\},$$

$$\mathcal{R}\bigtriangleup\mathcal{S}=\{R_i\cap S_i:\quad i\in I\},\qquad \mathcal{R}\bigtriangledown\mathcal{S}=\{R_i\cup S_i:\quad i\in I\}.$$

Remark 7.2. Note that thus, in particular, we have $\mathcal{R}\circledcirc\mathcal{S}\subset\mathcal{R}\circ\mathcal{S}$ and $\mathcal{R}\bigtriangledown\mathcal{S}\subset\mathcal{R}\vee\mathcal{S}$.

Moreover, if $\mathcal{R}$ is a relator on X, then by considering $\mathcal{R}=\{R\}_{R\in\mathcal{R}}$ and $\mathcal{R}^{-1}=\{R^{-1}\}_{R\in\mathcal{R}}$ we have

$$\mathcal{R}\circledcirc\mathcal{R}^{-1}=\{R\circ R^{-1}:\quad R\in\mathcal{R}\}\quad\text{and}\quad\mathcal{R}\bigtriangledown\mathcal{R}^{-1}=\{R\cup R^{-1}:\quad R\in\mathcal{R}\}.$$

Concerning the latter relators, we can easily prove the following

Theorem 7.3. *If $\mathcal{R}$ is a uniformly filtered relator on X, then*

$$\big(\mathcal{R}\circledcirc\mathcal{R}^{-1}\big)^*=\big(\mathcal{R}\circ\mathcal{R}^{-1}\big)^*\qquad\textit{and}\qquad\big(\mathcal{R}\bigtriangledown\mathcal{R}^{-1}\big)^*=\big(\mathcal{R}\vee\mathcal{R}^{-1}\big)^*.$$

Hint. In this case, for any $R,S\in\mathcal{R}$ there exists a $T\in\mathcal{R}$ such that $T\subset R\cap S$. Hence, it follows that $T\subset R$ and $T^{-1}\subset S^{-1}$, and thus $T\circ T^{-1}\subset R\circ S^{-1}$. Therefore, $\mathcal{R}\circ\mathcal{R}^{-1}\subset\big(\mathcal{R}\circledcirc\mathcal{R}^{-1}\big)^*$, and hence $\big(\mathcal{R}\circ\mathcal{R}^{-1}\big)^*\subset\big(\mathcal{R}\circledcirc\mathcal{R}^{-1}\big)^*$. Moreover, since $\mathcal{R}\circledcirc\mathcal{R}^{-1}\subset\mathcal{R}\circ\mathcal{R}^{-1}$, it is clear that the converse inclusion is always true.

Now, by writing $\mathcal{R}^{-1}$ in place of $\mathcal{R}$ in the assertions of Theorem 7.3, we can immediately get

Corollary 7.4. *If $\mathcal{R}$ is a uniformly filtered relator on X, then*

$$\big(\mathcal{R}^{-1}\circledcirc\mathcal{R}\big)^*=\big(\mathcal{R}^{-1}\circ\mathcal{R}\big)^*\qquad\textit{and}\qquad\big(\mathcal{R}^{-1}\bigtriangledown\mathcal{R}\big)^*=\big(\mathcal{R}^{-1}\vee\mathcal{R}\big)^*.$$

Moreover, in addition to Theorem 7.3, we can also easily establish the following

Theorem 7.5. *If $\mathcal{R}$ is a reflexive relator on X, then*

$$\big(\mathcal{R}\circledcirc\mathcal{R}^{-1}\big)^*\subset\big(\mathcal{R}\bigtriangledown\mathcal{R}^{-1}\big)^*\qquad\textit{and}\qquad\big(\mathcal{R}\circ\mathcal{R}^{-1}\big)^*\subset\big(\mathcal{R}\vee\mathcal{R}^{-1}\big)^*.$$

Hence, it is clear that we also have

Corollary 7.6. *If $\mathcal{R}$ is a reflexive relator on X, then*

$$\big(\mathcal{R}^{-1}\circledcirc\mathcal{R}\big)^*\subset\big(\mathcal{R}\bigtriangledown\mathcal{R}^{-1}\big)^*\qquad\textit{and}\qquad\big(\mathcal{R}^{-1}\circ\mathcal{R}\big)^*\subset\big(\mathcal{R}\vee\mathcal{R}^{-1}\big)^*.$$

Remark 7.7. Note that in Theorems 7.3 and 7.5 and their corollaries, we may again write any increasing $*$-absorbing operation $\Box$ in place of $*$.

In addition to Theorem 6.12, it is also worth proving the following

Theorem 7.8. *If* $\mathfrak{F}$ *is a normal increasing (decreasing) set-valued function for relators on* X *, and moreover* $\mathcal{R}$ *,* $\mathcal{S}$ *and* $\mathcal{U}$ *are relators on* X *, then the following assertions are equivalent:*

(1) $\mathfrak{F}_{\mathcal{U}} = \mathfrak{F}_{\mathcal{R}} \cup \mathfrak{F}_{\mathcal{S}} \quad \big(\mathfrak{F}_{\mathcal{U}} = \mathfrak{F}_{\mathcal{R}} \cap \mathfrak{F}_{\mathcal{S}}\big)$ *;*

(2) $\mathcal{U}^{\Box_{\mathfrak{F}}} = \big(\mathcal{R} \cup \mathcal{S}\big)^{\Box_{\mathfrak{F}}}$ *;* *(3)* $\mathcal{U}^{\Box_{\mathfrak{F}}} = \Big(\mathcal{R}^{\Box_{\mathfrak{F}}} \cup \mathcal{S}^{\Box_{\mathfrak{F}}}\Big)^{\Box_{\mathfrak{F}}}$ *;*

(4) $\mathcal{U} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \iff \mathcal{R} \cup \mathcal{S} \subset \mathcal{V}^{\Box_{\mathfrak{F}}}$ *for every relator* $\mathcal{V}$ *on* X*.*

Proof. Define $\mathcal{W} = \mathcal{R} \cup \mathcal{S}$. Then, by the increasingness and the normality of $\mathfrak{F}$ it is clear that $\mathfrak{F}_{\mathcal{W}} = \mathfrak{F}_{\mathcal{R}} \cup \mathfrak{F}_{\mathcal{S}}$. Moreover, by Theorems 2.5 and 2.3, it is clear that $\mathfrak{F}_{\mathcal{U}} = \mathfrak{F}_{\mathcal{W}}$ if and only if $\mathcal{U}^{\Box_{\mathfrak{F}}} = \mathcal{W}^{\Box_{\mathfrak{F}}}$. Therefore, the first part of the assertion (1) is equivalent to the assertion (2). Hence, since $\mathfrak{F}_{\mathcal{R}} = \mathfrak{F}_{\mathcal{R}^{\Box_{\mathfrak{F}}}}$ and $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{S}^{\Box_{\mathfrak{F}}}}$, is is clear that the assertions (2) and (3) are also equivalent.

On the other hand, if the assertion (2) holds and $\mathcal{V}$ is a relator on X, then by using Theorem 2.2 we can easily see that

$$\mathcal{U} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \iff \mathcal{U}^{\Box_{\mathfrak{F}}} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \iff \mathcal{W}^{\Box_{\mathfrak{F}}} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \iff \mathcal{W} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} .$$

Therefore, the assertion (4) also holds. Finally, if the assertion (4) holds, then by putting $\mathcal{U}$ and $\mathcal{W}$ in place of $\mathcal{V}$ in the inclusions of (4) we can immediately see that $\mathcal{W} \subset \mathcal{U}^{\Box_{\mathfrak{F}}}$ and $\mathcal{U} \subset \mathcal{W}^{\Box_{\mathfrak{F}}}$, and hence $\mathcal{U}^{\Box_{\mathfrak{F}}} = \mathcal{W}^{\Box_{\mathfrak{F}}}$. Therefore, the assertion (2) also holds.

Remark 7.9. From Theorem 7.8, by using Corollary 2.6, we can at once see that if $\Diamond$ is a refinement operation for relators on X, and moreover $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then

$$\big(\mathcal{R} \cup \mathcal{S}\big)^{\Diamond} = \big(\mathcal{R}^{\Diamond} \cup \mathcal{S}^{\Diamond}\big)^{\Diamond}.$$

Unfortunately, most of the basic unary operations for relators are not normal. However, for instance, it can be easily seen that the refinement operations $*$ and $\blacklozenge$ and the modification operations ∞ and ∂ are normal.

As an immediate consequence of Theorem 7.8 we can also state the following

Corollary 7.10. *If* $\mathfrak{F}$ *is a normal increasing (decreasing) set-valued function for relators on* X *, and moreover* $\mathcal{R}$ *and* $\mathcal{S}$ *are relators on* X*, then the following assertions are equivalent:*

(1) $\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}} \cup \mathfrak{F}_{\mathcal{R}^{-1}} \quad \big(\mathfrak{F}_{\mathcal{S}} = \mathfrak{F}_{\mathcal{R}} \cap \mathfrak{F}_{\mathcal{R}^{-1}}\big)$ *;*

(2) $\mathcal{S}^{\Box_{\mathfrak{F}}} = \big(\mathcal{R} \cup \mathcal{R}^{-1}\big)^{\Box_{\mathfrak{F}}}$ *;* *(3)* $\mathcal{S}^{\Box_{\mathfrak{F}}} = \Big(\mathcal{R}^{\Box_{\mathfrak{F}}} \cup \big(\mathcal{R}^{-1}\big)^{\Box_{\mathfrak{F}}}\Big)^{\Box_{\mathfrak{F}}}$ *;*

(4) $\mathcal{S} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \iff \mathcal{R} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \cap \big(\mathcal{V}^{\Box_{\mathfrak{F}}}\big)^{-1}$ *for every relator* $\mathcal{V}$ *on* X*.*

Proof. By Theorem 7.8, for every relator $\mathcal{V}$ on X, we have

$$\begin{aligned} \mathcal{S} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \iff \mathcal{R} \cup \mathcal{R}^{-1} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \iff \mathcal{R} \subset \mathcal{V}^{\Box_{\mathfrak{F}}}, \quad \mathcal{R}^{-1} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \iff \\ \mathcal{R} \subset \mathcal{V}^{\Box_{\mathfrak{F}}}, \quad \mathcal{R} \subset \big(\mathcal{V}^{\Box_{\mathfrak{F}}}\big)^{-1} \iff \mathcal{R} \subset \mathcal{V}^{\Box_{\mathfrak{F}}} \cap \big(\mathcal{V}^{\Box_{\mathfrak{F}}}\big)^{-1}. \end{aligned}$$

From Theorem 7.8, by Theorem 3.16, it is clear that in particular we also have the following

Theorem 7.11. *If $\mathcal{R}$, $\mathcal{S}$ and $\mathcal{U}$ are relators on X, then the following assertions are equivalent:*

(1) $\operatorname{Int}_{\mathcal{U}} = \operatorname{Int}_{\mathcal{R}} \cup \operatorname{Int}_{\mathcal{S}} \quad \big(\operatorname{Cl}_{\mathcal{U}} = \operatorname{Cl}_{\mathcal{R}} \cap \operatorname{Cl}_{\mathcal{S}} \big)$;

(2) $\mathcal{U}^{\#} = \big(\mathcal{R} \cup \mathcal{S} \big)^{\#}$; *(3)* $\mathcal{U}^{\#} = \big(\mathcal{R}^{\#} \cup \mathcal{S}^{\#} \big)^{\#}$;

(4) $\mathcal{U} \subset \mathcal{V}^{\#} \iff \mathcal{R} \cup \mathcal{S} \subset \mathcal{V}^{\#}$ *for every relator* $\mathcal{V}$ *on* X.

Hence, it is clear that in particular we also have

Corollary 7.12. *If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the following assertions are equivalent:*

(1) $\operatorname{Int}_{\mathcal{S}} = \operatorname{Int}_{\mathcal{R}} \cup \operatorname{Int}_{\mathcal{R}^{-1}} \quad \big(\operatorname{Cl}_{\mathcal{S}} = \operatorname{Cl}_{\mathcal{R}} \cap \operatorname{Cl}_{\mathcal{R}^{-1}} \big)$;

(2) $\mathcal{S}^{\#} = \big(\mathcal{R} \cup \mathcal{R}^{-1} \big)^{\#}$; *(3)* $\mathcal{S}^{\#} = \Big(\mathcal{R}^{\#} \cup \big(\mathcal{R}^{-1} \big)^{\#} \Big)^{\#}$;

(4) $\mathcal{S} \subset \mathcal{V}^{\#} \iff \mathcal{R} \subset \mathcal{V}^{\#} \cap \big(\mathcal{V}^{\#} \big)^{-1}$ *for every relator* $\mathcal{V}$ *on* X.

Remark 7.13. Note that, because of Theorem 6.12, we may naturally write $\mathcal{R} \wedge \mathcal{R}^{-1}$ or $\mathcal{R} \vartriangle \mathcal{R}^{-1}$ in place of $\mathcal{S}$ in Corollary 7.12.

Moreover, it is also worth noticing that some of the results of Sections 6 and 7 can be naturally extended to arbitrary families of relators.

8. Symmetric, transitive, filtered and compact relators

Definition 8.1. A relator $\mathcal{R}$ on X is called weakly (strongly) symmetric if $\rho_{\mathcal{R}}$ (each member of $\mathcal{R}$) is a symmetric relation.

Moreover, the relator $\mathcal{R}$ is called properly symmetric if $\mathcal{R} = \mathcal{R}^{-1}$. And if $\square$ is a unary operation for relators on X, then $\mathcal{R}$ is called $\square$–symmetric if the relator $\mathcal{R}^{\square}$ is properly symmetric.

Remark 8.2. We note that the relator $\mathcal{R}$ is properly symmetric if and only if $\mathcal{R} \subset \mathcal{R}^{-1}$, or equivalently $\mathcal{R}^{-1} \subset \mathcal{R}$.

Moreover, the relator $\mathcal{R}$ is, for instance, to be called proximally (quasi-proximally) symmetric if it is $\#$–symmetric ($\#\infty$–symmetric).

Concerning the latter notions, we shall only quote here the following two theorems which have been mostly established in [60].

Theorem 8.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi–proximally symmetric;*

(2) $\big(\mathcal{R}^{\infty} \big)^{-1} \subset \mathcal{R}^{\#}$; *(3)* $\big(\mathcal{R}^{\#\infty} \big)^{-1} \subset \mathcal{R}^{\infty\#}$;

(4) $\mathcal{R}$ *is* $\sharp$*–symmetric;* *(5)* $\mathcal{R}$ *is* $\infty\#$*–symmetric.*

Remark 8.4. In addition to this theorem, we can also state that the relator $\mathcal{R}$ is quasi-proximally symmetric if and only if the relator $\mathcal{R}^{\infty}$ is proximally symmetric.

Moreover, by calling a relator quasi-properly (pseudo-properly) symmetric if it is ∞–symmetric (∂–symmetric), we can also state that $\mathcal{R}$ is quasi-proximally symmetric if and only if $\mathcal{R}^{\#}$ is quasi-properly (pseudo-properly) symmetric.

Theorem 8.5. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi–proximally symmetric;*

(2) $\mathcal{R}$ *and* $\mathcal{R}^{-1}$ *are quasi–proximally equivalent;*

(3) $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^{-1}}$ $\left({\tau\!\!\!-}_{\mathcal{R}} = {\tau\!\!\!-}_{\mathcal{R}^{-1}} \right)$; *(4)* $\tau_{\mathcal{R}} = {\tau\!\!\!-}_{\mathcal{R}}$.

Remark 8.6. Note that, in contrast to the proximal symmetry, the topological symmetry is already a rather restrictive property.

Namely, by Theorem 4.5, a relator $\mathcal{R}$ is topologically symmetric if and only if $\mathcal{R}^{\wedge} = \{ \rho_{\mathcal{R}} \}^{\wedge}$, that is, $\mathcal{R}$ is topologically simple and weakly symmetric.

Therefore, a relator $\mathcal{R}$ has, in addition, to be called topologically semi-symmetric if $\mathcal{R}^{-1} \subset \mathcal{R}^{\wedge}$, or equivalently $\left(\mathcal{R}^{-1}\right)^{\wedge} \subset \mathcal{R}^{\wedge}$.

The importance of the binary operation $\circ$ lies mainly in the following

Definition 8.7. A relator $\mathcal{R}$ on X is called weakly (strongly) transitive if $\rho_{\mathcal{R}}$ (each member of $\mathcal{R}$) is transitive.

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ is called $\square$–transitive (strictly $\square$–transitive) if $\mathcal{R}^{\square\square} \subset \left(\mathcal{R}^{\square} \circ \mathcal{R}^{\square} \right)^{\square}$ $\left(\mathcal{R}^{\square\square} \subset \left(\mathcal{R}^{\square} \odot \mathcal{R}^{\square} \right)^{\square} \right)$.

Remark 8.8. Thus, the relator $\mathcal{R}$ is, for instance, to be called topologically transitive if $\mathcal{R}^{\wedge} \subset \left(\mathcal{R}^{\wedge} \circ \mathcal{R}^{\wedge} \right)^{\wedge}$.

Moreover, the relator $\mathcal{R}$ may, for instance, be called strongly topologically transitive if $\mathcal{R}^{\wedge} \subset \left(\mathcal{R}^{\#} \circ \mathcal{R}^{\wedge} \right)^{\wedge}$.

By using Theorems 2.2 and 6.6 and Corollary 6.7, we can easily establish the following

Theorem 8.9. *If $\mathcal{R}$ is a relator on X, then*

(1) $\mathcal{R}$ *is topologically transitive if and only if* $\mathcal{R} \subset \left(\mathcal{R}^{\wedge} \circ \mathcal{R} \right)^{\wedge}$;

(2) $\mathcal{R}$ *is strongly topologically transitive if and only if* $\mathcal{R} \subset \left(\mathcal{R} \circ \mathcal{R} \right)^{\wedge}$.

Remark 8.10. If $\mathcal{R}$ is a reflexive relator on X and $\square \in \{ *, \#, \wedge \}$, then by Theorems 6.3 and 6.9, we have $\left(\mathcal{R} \circ \mathcal{R} \right)^{\square} \subset \left(\mathcal{R} \vee \mathcal{R} \right)^{\square} = \mathcal{R}^{\square} \cap \mathcal{R}^{\square} = \mathcal{R}^{\square}$.

Therefore, in addition to Theorem 8.9, we can also state the following

Theorem 8.11. *If $\mathcal{R}$ is a reflexive relator on X, then*

(1) $\mathcal{R}$ *is topologically transitive if and only if* $\mathcal{R}^{\wedge} = \left(\mathcal{R}^{\wedge} \circ \mathcal{R} \right)^{\wedge}$;

(2) $\mathcal{R}$ *is strongly topologically transitive if and only if* $\mathcal{R}^{\wedge} = \left(\mathcal{R} \circ \mathcal{R} \right)^{\wedge}$.

Moreover, to let the reader feel the appropriateness of the above concepts, we can also state the following theorem of [60].

Theorem 8.12. *If $\mathcal{R}$ is a reflexive relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is quasi–topological;* *(2)* $\mathcal{R}$ *is topologically transitive;*

(3) $\mathcal{R}^{\wedge}$ *is quasi–topological;* *(4)* $\mathcal{R}^{\wedge}$ *is strictly proximally transitive.*

Remark 8.13. In [60], it was also proved that a relator $\mathcal{R}$ is topological if and only if the relator $\mathcal{R}^{\wedge}$ is proximal.

The importance of the binary operation $\wedge$ lies mainly in the following

Definition 8.14. A relator $\mathcal{R}$ on X is called properly filtered if $\mathcal{R} = \mathcal{R} \wedge \mathcal{R}$.

Moreover, if $\square$ is a unary operation on relators on X, then the relator $\mathcal{R}$ called $\square$–filtered if the relator $\mathcal{R}^{\square}$ is properly filtered.

Remark 8.15. Note that, by Remark 6.2, we always have $\mathcal{R} \subset \mathcal{R} \wedge \mathcal{R}$. Therefore, the relator $\mathcal{R}$ is properly filtered if and only if $\mathcal{R} \wedge \mathcal{R} \subset \mathcal{R}$.

Moreover, the relator $\mathcal{R}$ is, for instance, to be called uniformly, proximally and topologically filtered if it is $\square$–filtered with $\square = *$, $\#$ and $\wedge$, respectively.

Concerning the latter notions, we shall only quote here the following two theorems of [60].

Theorem 8.16. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is uniformly filtered;* *(2)* $\mathcal{R} \wedge \mathcal{R} \subset \mathcal{R}^{*}$; *(3)* $\mathcal{R}^{*} = \big(\mathcal{R} \wedge \mathcal{R}\big)^{*}$.

Theorem 8.17. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is topologically filtered;* *(2)* $\mathcal{R} \wedge \mathcal{R} \subset \mathcal{R}^{\wedge}$; *(3)* $\mathcal{R}^{\wedge} = \big(\mathcal{R} \wedge \mathcal{R}\big)^{\wedge}$.

Remark 8.18. Unfortunately, an analogue of the above theorems fails to hold for the proximal filteredness taken in the sense of Definition 8.14.

Therefore, a relator $\mathcal{R}$ has, in addition, to be called weakly proximally filtered if $\mathcal{R} \wedge \mathcal{R} \subset \mathcal{R}^{\#}$, or equivalently $\mathcal{R}^{\#} = \big(\mathcal{R} \wedge \mathcal{R}\big)^{\#}$ $\big(\operatorname{Int}_{\mathcal{R}} = \operatorname{Int}_{\mathcal{R} \wedge \mathcal{R}}\big)$.

Moreover, a relator $\mathcal{R}$ has to be called properly proximally filtered if for any $A \subset X$ and $R, S \in \mathcal{R}$ there exists $T \in \mathcal{R}$ such that $T(A) \subset R(A) \cap S(A)$, or equivalently $\operatorname{Int}_{\mathcal{R}}(A \cap B) = \operatorname{Int}_{\mathcal{R}}(A) \cap \operatorname{Int}_{\mathcal{R}}(B)$ for all $A, B \subset X$.

The importance of the various unary operations for relators can also be well illustrated by the following

Definition 8.19. A relator $\mathcal{R}$ on X is called properly compact if for each $R \in \mathcal{R}$ there exists a finite subset A of X such that $X = R(A)$.

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ is called $\square$–compact if the relator $\mathcal{R}^{\square}$ is properly compact.

Remark 8.20. In particular, the relator $\mathcal{R}$ is called topologically compact (quasi-topologically compact) if it is $\wedge$–compact ($\wedge\infty$–compact).

The appropriateness of the above definitions is apparent from the following two theorems of [69].

Theorem 8.21. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly compact.

(2) each directed net in $X(\mathcal{R})$ is adherence Cauchy;

(3) each directed universal net in $X(\mathcal{R})$ is convergence Cauchy.

Remark 8.22. A net x in a relator space $X(\mathcal{R})$ is called convergence (adherence) Cauchy if it is convergent (adherent) in each of the spaces $X(R)$, where $R \in \mathcal{R}$.

In [63], it was proved that a net x in the relator space $X(\mathcal{R})$ is convergent (adherent) if and only if it is convergence (adherence) Cauchy in the space $X(\mathcal{R}^{\wedge})$.

Therefore, the following theorem is actually an immediate consequence of the above theorem.

Theorem 8.23. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically compact.

(2) each directed net in $X(\mathcal{R})$ is adherent;

(3) each directed universal net in $X(\mathcal{R})$ is convergent.

Hence, by using Theorems 3.14 and Theorem 5.6, it can be easily shown that the following more familiar theorem is also true.

Theorem 8.24. *If $\mathcal{R}$ is a topological relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically compact;

(2) each open cover of $X(\mathcal{R})$ has a finite subcover;

(3) each centered family of closed subsets of $X(\mathcal{R})$ has a nonvoid intersection.

The topologically compact relators are closely related to the Lebesgue ones [59].

Definition 8.25. If $\mathcal{R}$ and $\mathcal{S}$ are relators on X, then the relator $\mathcal{S}$ is said to be properly refined by $\mathcal{R}$ if for each $S \in \mathcal{S}$ there exists a function f on X to X such that $S \circ f \in \mathcal{R}$.

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{S}$ is said to be $\square$–refined by $\mathcal{R}$ if it is properly refined by $\mathcal{R}^{\square}$.

Remark 8.26. In particular, the relator $\mathcal{S}$ is said to be uniformly refined by $\mathcal{R}$ if it is $*$–refined by $\mathcal{R}$.

Moreover, the relator $\mathcal{R}$ is called a Lebesgue relator if its topological refinenement $\mathcal{R}^{\wedge}$ is uniformly refined by $\mathcal{R}$.

The appropriateness of the latter defintions is apparent from the following generalization of Lebesgue's covering theorem [59].

Theorem 8.27. *If $\mathcal{R}$ is a uniformly filtered, strongly topologically transitive and topologically compact relator on X, then $\mathcal{R}$ is a Lebesgue relator.*

Remark 8.28. The importance of Lebesgue relators lies mainly in the fact that they are both convergence and adherence complete.

Namely, the topologically compact relators are, in general, only directedly convergence-adherence complete in the sense that each directed convergence Cauchy net in $X(\mathcal{R})$ is adherent.

9. Mild continuities of relations in relator spaces

Definition 9.1. If F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square$ is a unary operation for relators on X, then the relation F is said to be $\square$–continuous, or more precisely mildly $\square$–continuous [71] if

$$\left(F^{-1} \circ \mathcal{S}^{\square} \circ F \right)^{\square} \subset \mathcal{R}^{\square\square}.$$

Remark 9.2. Now, the relation F may be naturally called properly continuous if it is $\square$–continuous with $\square$ being the identity operation for relators.

Moreover, the relation F may, for instance, be called uniformly and paratopologically continuous if it is $\square$–continuous with $\square = *$ and $\square = \vartriangle$, respectively.

By Theorem 2.2, we evidently have the following specialization of Definition 9.1.

Theorem 9.3. *If F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square$ is refinement operation for relators on X, then the following assertions are equivalent:*

(1) F is $\square$–continuous; *(2) $F^{-1} \circ \mathcal{S}^{\square} \circ F \subset \mathcal{R}^{\square}$.*

Remark 9.4. Therefore, in this case, F is $\square$–continuous if and only if F is a properly continuous as a relation on $X(\mathcal{R}^{\square})$ to $Y(\mathcal{S}^{\square})$.

Moreover, as some further specializations of Definition 9.1, we can also prove the following theorems.

Theorem 9.5. *If F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$ and $\square \in \{ *, \# \}$, then the following assertions are equivalent:*

(1) F is $\square$–continuous; *(2) $F^{-1} \circ \mathcal{S} \circ F \subset \mathcal{R}^{\square}$.*

Hint. If $V \in \mathcal{S}^{\#}$, then for each $A \subset X$ there exists an $S \in \mathcal{S}$ such that $S\big(F(A)\big) \subset V\big(F(A)\big)$. Hence, $F^{-1}\big(S\big(F(A)\big)\big) \subset F^{-1}\big(V\big(F(A)\big)\big)$, and thus $\big(F^{-1} \circ S \circ F\big)(A) \subset \big(F^{-1} \circ V \circ F\big)(A)$. Moreover, if the assertion (2) holds with $\square = \#$, then $F^{-1} \circ S \circ F \in \mathcal{R}^{\#}$. Therefore, there exists an $R \in \mathcal{R}$ such that $R(A) \subset \big(F^{-1} \circ S \circ F\big)(A)$. Consequently, we also have $R(A) \subset \big(F^{-1} \circ V \circ F\big)(A)$. Hence, it is clear that $F^{-1} \circ V \circ F \in \mathcal{R}^{\#}$, and thus $F^{-1} \circ \mathcal{S}^{\#} \circ F \subset \mathcal{R}^{\#}$. Thus, by Theorem 9.3, the assertion (1) also holds with $\square = \#$.

Theorem 9.6. *If* f *is a function on one relator space* $X(\mathcal{R})$ *to another* $Y(\mathcal{S})$ *and* $\Box \in \{\wedge, \bullet\}$, *then the following assertions are equivalent:*

(1) f *is* $\Box$*–continuous;* (2) $f^{-1} \circ \mathcal{S} \circ f \subset \mathcal{R}^{\Box}$.

Hint. If the assertion (2) holds with $\Box = \bullet$, then for each $S \in \mathcal{S}$ we have $f^{-1} \circ S \circ f \in \mathcal{R}^{\bullet}$. Hence, since $\mathcal{R}^{\bullet} = \{\rho_{\mathcal{R}}^{-1}\}^{*} = \{\bigcap \mathcal{R}\}^{*}$, it follows that $\bigcap \mathcal{R} \subset f^{-1} \circ S \circ f$. Therefore, we also have $\bigcap \mathcal{R} \subset \bigcap_{S \in \mathcal{S}} f^{-1} \circ S \circ f$. Hence, by using that $\bigcap_{S \in \mathcal{S}} f^{-1} \circ S \circ f = f^{-1} \circ \left(\bigcap \mathcal{S}\right) \circ f$, we can infer that $\bigcap \mathcal{R} \subset f^{-1} \circ \left(\bigcap \mathcal{S}\right) \circ f$. Therefore, we also have $f^{-1} \circ \left(\bigcap \mathcal{S}\right) \circ f \in \left(\bigcap \mathcal{R}\right)^{*}$. Hence, by Theorem 9.5, it is clear that $f^{-1} \circ \{\bigcap \mathcal{S}\}^{*} \circ f \subset \left(\bigcap \mathcal{R}\right)^{*}$. Therefore, we also have $f^{-1} \circ \mathcal{S}^{\bullet} \circ f \subset \mathcal{R}^{\bullet}$. Thus, by Theorem 9.3, the assertion (1) also holds with $\Box = \bullet$.

Theorem 9.7. *If* f *is a function on one relator space* $X(\mathcal{R})$ *onto another* $Y(\mathcal{S})$ *and* $\Box \in \{\vartriangle, \blacktriangle, \blacklozenge\}$, *then the following assertions are equivalent:*

(1) f *is* $\Box$*–continuous;* (2) $f^{-1} \circ \mathcal{S} \circ f \subset \mathcal{R}^{\Box}$.

Hint. If $\mathcal{S} = \{Y^2\}$, then $\mathcal{S}^{\blacklozenge} = \{Y^2\}$. Hence, by Theorem 9.3, it is clear that the implication (2) $\Longrightarrow$ (1) holds true with $\Box = \blacklozenge$.

While, if $\mathcal{S} \neq \{Y^2\}$, then there exists an $S \in \mathcal{S}$ such that $S \neq Y^2$. Therefore, there exist $y, z \in Y$ such that $(y, z) \notin S$, and hence $z \notin S(y)$. Moreover, since $Y = f(X)$, there exist $u, v \in X$ such that $y = f(u)$ and $z = f(v)$. Therefore, we also have $f(v) \notin S\big(f(u)\big)$. Hence, it follows that $v \notin f^{-1}\big(S\big(f(u)\big)\big)$, and thus $v \notin \big(f^{-1} \circ S \circ f\big)(u)$. Therefore, $(u, v) \notin f^{-1} \circ S \circ f$, and thus $f^{-1} \circ S \circ f \neq X^2$. On the other hand, if the assertion (2) holds with $\Box = \blacklozenge$, then $f^{-1} \circ S \circ f \in \mathcal{R}^{\blacklozenge}$. Therefore, we necessarily have $\mathcal{R}^{\blacklozenge} = \mathcal{P}\big(X^2\big)$. And thus, the assertion (1) also holds with $\Box = \blacklozenge$.

Remark 9.8. Note that if, for instance, $X = \{0, 1\}$ and $f = X \times \{0\}$, then $f^{-1} \circ \Delta_X \circ f = X^2 \in \{\Delta_X\}^{\vartriangle}$, but f is not a $\vartriangle$–continuous function of $X\big(\Delta_X\big)$ into itself. Namely, if $V = X^2 \setminus \Delta_X$, then $V \in \{\Delta_X\}^{\vartriangle}$, but $f^{-1} \circ V \circ f = \emptyset \notin \{\Delta_X\}^{\vartriangle}$.

Moreover, it is also worth noticing that an analogue of Theorem 9.7 does not hold for the operation $\bigstar$ since it is not idempotent by [45, Example 7.1].

The appropriateness of Definition 9.1 is apparent from the following particular cases of the results of [56], [60] and [71].

Theorem 9.9. *If* f *is a function on one relator space* $X(\mathcal{R})$ *to another* $Y(\mathcal{S})$, *then the following assertions are equivalent:*

(1) f *is uniformly continuous;*

(2) $x \in \operatorname{Lim}_{\mathcal{R}}(y)$ *implies* $f \circ x \in \operatorname{Lim}_{\mathcal{S}}(f \circ y)$;

(3) $x \in \operatorname{Adh}_{\mathcal{R}}(y)$ *implies* $f \circ x \in \operatorname{Adh}_{\mathcal{S}}(f \circ y)$.

Theorem 9.10. *If F is a relation on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) *F is proximally continuous;*

(2) $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ *implies* $F(A) \in \operatorname{Cl}_{\mathcal{S}}\big(F(B)\big)$*;*

(3) $F(A) \in \operatorname{Int}_{\mathcal{S}}(B)$ *implies* $A \in \operatorname{Int}_{\mathcal{R}}\big(F^{-1}(B)\big)$.

Theorem 9.11. *If f is a function on one relator space $X(\mathcal{R})$ to another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) *f is topologically continuous;*

(2) $x \in \lim_{\mathcal{R}}(y)$ *implies* $f(x) \in \lim_{\mathcal{S}}(f \circ y)$*;*

(3) $x \in \operatorname{adh}_{\mathcal{R}}(y)$ *implies* $f(x) \in \operatorname{adh}_{\mathcal{S}}(f \circ y)$.

Theorem 9.12. *If f is a function of one relator space $X(\mathcal{R})$ into another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) *f is topologically continuous;*

(2) $a \in \operatorname{cl}_{\mathcal{R}}(B)$ *implies* $f(a) \in \operatorname{cl}_{\mathcal{S}}\big(f(B)\big)$*;*

(3) $f(a) \in \operatorname{int}_{\mathcal{S}}(B)$ *implies* $a \in \operatorname{int}_{\mathcal{R}}\big(f^{-1}(B)\big)$.

Corollary 9.13. *If f is a function of an arbitrary relator space $X(\mathcal{R})$ into a topological one $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) *f is topologically continuous;*

(2) $U \in \mathcal{T}_{\mathcal{S}}$ *implies* $f^{-1}(U) \in \mathcal{T}_{\mathcal{R}}$*;* *(3)* $V \in \mathcal{F}_{\mathcal{S}}$ *implies* $f^{-1}(V) \in \mathcal{F}_{\mathcal{R}}$.

Remark 9.14. If f is a function of an arbitrary relator space $X(\mathcal{R})$ into a proximal one $Y(\mathcal{S})$, then we can also state that f is proximally continuous if and only if $U \in \tau_{\mathcal{S}}$ $\big(V \in \bar{\tau}_{\mathcal{S}}\big)$ implies $f^{-1}(U) \in \tau_{\mathcal{R}}$ $\big(f^{-1}(V) \in \bar{\tau}_{\mathcal{R}}\big)$.

Theorem 9.15. *If f is a function of one relator space $X(\mathcal{R})$ onto another $Y(\mathcal{S})$, then the following assertions are equivalent:*

(1) *f is paratopologically continuous;*

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *implies* $f(A) \in D_{\mathcal{S}}$*;* *(3)* $B \in \mathcal{E}_{\mathcal{S}}$ *implies* $f^{-1}(B) \in \mathcal{E}_{\mathcal{R}}$.

Remark 9.16. By using Theorem 3.7, it can be easily seen that the assertions (2) and (3) are equivalent for any relation f on $X(\mathcal{R})$ to $Y(\mathcal{S})$.

However, the implications $(1) \Longrightarrow (2)$ and $(2) \Longrightarrow (1)$ are not, in general, true. Therefore, it is of some importance to point out that the following theorem is true.

Theorem 9.17. *If F is a paratopologically continuous relation on a total relator space $X(\mathcal{R})$ to an arbitrary one $Y(\mathcal{S})$, then $F(A) \in \mathcal{D}_{\mathcal{S}}$ for all $A \in \mathcal{D}_{\mathcal{R}}$.*

Proof. If this not the case, then there exists $A \in \mathcal{D}_{\mathcal{R}}$ such that $F(A) \notin \mathcal{D}_{\mathcal{S}}$. Then, by Theorem 3.7, we necesarily have $Y \setminus F(A) \in \mathcal{E}_{\mathcal{S}}$. Hence, by defining $V = Y \times \big(Y \setminus F(A)\big)$ and $U = F^{-1} \circ V \circ F$, we can at once see that $V \in \mathcal{S}^{\triangle}$, and thus $U \in \mathcal{R}^{\triangle}$. Moreover, we can also at once see that

$$U(x) = F^{-1}\big(V\big(F(x)\big) = F^{-1}\big(Y \setminus F(A)\big)$$

for all $x \in X$ with $F(x) \neq \emptyset$. Therefore, if $u \in U(x)$ for some $x \in X$, then we necessarily have $F(u) \cap \big(Y \setminus F(A)\big) \neq \emptyset$. Thus, there exists $w \in F(u)$ such that $w \notin F(A)$, i.e., $w \notin F(a)$ for all $a \in A$. This shows that $U(x) \cap A = \emptyset$, and hence $A \notin \mathcal{D}_{\mathcal{R}}$, which is a contradiction. Therefore, we actually have $U = \emptyset$, and hence $\emptyset \in \mathcal{R}^{\Delta}$. Thus, by Theorem 5.3, the relator $\mathcal{R}$ cannot be total, which is again a contradiction.

Now, as an immediate consequence of Theorems 5.3 and 9.17, we can also state

Corollary 9.18. *If F is a paratopologically continuous relation on a total relator space $X(\mathcal{R})$ to an arbitrary one $Y(\mathcal{S})$, then $F(X) \in \mathcal{D}_{\mathcal{S}}$, and thus in particular $Y(\mathcal{S})$ is also total.*

Hence, it is clear that in particular we also have

Corollary 9.19. *If F is a paratopologically continuous relation on a total relator space $X(\mathcal{R})$ to an arbitrary one $Y(\mathcal{S})$ such that $F(X) \in \mathcal{F}_{\mathcal{S}}$, then $Y = F(X)$.*

Moreover, by noticing that $\mathcal{F}_{\mathcal{S}} = \mathcal{P}(Y)$ whenever $\Delta_Y \in \mathcal{S}^{\wedge}$, and moreover $\operatorname{card}(X) < \operatorname{card}(Y)$ whenever there is a function of X onto Y, we can also state

Corollary 9.20. *If $\operatorname{card}(X) < \operatorname{card}(Y)$, and $\mathcal{R}$ and $\mathcal{S}$ are relators on X and Y, respectively, such that $\mathcal{R}$ is total and $\Delta_Y \in \mathcal{S}^{\wedge}$, then there is no paratopologically continuous function of $X(\mathcal{R})$ into $Y\big(\mathcal{S}\big)$.*

10. Some basic properties of the Davis-Pervin relations

Definition 10.1. For each $A \subset X$, the relation

$$R_A = A^2 \cup (X \setminus A) \times X$$

is called the Davis–Pervin relation on X generated by A.

Remark 10.2. Namely, the relations R_A were first used by Davis [7] and Pervin [47] in their uniformization procedures of topological spaces.

In the sequel, we shall often need the following simple propositions about the inverses, complements and images of the relations R_A.

Proposition 10.3. *If $A \subset X$, then R_A is a preorder on X such that*

$$R_A^{-1} = R_{X \setminus A} \qquad \textit{and} \qquad R_A^{c} = A \times (X \setminus A).$$

Proposition 10.4. *If $A, B \subset X$, then $R_A(B) = \emptyset$ if $B = \emptyset$,*

$$R_A(B) = A \quad \textit{if} \quad \emptyset \neq B \subset A \qquad \textit{and} \qquad R_A(B) = X \quad \textit{if} \quad B \not\subset A.$$

Remark 10.5. The relations R_A are important particular cases of the relations $R_{(A,B)} = A \times B \cup (X \setminus A) \times X$ considered first by Császár [6, pp. 42] and Hunsaker and Lindgren [14] for some $A \subset B \subset X$.

Moreover, the following theorem is an important particular case of [60, Theorems 2.6.1 and 2.9.1]. However, since we are now not interested in the relations $R_{(A,B)}$, it seems appropriate to provide here a direct proof.

Theorem 10.6. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^*$*;* *(2)* $A \in \tau_{\mathcal{R}}$*.*

Proof. If the assertion (1) holds, then there exists $R \in \mathcal{R}$ such that $R \subset R_A$. Hence, it follows that $R(A) \subset R_A(A) = A$. Therefore, the assertion (2) also holds.

While, if the assertion (2) holds, then there exists $R \in \mathcal{R}$ such that $R(A) \subset A$. Hence, it follows that $R(x) \subset R(A) \subset A = R_A(x)$ for all $x \in A$. Moreover, it is clear that $R(x) \subset X = R_A(x)$ for all $x \in X \setminus A$. Therefore, $R \subset R_A$, and thus the assertion (1) also holds.

Corollary 10.7. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^*$*;* *(2)* $R_A \in \mathcal{R}^{\#}$*.*

Proof. By Theorems 10.6, 4.2 (7) and 2.8, it is clear that

$$R_A \in \mathcal{R}^* \iff A \in \tau_{\mathcal{R}} \iff A \in \tau_{\mathcal{R}^{\#}} \iff R_A \in \mathcal{R}^{\#*} \iff R_A \in \mathcal{R}^{\#}.$$

From Theorem 10.6, we can also easily get the following more particular theorems.

Theorem 10.8. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\wedge}$*;* *(2)* $A \in \mathcal{T}_{\mathcal{R}}$*.*

Proof. By Theorems 2.8 and 10.6 and Corollary 5.15, it is clear that

$$R_A \in \mathcal{R}^{\wedge} \iff R_A \in \mathcal{R}^{\wedge *} \iff A \in \tau_{\mathcal{R}^{\wedge}} \iff A \in \mathcal{T}_{\mathcal{R}}.$$

Theorem 10.9. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *such that* $A \neq \emptyset$*, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\vartriangle}$*;* *(2)* $A \in \mathcal{E}_{\mathcal{R}}$*.*

Proof. By Theorems 2.8 and 10.6 and Corollary 5.17, it is clear that

$$R_A \in \mathcal{R}^{\vartriangle} \iff R_A \in \mathcal{R}^{\vartriangle *} \iff A \in \tau_{\mathcal{R}^{\vartriangle}} \iff A \in \mathcal{E}_{\mathcal{R}}.$$

Theorem 10.10. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\bullet}$*;* *(2)* $A \cap \rho_{\mathcal{R}}(X \setminus A) = \emptyset$*.*

Proof. By Theorems 2.8 and 10.6 and Corollary 5.19, it is clear that

$$R_A \in \mathcal{R}^{\bullet} \iff R_A \in \mathcal{R}^{\bullet *} \iff A \in \tau_{\mathcal{R}^{\bullet}} \iff A \cap \rho_{\mathcal{R}}(X \setminus A) = \emptyset.$$

Theorem 10.11. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $A \neq \emptyset$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\bigstar}$; *(2)* $X \neq \rho_{\mathcal{R}}(X \setminus A)$.

Proof. By Theorems 2.8 and 10.6 and Corollary 5.21, it is clear that

$$R_A \in \mathcal{R}^{\bigstar} \iff R_A \in \mathcal{R}^{\bigstar *} \iff A \in \tau_{\mathcal{R}^{\bigstar}} \iff X \neq \rho_{\mathcal{R}}(X \setminus A).$$

Theorem 10.12. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $A \neq \emptyset$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\blacktriangle}$; *(2)* $E_{\mathcal{R}} \subset A$.

Proof. By Theorems 2.8 and 10.6 and Corollary 5.22, it is clear that

$$R_A \in \mathcal{R}^{\blacktriangle} \iff R_A \in \mathcal{R}^{\blacktriangle *} \iff A \in \tau_{\mathcal{R}^{\blacktriangle}} \iff E_{\mathcal{R}} \subset A.$$

Remark 10.13. Since $\mathcal{R}^{\bigstar} = \mathcal{R}^{\bullet \vartriangle}$ and $\mathcal{R}^{\blacktriangle} = \mathcal{R}^{\vartriangle \bullet}$, Theorems 10.11 and 10.12 can also be proved with the help of Theorems 10.9 and 10.10.

For this, it is enough to note only that

$$\mathcal{E}_{\mathcal{R}^{\bullet}} = \mathcal{E}_{\{\rho_{\mathcal{R}}^{-1}\}^*} = \mathcal{E}_{\{\rho_{\mathcal{R}}^{-1}\}} = \{ A \subset X : \quad X \neq \rho_{\mathcal{R}}(X \setminus A) \}$$

and $\rho_{\mathcal{R}^{\vartriangle}} = \rho_{\mathcal{R}^{\vartriangle \bullet}} = \rho_{\mathcal{R}^{\blacktriangle}} = \rho_{\{X \times E_{\mathcal{R}}\}^*} = \rho_{\{X \times E_{\mathcal{R}}\}} = E_{\mathcal{R}} \times X$.

Theorem 10.14. *If $\mathcal{R}$ is a relator on X and $A \subset X$ such that $\emptyset \neq A \neq X$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\blacklozenge}$; *(2)* $\mathcal{R} \neq \{X^2\}$.

Proof. If the assertion (2) does not hold, then we have $\mathcal{R}^{\blacklozenge} = \{X^2\}$. Hence, since $R_A \neq X^2$, it is clear that the assertion (1) does not also holds.

On the other hand, if the assertion (2) holds, then we have $\mathcal{R}^{\blacklozenge} = \mathcal{P}(X^2)$. Therefore, the assertion (1) also holds.

Finally, we note that, in contrast to Theorem 10.10, we can also prove the following

Theorem 10.15. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\star}$; *(2)* $\operatorname{card}(A) \neq 1$ *or* $A \in \tau_{\mathcal{R}}$ $\big(A \in \mathcal{T}_{\mathcal{R}}\big)$.

Proof. By a reformulation of the definition of $\mathcal{R}^{\star}$, and Proposition 10.3, it is clear that $R_A \in \mathcal{R}^{\star}$ if and only if

$$\operatorname{cl}_{\mathcal{R}}\big(X \setminus \{x\}\big) \subset R_A^{-1}\big(X \setminus \{x\}\big) = R_{X \setminus A}\big(X \setminus \{x\}\big)$$

for all $x \in X$. Hence, by noticing that

$$R_{X \setminus A}\big(X \setminus \{x\}\big) = \begin{cases} \emptyset & \text{if} \quad X = \{x\}, \\ X \setminus A & \text{if} \quad A \subset \{x\} \neq X, \\ X & \text{if} \quad A \setminus \{x\} \neq \emptyset, \end{cases}$$

it is not hard to check that the required assertions are also equivalent.

Remark 10.16. In this respect, it is also worth noticing that $R_A \in \mathcal{R}^{\partial}$ if and only if $R_A \in \mathcal{R}$. Thus, in particular, $R_A \in \mathcal{R}^{\sharp}$ $\left(R_A \in \mathcal{R}^{\curlywedge}\right)$ if and only if $R_A \in \mathcal{R}^{\#}$ $\left(R_A \in \mathcal{R}^{\wedge}\right)$.

Moreover, in addition to Corollary 10.7, we can also prove the following

Theorem 10.17. *If* $\mathcal{R}$ *is a relator on* X, $A \subset X$ *and* $\square \in \{*, \#\}$, *then the following assertions are equivalent:*

(1) $R_A \in \mathcal{R}^{\square}$; *(2)* $R_A \in \mathcal{R}^{\square\infty}$; *(3)* $R_A \in \mathcal{R}^{\infty\square}$.

Proof. By Proposition 10.3, is clear that $(1) \Longrightarrow (2)$. Moreover, by the inclusions $\mathcal{R}^{\square\infty} \subset \mathcal{R}^{\infty\square} \subset \mathcal{R}^{\square}$, is clear that implications $(2) \Longrightarrow (3) \Longrightarrow (1)$ are also true.

Remark 10.18. Note that in Remark 10.16 and Theorem 10.17 we may write any preorder relation in place of R_A.

11. Symmetrizations of the Davis-Pervin relations

Definition 11.1. For each $A \subset X$, the relation

$$S_A = R_A \cap R_A^{-1}$$

is called the symmetrization of the Davis–Pervin relation R_A.

Concerning the relations S_A, we can easily establish the following propositions.

Proposition 11.2. *If* $A \subset X$, *then* S_A *is an equivalence on* X *such that*

$$S_A = A^2 \cup (X \setminus A)^2 \qquad \text{and} \qquad S_A^c = A \times (X \setminus A) \cup (X \setminus A) \times A.$$

Proposition 11.3. *If* $A, B \subset X$, *then*

$$S_A(B) = \emptyset \quad \text{if} \quad B = \emptyset, \qquad S_A(B) = X \setminus A \quad \text{if} \quad \emptyset \neq B \subset X \setminus A,$$

$$S_A(B) = A \quad \text{if} \quad \emptyset \neq B \subset A, \qquad S_A(B) = X \quad \text{if} \quad B \not\subset A \text{ and } B \not\subset X \setminus A.$$

Proposition 11.4. *If* $A \subset X$, *then*

$$\operatorname{Int}_{S_A} = \operatorname{Int}_{R_A} \cup \operatorname{Int}_{R_A^{-1}} \qquad \text{and} \qquad \operatorname{Cl}_{S_A} = \operatorname{Cl}_{R_A} \cap \operatorname{Cl}_{R_A^{-1}}.$$

Proof. By the corresponding definitions and Propositions 11.3 and and 10.3, it is clear that for any $B, C \subset X$ we have

$$B \in \operatorname{Int}_{S_A}(C) \iff S_A(B) \subset C \iff R_A(B) \subset C \quad \text{or} \quad R_{X \setminus A}(B) \subset C \iff$$
$$B \in \operatorname{Int}_{R_A}(C) \quad \text{or} \quad B \in \operatorname{Int}_{R_A^{-1}}(C) \iff B \in \left(\operatorname{Int}_{R_A} \cup \operatorname{Int}_{R_A^{-1}}\right)(C).$$

Therefore, the first assertion of the theorem is true. The second assertion of the theorem can be easily derived from the first one by using Theorem 3.1.

Moreover, concerning the relations S_A, we can also easily prove the following

Theorem 11.5. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$; *(2)* $R_A \in \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^*$.

Proof. If the assertion (1) holds, then there exists $R \in \mathcal{R}$ such that $R \subset S_A$. Hence, it follows that $R \cup R^{-1} \subset S_A \cup S_A^{-1} = S_A \subset R_A$. Therefore, the assertion (2) also holds.

While, if the assertion (2) holds, then there exists $R \in \mathcal{R}$ such that $R \cup R^{-1} \subset R_A$. Hence, it follows that $R \subset R_A \cap R_A^{-1} = S_A$. Therefore, the assertion (1) also holds.

From Theorem 11.5, by Theorem 7.3, it is clear that we also have

Corollary 11.6. *If $\mathcal{R}$ is a uniformly filtered relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$; *(2)* $R_A \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*$.

Moreover, analogously to Theorem 11.5, we can also easily prove the following

Theorem 11.7. *If $\mathcal{R}$ is a reflexive relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$; *(2)* $R_A \in \big(\mathcal{R} \odot \mathcal{R}^{-1}\big)^*$; *(3)* $R_A \in \big(\mathcal{R}^{-1} \odot \mathcal{R}\big)^*$.

Proof. If the assertion (1) holds, then there exists $R \in \mathcal{R}$ such that $R \subset S_A$. Hence, by Proposition 11.2, it is clear that $R \circ R^{-1} \subset S_A \circ S_A^{-1} = S_A \subset R_A$ and $R^{-1} \circ R \subset S_A^{-1} \circ S_A = S_A \subset R_A$. Therefore, the assertions (2) and (3) also hold.

While if the assertion (2) or (3) hold, then by Theorem 7.5 and Corollary 7.6, we have $R_A \in \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^*$. Therefore, by Theorem 11.5, the assertion (1) also holds.

Remark 11.8. Note that the implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) do not require the relator $\mathcal{R}$ to be reflexive.

Therefore, the inclusion $R_A \in \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^*$ implies $R_A \in \big(\mathcal{R} \odot \mathcal{R}^{-1}\big)^*$ and $R_A \in \big(\mathcal{R}^{-1} \odot \mathcal{R}\big)^*$ even if the relator $\mathcal{R}$ is not reflexive.

From Theorem 11.7, by Theorem 7.3 and Corollary 7.4, it is clear that we also have

Corollary 11.9. *If $\mathcal{R}$ is a uniformly filtered reflexive relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$; *(2)* $R_A \in \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^*$; *(3)* $R_A \in \big(\mathcal{R}^{-1} \circ \mathcal{R}\big)^*$.

On the other hand from Proposition 11.4, by Corollaries 7.12 and 10.7, it is clear that we also have the following

Theorem 11.10. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\#}$; *(2)* $R_A \in \mathcal{R}^* \cap \big(\mathcal{R}^*\big)^{-1}$; *(3)* $R_A \in \mathcal{R}^{\#} \cap \big(\mathcal{R}^{\#}\big)^{-1}$.

Remark 11.11. Note that if $X = \{0, 1\}$, $A = \{0\}$ and $\mathcal{R} = \{R_A, R_A^{-1}\}$, then $R_A \in \mathcal{R}^* \cap \big(\mathcal{R}^*\big)^{-1}$, but $S_A \notin \mathcal{R}^*$. Therefore, an analogue of Theorem 11.10 does not, in general, hold for the operation $*$.

However, as an immediate consequence of Theorem 11.10, we can also state that if $\square$ is a $\#$–invariant operation for relators on X, then for any set $A \subset X$ and any relator $\mathcal{R}$ on X we have $S_A \in \mathcal{R}^\square$ if and only if $R_A \in \mathcal{R}^\square \cap \big(R^\square\big)^{-1}$.

From Theorem 11.10, by using Theorem 10.6, we can also quite easily get the following

Theorem 11.12. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\#$; *(2)* $A \in \tau_{\mathcal{R}} \cap \bar{\tau}_{\mathcal{R}}$.

Proof. If $S_A \in \mathcal{R}^\#$, then by Theorem 11.10 we also have $R_A \in \mathcal{R}^* \cap (\mathcal{R}^*)^{-1}$. This implies that $R_A \in \mathcal{R}^*$ and $R_A \in (\mathcal{R}^*)^{-1}$, i.e., $R_{X\setminus A} \in \mathcal{R}^*$. Hence, by Theorem 10.6, it follows that $A \in \tau_{\mathcal{R}}$ and $X \setminus A \in \tau_{\mathcal{R}}$, i.e., $A \in \bar{\tau}_{\mathcal{R}}$. Therefore, the implication $(1) \Longrightarrow (2)$ is true. The converse implication can be proved quite similarly, by reversing the above argument.

In addition to Theorem 11.12, it is also worth proving the following

Theorem 11.13. *If $\mathcal{R}$ is a relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\#$; *(2)* $R_A \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*$.

Proof. If the assertion (1) holds, then by Theorem 11.12 we have $A \in \tau_{\mathcal{R}}$ and $A \in \bar{\tau}_{\mathcal{R}}$, and hence $A \in \tau_{\mathcal{R}^{-1}}$. Hence, by Theorem 10.6, it follows that $R_A \in \mathcal{R}^*$ and $R_A \in \big(\mathcal{R}^{-1}\big)^*$. Therefore, by Theorem 6.9, we have $R_A \in \big(\mathcal{R} \vee \mathcal{R}^{-1}\big)^*$. That is, the assertion (2) also holds.

The converse implication $(2) \Longrightarrow (1)$ can be proved quite similarly, by reversing the above argument.

Now, as an immediate consequence of Theorem 11.13 and Corollary 11.6, we can also state

Corollary 11.14. *If $\mathcal{R}$ is a uniformly filtered relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^*$; *(2)* $S_A \in \mathcal{R}^\#$.

Moreover, analogously to the Theorem 11.13, we can also prove the following

Theorem 11.15. *If $\mathcal{R}$ is a reflexive relator on X and $A \subset X$, then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\#$; *(2)* $R_A \in \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^*$; *(3)* $R_A \in \big(\mathcal{R}^{-1} \circ \mathcal{R}\big)^*$.

Proof. If the assertion (1) holds, then as in the proof of Theorem 11.9 we have $R_A \in \mathcal{R}^*$ and $R_A \in \big(\mathcal{R}^{-1}\big)^*$. Hence, by Proposition 10.3 and Theorem 6.5, it is clear that

$$R_A = R_A \circ R_A \in \mathcal{R}^* \circ \big(\mathcal{R}^{-1}\big)^* \subset \Big(\mathcal{R}^* \circ \big(\mathcal{R}^{-1}\big)^*\Big)^* = \big(\mathcal{R} \circ \mathcal{R}^{-1}\big)^*$$

and

$$R_A = R_A \circ R_A \in \left(\mathcal{R}^{-1}\right)^* \circ \mathcal{R}^* \subset \Big(\left(\mathcal{R}^{-1}\right)^* \circ \mathcal{R}^*\Big)^* = \left(\mathcal{R}^{-1} \circ \mathcal{R}\right)^*.$$

That is, the assertions (2) and (3) also hold.

While if the assertion (2) or (3) holds, then by Theorem 7.5 and Corollary 7.6 we have $R_A \in \left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^*$. Therefore, by Theorem 11.13, the assertion (1) also holds.

Remark 11.16. Note that the implications (1) $\Longrightarrow$ (2) and (1) $\Longrightarrow$ (3) do not require the relator $\mathcal{R}$ to be reflexive.

Therefore, the inclusion $R_A \in \left(\mathcal{R} \vee \mathcal{R}^{-1}\right)^*$ implies $R_A \in \left(\mathcal{R} \circ \mathcal{R}^{-1}\right)^*$ and $R_A \in \left(\mathcal{R}^{-1} \circ \mathcal{R}\right)^*$ even if the relator $\mathcal{R}$ is not reflexive.

From Theorem 11.12, we can also easily get the following more particular theorems.

Theorem 11.17. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\wedge$; *(2)* $A \in \mathcal{T}_\mathcal{R} \cap \mathcal{F}_\mathcal{R}$.

Proof. By Theorems 2.8 and 11.12 and Corollary 5.15, it is clear that

$$S_A \in \mathcal{R}^\wedge \iff S_A \in \mathcal{R}^{\wedge\#} \iff A \in \tau_{\mathcal{R}^\wedge} \cap \digamma_{\mathcal{R}^\wedge} \iff A \in \mathcal{T}_\mathcal{R} \cap \mathcal{F}_\mathcal{R}.$$

Theorem 11.18. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *such that* $\emptyset \neq A \neq X$, *then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\vartriangle$; *(2)* $A \in \mathcal{E}_\mathcal{R} \setminus \mathcal{D}_\mathcal{R}$.

Proof. By Theorems 2.8 and 11.12, and Corollary 5.17, it is clear that

$$S_A \in \mathcal{R}^\vartriangle \iff S_A \in \mathcal{R}^{\vartriangle\#} \iff A \in \tau_{\mathcal{R}^\vartriangle} \cap \digamma_{\mathcal{R}^\vartriangle} \iff$$
$$A \in \mathcal{E}_\mathcal{R} \cap \left(\mathcal{P}(X) \setminus \mathcal{D}_\mathcal{R}\right) \iff A \in \mathcal{E}_\mathcal{R} \setminus \mathcal{D}_\mathcal{R}.$$

Theorem 11.19. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^\bullet$; *(2)* $A \cap \rho_\mathcal{R}(X \setminus A) = \emptyset$ *and* $(X \setminus A) \cap \rho_\mathcal{R}(A) = \emptyset$.

Proof. By Theorems 2.8 and 11.12 and Corollary 5.19, it is clear that

$$S_A \in \mathcal{R}^\bullet \iff S_A \in \mathcal{R}^{\bullet\#} \iff A \in \tau_{\mathcal{R}^\bullet}, \quad A \in \digamma_{\mathcal{R}^\bullet} \iff$$
$$A \cap \rho_\mathcal{R}(X \setminus A) = \emptyset, \quad (X \setminus A) \cap \rho_\mathcal{R}(A) = \emptyset.$$

Theorem 11.20. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *such that* $\emptyset \neq A \neq X$, *then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\bigstar}$; *(2)* $X \neq \rho_{\mathcal{R}}(A)$ *and* $X \neq \rho_{\mathcal{R}}(X \setminus A)$.

Proof. By Theorems 2.8 and 11.12 and Corollary 5.21, it is clear that

$$S_A \in \mathcal{R}^{\bigstar} \iff S_A \in \mathcal{R}^{\bigstar\#} \iff A \in \tau_{\mathcal{R}^{\bigstar}}, \quad A \in \bar{\tau}_{\mathcal{R}^{\bigstar}} \iff X \neq \rho_{\mathcal{R}}(X \setminus A), \quad X \neq \rho_{\mathcal{R}}(A).$$

Theorem 11.21. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *such that* $\emptyset \neq A \neq X$, *then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\blacktriangle}$; *(2)* $E_{\mathcal{R}} = \emptyset$.

Proof. By Theorems 2.8 and 11.12, Corollary 5.23 and Theorem 3.8, it is clear that

$$S_A \in \mathcal{R}^{\blacktriangle} \iff S_A \in \mathcal{R}^{\blacktriangle\#} \iff A \in \tau_{\mathcal{R}^{\blacktriangle}}, \quad A \in \bar{\tau}_{\mathcal{R}^{\blacktriangle}} \iff E_{\mathcal{R}} \subset A, \quad A \subset D_{\mathcal{R}} \iff E_{\mathcal{R}} \subset A, \quad E_{\mathcal{R}} \subset X \setminus A \iff E_{\mathcal{R}} = \emptyset.$$

Finally, we note that analogously to Theorems 10.14 and 10.15, the following two theorems are also true.

Theorem 11.22. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$ *such that* $\emptyset \neq A \neq X$, *then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\blacklozenge}$; *(2)* $\mathcal{R} \neq \{X^2\}$.

Theorem 11.23. *If* $\mathcal{R}$ *is a relator on* X *and* $A \subset X$, *then the following assertions are equivalent:*

(1) $S_A \in \mathcal{R}^{\star}$; *(2)* $\big(\operatorname{card}(A) \neq 1$ *or* $A \in \tau_{\mathcal{R}}$ $\big(A \in \mathcal{T}_{\mathcal{R}}\big)\big)$ *and* $\big(\operatorname{card}(X \setminus A) \neq 1$ *or* $A \in \bar{\tau}_{\mathcal{R}}$ $\big(A \in \mathcal{F}_{\mathcal{R}}\big)\big)$.

Proof. By Proposition 11.4, it is clear that $\sigma_{S_A} = \sigma_{R_A} \cup \sigma_{R_A^{-1}}$. Hence, by Theorem 3.16, Corollary 7.10 and Proposition 10.3, it is clear that

$$S_A \in \mathcal{R}^{\star} \iff R_A \in \mathcal{R}^{\star} \cap \big(\mathcal{R}^{\star}\big)^{-1} \iff R_A \in \mathcal{R}^{\star}, \quad R_{X \setminus A} \in \mathcal{R}^{\star}.$$

Therefore, by Theorems 10.15 and 3.6, the required assertions are also equivalent.

Remark 11.24. Moreover, note that, by Remark 10.18, we can write S_A in place of R_A in Remark 10.16 and Theorem 10.17.

12. WELL-CHAINEDNESS OF ARBITRARY RELATORS

Definition 12.1. A relator $\mathcal{R}$ on X will be called properly well-chained or chain-connected if $\mathcal{R}^{\infty} = \{X^2\}$.

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ will be called $\square$–well-chained if the relator $\mathcal{R}^{\square}$ is properly well-chained.

Remark 12.2. The condition $\mathcal{R}^{\infty} = \{ X^2 \}$, in a detailed form, means only that for every $R \in \mathcal{R}$ we have $X^2 = R^{\infty} = \Delta_X \cup \bigcup_{n=1}^{\infty} R^n$. That is, for every $x, y \in X$, with $x \neq y$, there exists an $n \in \mathbb{N}$ such that $(x, y) \in R^n$. That is, there exists a family $(x_i)_{i=0}^{n}$ in X such that $x_o = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R$ for all $i = 1, \dots, n$.

Therefore, our present definition of proper well-chainedness is a straightforward generalization of Cantor's chain-connectedness. (See, for instance, Thron [72, p. 29] and Wilder [76, p. 721].)

Preliminary forms of some of the following theorems, for Weil uniformities and reflexive relators, have already proved by Levine [24] and Kurdics and Száz [22], respectively. However, for the readers convenience, we shall now give some improved proofs.

Theorem 12.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) $R_A \notin \mathcal{R}^*$ for every proper nonvoid subset A of X;

(3) $R \notin \mathcal{R}^*$ for every proper preorder relation R on X;

(4) $R \notin \mathcal{R}^*$ for every proper nonvoid transitive relation R on X.

Proof. If $A \subset X$ such that $R_A \in \mathcal{R}^*$, then there exists an $R \in \mathcal{R}$ such that $R \subset R_A$. Hence, it follows that $R^{\infty} \subset R_A^{\infty} = R_A$. Moreover, if the assertion (1) holds, then we have $R^{\infty} = X^2$. Therefore, we also have $R_A = X^2$. This implies that $A = \emptyset$ or $A = X$. Therefore, the assertion (2) also holds.

While, if R is a preorder relation on X and $A = R(x)$ for some $x \in X$, then $x \in A$ and $R(A) = R\big(R(x)\big) \subset R(x) = A$. Therefore, if $R \in \mathcal{R}^*$ holds, then $A \in \tau_{\mathcal{R}^*} = \tau_{\mathcal{R}}$ also holds. Hence, by Theorem 10.6, it follows that $R_A \in \mathcal{R}^*$. Therefore, if the assertion (2) holds, then since $A \neq \emptyset$ we necessarily have $A = X$, and thus $R(x) = X$. Hence, it is clear that $R = X^2$, and thus the assertion (3) also holds.

On the other hand, if R is a transitive relation on X, then $S = \Delta_X \cup R$ is a preorder relation on X such that $R \subset S$. Therefore, if $R \in \mathcal{R}^*$, then we also have $S \in \mathcal{R}^*$. Hence, if the assertion (3) holds, we can infer that $S = X^2$, and thus $X^2 = \Delta_X \cup R$. Therefore, if $u \in X$ and $v \in X \setminus \{u\}$, then we necessarily have $(u, v) \in R$ and $(v, u) \in R$. Hence, by the transitivity of R, it follows that $(u, u) \in R$. Therefore, if $\operatorname{card}(X) > 1$, then we necessarily have $R = X^2$ even if R was not supposed to be nonvoid. Therefore, the assertion (4) also holds. Namely, if $\operatorname{card}(X) = 1$, then $\emptyset$ and X^2 are the only relations on X.

Finally, to complete the proof, we note that if $R \in \mathcal{R}$, then R^{∞} is, in particular, a nonvoid transitive relation on X. Therefore, if the assertion (4) holds, then we necessarily have $R^{\infty} = X^2$. And thus, the assertion (1) also holds.

Remark 12.4. The assertion (3) of Theorem 12.3 can be briefly verbalized by saying that X^2 is the only preorder relation being contained in $\mathcal{R}^*$.

A simple application of the assertion (4) of Theorem 12.3 gives the following

Corollary 12.5. *If $\mathcal{R}$ is a properly well-chained relator on X and* $\operatorname{card}(X) > 1$*, then $\mathcal{R}$ is a total relator on X.*

Proof. If this not the case, then there exist $x \in X$ and $R \in \mathcal{R}$ such that $R(x) = \emptyset$. Hence, it is clear that $S = (X \setminus \{x\}) \times X$ is a proper nonvoid transitive relation on X such that $R \subset S$, and thus $S \in \mathcal{R}^*$. And this is a contradiction by Theorem 12.3.

The following simple proposition shows that the extra cardinality condition on X cannot be omitted from the above corollary.

Proposition 12.6. *If X is a nonvoid set, then the following assertions are equivalent:*

(1) $\operatorname{card}(X) = 1$*;*

(2) $\{\emptyset\}$ is a properly well-chained relator on X;

(3) every relator $\mathcal{R}$ on X is a properly well-chained.

Now, by using Theorem 12.3 and Corollary 12.5, we can also easily establish a useful reformulation of Definition 12.1

Proposition 12.7. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained; *(2) $X^2 = \bigcup_{n=1}^{\infty} R^n$ for all $R \in \mathcal{R}$.*

Proof. If $R \in \mathcal{R}$, then it is clear that $S = \bigcup_{n=1}^{\infty} R^n$ is a transitive relation on X such that $R \subset S$, and hence $S \in \mathcal{R}^*$. Moreover, if the assertion (1) holds, then by Corollary 12.5 in particular we have $R \neq \emptyset$, and hence $S \neq \emptyset$. Therefore, by Theorem 12.3, we necessarily have $S = X^2$, and thus the assertion (2) also holds. Now, since the converse implication $(2) \Longrightarrow (1)$ is quite obvious, the proof is complete.

Moreover, as an immediate consequence of Theorems 12.3 and 10.6, we can also state the following

Theorem 12.8. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$; *(3) $\mathcal{F}_{\mathcal{R}} = \{\emptyset, X\}$.*

Proof. Namely, by Theorem 12.3, we have (1) if and only if $R_A \notin \mathcal{R}^*$ for all proper nonvoid subset A of X. Moreover, by Theorem 10.6, for any $A \subset X$ we have $R_A \notin \mathcal{R}^*$ if and only if $A \notin \mathcal{T}_{\mathcal{R}}$. Therefore, the assertions (1) and (2) are equivalent. Moreover, by Theorem 3.6, it is clear that the assertions (2) and(3) are also equivalent.

Remark 12.9. The assertion (2) of Theorem 12.8 can be briefly verbalized by saying that no proper nonvoid subset of $X(\mathcal{R})$ is proximally open.

From Theorem 12.8, by the definitions of the families $\mathcal{T}_{\mathcal{R}}$ and $\mathcal{F}_{\mathcal{R}}$, it is clear that the proper well-chainedness of a relator $\mathcal{R}$ can also be expressed in terms of the relations $\operatorname{Int}_{\mathcal{R}}$ and $\operatorname{Cl}_{\mathcal{R}}$.

Therefore, it is rather surprising that the following theorem has formerly been overlooked by the authors of the papers [22] and [23].

Theorem 12.10. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly well-chained;*

(2) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ *implies* $A \not\subset B$ *for all* $A, B \subset X$ *with* $A \neq X$ *and* $B \neq \emptyset$;

(3) $X = A \cup B$ *implies* $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ *for all* $A, B \subset X$ *with* $A \neq \emptyset$ *and* $B \neq \emptyset$.

Proof. If the assertion (1) holds, then by Theorem 12.8 we have $\tau_{\mathcal{R}} = \{\emptyset, X\}$. Moreover, if the assertion (2) does not hold, then there exist $A, B \subset X$, with $A \neq X$ and $B \neq \emptyset$, such that $B \in \operatorname{Int}_{\mathcal{R}}(A)$ and $A \subset B$. Hence, by the corresponding definitions, it is clear that $A \in \operatorname{Int}_{\mathcal{R}}(A)$ and $B \in \operatorname{Int}_{\mathcal{R}}(B)$, and thus $A, B \in \tau_{\mathcal{R}}$. Hence, since $\tau_{\mathcal{R}} = \{\emptyset, X\}$ and $A \neq X$ and $B \neq \emptyset$, we can infer that $A = \emptyset$ and $B = X$. Therefore, we actually have $X \in \operatorname{Int}_{\mathcal{R}}(\emptyset)$, and hence $\emptyset \in \mathcal{R}$. Hence, by the assertion (1) and Proposition 12.6, it follows that $\operatorname{card}(X) = 1$, which is a contradiction. Therefore, the implication (1) $\Longrightarrow$ (2) is true.

Now, to prove the converse implication (2) $\Longrightarrow$ (1), we note that if $A \in \tau_{\mathcal{R}}$, then $A \in \operatorname{Int}_{\mathcal{R}}(A)$. Therefore, if the assertion (2) holds then we necessarily have $A = X$ or $A = \emptyset$. Consequently, $\tau_{\mathcal{R}} = \{\emptyset, X\}$, and thus by Theorem 12.8, the assertion (1) also holds.

Finally, to complete the proof, we note that the equivalence of the assertions (2) and (3) is immediate from Theorem 3.1. Namely, for $A, B \subset X$, the conditions $X \setminus A \subset B$ and $X = A \cup B$ are equivalent.

Remark 12.11. By Proposition 12.6, it is clear that not only the equivalence of the assertions (2) and (3), but also the implications (2) $\Longrightarrow$ (1) and (3) $\Longrightarrow$ (1) are true without the extra cardinality condition on X.

However, if $\operatorname{card}(X) = 1$ and $\mathcal{R} = \{\emptyset\}$, and moreover $A = B = X$, then $X = A \cup B$, with $A \neq \emptyset$ and $B \neq \emptyset$, such that $B \notin \operatorname{Cl}_{\mathcal{R}}(A)$. Therefore, in this case, the converses of the above implications fail to hold.

Now, as an immediate consequence of Theorems 12.8, 12.10 and 3.9, we can also state the following

Theorem 12.12. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly well-chained;*

(2) for each proper nonvoid subset A *of* X *there exist nets* x *and* y *in* A *and* $X \setminus A$, *respectively, such that* $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$;

(3) for any two nonvoid subsets A *and* B *of* X, *with* $X = A \cup B$, *there exist nets* x *and* y *in* A *and* B, *respectively, such that* $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$.

Remark 12.13. Later, we shall see that the proper well-chainedness of a relator $\mathcal{R}$ cannot, in general, be expressed in terms of the relations $\operatorname{cl}_{\mathcal{R}}$ or $\lim_{\mathcal{R}}$.

Therefore, it is of some interest to point out that we still have the following

Theorem 12.14. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly well-chained;* *(2)* $\rho_{\mathcal{R}^\infty} = X^2$.

Proof. Note that, by Theorem 3.3, we have $\rho_{\mathcal{R}^\infty}^{-1} = \bigcap \mathcal{R}^\infty$. Therefore, the equality $\mathcal{R}^\infty = \{ X^2 \}$ can hold if and only if $\rho_{\mathcal{R}^\infty}^{-1} = X^2$, that is, $\rho_{\mathcal{R}^\infty} = X^2$.

Moreover, as an immediate consequence of Definition 12.1 and the inversion compatibility of the operation ∞, we can also at once state

Theorem 12.15. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly well-chained;* *(2)* $\mathcal{R}^{-1}$ *is properly well-chained.*

Finally, as some immediate consequences of the corrresponding definitions and Theorems 12.3, 12.15 and 2.8, we can also state the following two theorems.

Theorem 12.16. *If $\mathcal{R}$ is a relator on X and $\Box$ is an $*$–invariant operation on relators, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\Box$*–well-chained;*

(2) $R_A \notin \mathcal{R}^\Box$ *for every proper nonvoid subset A of X;*

(3) $R \notin \mathcal{R}^\Box$ *for every proper preorder relation R on X;*

(4) $R \notin \mathcal{R}^\Box$ *for every proper nonvoid transitive relation R on X.*

Theorem 12.17. *If $\mathcal{R}$ is a relator on X and $\Box$ is an inversion compatible operation on relators, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\Box$*–well-chained;* *(2)* $\mathcal{R}^{-1}$ *is* $\Box$*–well-chained.*

13. Well–chainedness of refinement relators

Definition 13.1. A relator $\mathcal{R}$ on X will be called uniformly, proximally, topologically, paratopologically, infinitesimally, ultrainfinitesimally, parainfinitesimally, and ultimately well-chained if it is $\Box$–well-chained with $\Box = *, \#, \wedge, \vartriangle, \bullet, \bigstar, \blacktriangle$, and $\blacklozenge$, respectively.

Remark 13.2. From the inclusion relations of the above operations, it is clear that 'paratopologically or infinitesimally well-chained' $\Longrightarrow$ 'topologically well-chained' $\Longrightarrow$ 'proximally well-chained' $\Longrightarrow$ 'uniformly well-chained' $\Longrightarrow$ 'properly well-chained'. And 'ultimately well-chained' $\Longrightarrow$ 'parainfinitesimally well-chained' $\Longrightarrow$ 'ultrainfinitesimally wel-chained' $\Longrightarrow$ 'paratopologically and infinitesimally well-chained'.

Moreover, in addition to the corresponding particular cases of Theorems 12.16 and 12.17, we can also easily establish the following theorems.

Theorem 13.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly well-chained;

(2) $\mathcal{R}$ is uniformly well-chained; *(3) $\mathcal{R}$ is proximally well-chained.*

Proof. Note that, by Theorem 4.2 (2), we have $\tau_{\mathcal{R}} = \tau_{\mathcal{R}^*} = \tau_{\mathcal{R}^{\#}}$. Therefore, by Theorem 12.8, the required assertions are also equivalent.

Remark 13.4. Later we shall see that 'proximally well-chained' $\not\Longrightarrow$ 'topologically well-chained' $\not\Longrightarrow$ 'infintesimally well-chained' $\not\Longrightarrow$ 'paratopologically well-chained'.

But, 'paratopologically well-chained' $\Longrightarrow$ 'ultimately well-chained'. Therefore, 'paratopologically well-chained' is equivalent to 'ultrainfinitesimaly, parainfinitesimally and ultimately well-chained'. Moreover, 'paratoplogically well-chained' $\Longrightarrow$ 'infinitesimally well-chained'.

Theorem 13.5. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically well-chained;

(2) $\mathcal{T}_{\mathcal{R}} = \{ \emptyset , X \}$; *(3) $\mathcal{F}_{\mathcal{R}} = \{ \emptyset , X \}$.*

Proof. By Theorem 12.8, we have (1) if and only if $\tau_{\mathcal{R}^\wedge} = \{ \emptyset , X \}$. Moreover, by Corollary 5.15, we also have $\tau_{\mathcal{R}^\wedge} = \mathcal{T}_{\mathcal{R}}$. Therefore, the assertions (1) and (2) are equivalent. Moreover, by Theorem 3.6, the assertions (2) and (3) are also equivalent.

Theorem 13.6. *If $\mathcal{R}$ is a relator on X and $\operatorname{card}(X) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically well-chained;

(2) $B \subset \operatorname{int}_{\mathcal{R}}(A)$ implies $A \not\subset B$ for all $A , B \subset X$ with $A \neq X$ and $B \neq \emptyset$;

(3) $X = A \cup B$ implies $B \cap \operatorname{cl}_{\mathcal{R}}(A) \neq \emptyset$ for all $A , B \subset X$ with $A \neq \emptyset$ and $B \neq \emptyset$.

Proof. By Theorem 2.8, it is clear that $\mathcal{R}$ is topologically well-chained if and only if $\mathcal{R}^\wedge$ is proximally well chained. Moreover, by Theorem 5.14, for any $A , B \subset X$, we have $B \in \operatorname{Int}_{\mathcal{R}^\wedge}(A)$ if and only if $B \subset \operatorname{int}_{\mathcal{R}}(A)$. Therefore, by Theorem 12.10, the assertions (1) and (2) are equivalent. Moreover, by Theorem 3.1, it is clear that the assertions (2) and (3) are also equivalent.

Theorem 13.7. *If $\mathcal{R}$ is a relator on X and $\operatorname{card}(X) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically well-chained;

(2) for each poper nonvoid subset A there exists a net x in A and a point y in $X \setminus A$ such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x) \big)$;

(3) for any two nonvoid subsets A and B of X, with $X = A \cup B$, there exists a net x in A and a point y in B such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x) \big)$.

Remark 13.8. Later we shall see that the inverse of a topologically well-chained relator need not be topologically well-chained.

Concerning paratopological well-chainedness, we first prove the following analogue of Theorem 13.5.

Theorem 13.9. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically well-chained;*

(2) $\mathcal{E}_{\mathcal{R}} = \{ X \}$*;* *(3)* $\mathcal{D}_{\mathcal{R}} = \mathcal{P}(X) \setminus \{ \emptyset \}$.

Proof. By Corollary 5.17, we have $\tau_{\mathcal{R}^{\Delta}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\}$. Moreover, if the assertion (1) holds, then by Theorem 12.8 and Corollary 12.5 we also have $\tau_{\mathcal{R}^{\Delta}} = \{ \emptyset, X \}$ and $\emptyset \notin \mathcal{E}_{\mathcal{R}}$. Therefore, the assertion (2) also holds.

On the other hand, if the assertion (2) holds, then again by Corollary 5.17, we have $\tau_{\mathcal{R}^{\Delta}} = \mathcal{E}_{\mathcal{R}} \cup \{\emptyset\} = \{ \emptyset, X \}$. And thus, by Theorem 12.8, the assertion (1) also holds even if the condition $\operatorname{card}(X) > 1$ is not assumed.

Finally, to complete the proof, we note that the equivalence of the assertions (2) and (3) is immediate from Theorem 3.7.

The latter theorem allows us to easily prove that paratopologically well-chained relators need not actually be studied since we have the following

Theorem 13.10. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically well-chained;* *(2)* $\mathcal{R} = \{ X^2 \}$.

Proof. If $R \in \mathcal{R}$ and $x \in X$, then $R(x) \in \mathcal{E}_{\mathcal{R}}$. Therefore, if the assertion (1) holds, then by Theorem 13.9 we necessarily have $R(x) = X$. Hence, it is clear that $R = X^2$, and thus the assertion (2) also holds.

On the other hand, if the assertion (2) holds, then by the corresponding definitions it is clear that $\mathcal{E}_{\mathcal{R}} = \{X\}$. Therefore, again by Theorem 13.9, the assertion (1) also holds.

Remark 13.11. Note that if $\operatorname{card}(X) = 1$, then by Proposition 12.6 any relator on X is paratopologically well chained.

Moreover, if $\operatorname{card}(X) = 1$ and $\mathcal{R}$ is a relator on X, then we actually have $\mathcal{E}_{\mathcal{R}} = \{X\}$ if and only if $\emptyset \notin \mathcal{R}$.

From Theorem 13.10, it is clear that, in contrast to Remark 13.8, we have

Corollary 13.12. *A relator* $\mathcal{R}$ *on* X *is paratopologically well-chained if and only if its inverse* $\mathcal{R}^{-1}$ *is paratopologically well-chained.*

Moreover, by using Theorem 13.10, we can also easily establish the following

Theorem 13.13. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically well-chained;*

(2) $\operatorname{int}_{\mathcal{R}}(A) = \emptyset$ $\big(\operatorname{Int}_{\mathcal{R}}(A) = \{ \emptyset \} \big)$ *for all* $A \subset X$ *with* $A \neq X$*;*

(3) $\operatorname{cl}_{\mathcal{R}}(A)=X$ $\big(\operatorname{Cl}_{\mathcal{R}}(A)=\mathcal{P}(X)\setminus\{\emptyset\}\big)$ *for all* $A\subset X$ *with* $A\neq\emptyset$.

Hint. Note that if the assertion (2) holds, then $x\notin\operatorname{int}_{\mathcal{R}}(A)$ for all $x\in X$ and $A\subset X$ with $A\neq X$. This implies that $R(x)\not\subset A$ for all $R\in\mathcal{R}$, $x\in X$ and $A\subset X$ with $A\neq X$. Therefore, we necessarily have $R(x)=X$ for all $R\in\mathcal{R}$ and $x\in X$. Consequently, $\mathcal{R}=\{X^2\}$, and thus the assertion (1) also holds.

Theorem 13.14. *If* $\mathcal{R}$ *is a relator on* X, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically well-chained;* *(2)* $\mathcal{R}$ *is ultrainitesimally well-chained;*

(3) $\mathcal{R}$ *is parainfinitesimally well-chained;* *(4)* $\mathcal{R}$ *is ultimately well-chained.*

Proof. If the assertion (1) holds and $\operatorname{card}(X)>1$, then by Theorem 13.10 we have $\mathcal{R}=\{X^2\}$, and hence $\mathcal{R}^{\blacklozenge}=\{X^2\}$. Therefore, by the corresponding definitions, the assertion (4) also holds.

Now, by Remarks 13.11 and 13.2, it is clear that required assertions are equivalent even if $\operatorname{card}(X)=1$. Moreover, we have the following

Corollary 13.15. *If* $\mathcal{R}$ *is a paratopologically well-chained relator on* X, *then* $\mathcal{R}$ *is, in particular, infinitesimally well-chained.*

In this respect, it is also worth mentioning that analogously to Theorem 12.14 we also have

Theorem 13.16. *If* $\mathcal{R}$ *is a relator on* X, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is infinitesimally well-chained;* *(2)* $\rho_{\mathcal{R}}^{\infty}=X^2$.

Proof. Recall that $\mathcal{R}^{\bullet}=\{\rho_{\mathcal{R}}^{-1}\}^{*}$. Therefore, the relator $\mathcal{R}$ is infinitesimally well-chained if and only if the relator $\{\rho_{\mathcal{R}}^{-1}\}$ is uniformly well-chained. That is, by Theorems 13.3 and 12.15, the relator $\{\rho_{\mathcal{R}}\}$ is properly well-chained. Therefore, the assertions (1) and (2) are equivalent.

From Theorems 12.8 and 12.10, by Corollary 5.19 and Theorem 5.18, it is clear that we also have the following

Theorem 13.17. *If* $\mathcal{R}$ *is a relator on* X, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is infinitesimally well-chained;*

(2) $A\cap\rho_{\mathcal{R}}(X\setminus A)\neq\emptyset$ *for every proper nonvoid subset* A *of* X;

(3) $A\cap\rho_{\mathcal{R}}(B)\neq\emptyset$ *for any two nonvoid subsets* A *and* B *of* X *with* $X=A\cup B$.

Finally, by calling a relator σ–infinitesimally well-chained if it is $\star$–well-chained, we can also prove the following

Theorem 13.18. *If* $\mathcal{R}$ *is a relator on* X, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* σ*–infinitesimally well-chained;*

(2) $\operatorname{card}(X) < 3$ *and* $\tau_{\mathcal{R}} = \{\emptyset, X\}$ $\big(\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}\big)$.

Proof. By Theorem 12.3, we have (1) if and only if $R_A \notin \mathcal{R}^\star$ for every proper nonvoid subset A of X. Moreover, by Theorem 10.15, for any $A \subset X$ we have $R_A \notin \mathcal{R}^\star$ if and only if $\operatorname{card}(A) = 1$ and $A \notin \tau_{\mathcal{R}}$ $\big(A \notin \mathcal{T}_{\mathcal{R}}\big)$. Therefore, the required assertions are also equivalent.

Remark 13.19. From the equality $\mathcal{R}^\infty = \mathcal{R}^{\infty\infty}$ we can at once see that the relator $\mathcal{R}$ is ∞–well-chained if and only if it is properly well-chained. Therefore, the 'quasi well-chainedness properties' of relators need not be studied.

Moreover, from Theorem 12.6, by Theorem 10.17 and Remark 10.16, we can at once see that the 'almost uniform (almost proximal) and the superproximal (supertopological) well-chainedness properties' of relators need not also be studied.

However, note that a relator $\mathcal{R}$ on X may be naturally called properly well-chained at a point x of X if $R^\infty(x) = X$ for all $R \in \mathcal{R}$. Therefore, localized forms of the corresponding well-chainedness properties may also be investigated.

14. Connectedness of arbitrary relators

Definition 14.1. A relator $\mathcal{R}$ on X will be called properly connected if the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is properly well-chained.

Moreover, if $\square$ is a unary operation for relators on X, then the relator $\mathcal{R}$ will be called $\square$–connected if the relator $\mathcal{R}^{\square}$ is properly connected.

Remark 14.2. The appropriateness of the above apparently very strange definition should have already been quite obvious from the results of Kurdics [19]. However, despite this, it has later been still overlooked even by Kurdics and Száz [23].

The proper connectedness of $\mathcal{R}$, i.e., the condition $\big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^\infty = \{X^2\}$, by Remark 12.2, means only that for every $x, y \in X$, with $x \neq y$, and every $R \in \mathcal{R}$ there exist a finite family $(x_i)_{i=0}^n$ in X such that $x_o = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R \cup R^{-1}$, i.e., $(x_{i-1}, x_i) \in R$ or $(x_i, x_{i-1}) \in R$ for all $i = 1, \dots, n$.

Moreover, as a close analogue of Theorem 12.3, we can also easily prove the following

Theorem 14.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly connected;*

(2) $S_A \notin \mathcal{R}^*$ *for every proper nonvoid subset* A *of* X*;*

(3) $S \notin \mathcal{R}^*$ *for every proper equivalence relation* S *on* X*;*

(4) $S \notin \mathcal{R}^*$ *for every proper nonvoid symmetric and transitive relation* S *on* X.

Proof. From Definition 14.1 and Theorem 12.3, we can at once see that the assertion (1) holds if and only if $R_A \notin \big(\mathcal{R} \triangledown \mathcal{R}^{-1}\big)^*$ for all proper nonvoid subset A of X. Moreover, from Theorem 11.5, we know that for any $A \subset X$ we have

$R_A \notin \left(\mathcal{R} \triangledown \mathcal{R}^{-1}\right)^*$ if and only if $S_A \notin \mathcal{R}^*$. Therefore, the assertions (1) and (2) are equivalent.

On the other hand, it is clear that the implications $(4) \Longrightarrow (3) \Longrightarrow (2)$ are true. Namely, S_A is a proper equivalence relation on X whenever A is a proper nonvoid subset of X. Therefore, to complete proof, we need only show that the implication $(1) \Longrightarrow (4)$ is also true.

For this, note that if the assertion (4) does not hold, then there exists a proper nonvoid symmetric and transitive relation S on X such that $S \in \mathcal{R}^*$. Therefore, there exists an $R \in \mathcal{R}$ such that $R \subset S$. Hence, it follows that $R \cup R^{-1} \subset S \cup S^{-1} = S$. Therefore, $S \in \left(\mathcal{R} \triangledown \mathcal{R}^{-1}\right)^*$. And thus, by Theorem 12.3 and Definition 14.1, the assertion (1) does not also holds.

Remark 14.4. The assertion (3) of Theorem 14.3 can be briefly verbalized by saying that X^2 is the only equivalence relation being contained in $\mathcal{R}^*$.

Now, as a useful consequence of Theorem 14.3, we can also state the following

Theorem 14.5. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(Y) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly connected;*

(2) $f^{-1} \circ f \notin \mathcal{R}^*$ *for every non-constant function* f *of* X *into* Y*.*

Proof. If the assertion (2) does not hold, then there exists a non-constant function f of X into Y such that $f^{-1} \circ f \in \mathcal{R}^*$. Hence, since

$$f^{-1} \circ f = \left\{(u, v) \in X^2 : \quad f(u) = f(v)\right\}$$

is a proper equivalence relation on X, Theorem 14.3 shows that the assertion (1) does not also holds. Therefore, the implication $(1) \Longrightarrow (2)$ is true.

While, if the assertion (1) does not hold, then by Theorem 14.3 there exists a proper nonvoid subset A of X such that $S_A \in \mathcal{R}^*$. Hence, by choosing $y, z \in Y$ such that $y \neq z$, and defining a function f on X such that $f(x) = y$ for all $x \in A$ and $f(x) = z$ for all $x \in X \setminus A$, we can at once see that $f^{-1} \circ f = S_A \in \mathcal{R}^*$. That is, the assertion (2) does not also hold. Therefore, the implication $(2) \Longrightarrow (1)$ is also true.

Hence, by Theorems 9.5, it is clear that we also have the following

Theorem 14.6. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(Y) > 1$*, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly connected;*

(2) every uniformly continuous function f *of* $X(\mathcal{R})$ *into* $Y(\Delta_Y)$ *is constant.*

Proof. Note that if f is a function of X into Y, then $f^{-1} \circ f = f^{-1} \circ \Delta_Y \circ f$. Moreover, by Theorem 9.5, f is a uniformly continuous function of $X(\mathcal{R})$ into $Y(\Delta_Y)$ if and only if $f^{-1} \circ \Delta_Y \circ f \in \mathcal{R}^*$. Therefore, the assertion (2) of Theorem 14.6 is equivalent to that of Theorem 14.5.

From Theorem 14.3, we can also easily get the following

Theorem 14.7. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly connected;*

(2) for each proper nonvoid subset A *of* X *there exists a net* (x, y) *in* $A \times (X \setminus A) \cup (X \setminus A) \times A$ *such that* $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$*;*

(3) for any two nonvoid subsets A *and* B *of* X, *with* $X = A \cup B$, *there exists a net* (x, y) *in* $A \times B \cup B \times A$ *such that* $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$.

Hint. If the assertion (1) holds, then by Theorem 14.3, for each proper nonvoid subset A of X, we have $S_A \notin \mathcal{R}^*$. Therefore, for each $R \in \mathcal{R}$, there exists a pair $(y_R, x_R) \in R$ such that $(y_R, x_R) \notin S_A$, and hence by Proposition 11.2 $(y_R, x_R) \in A \times (X \setminus A) \cup (X \setminus A) \times A$. Now, by defining $x = (x_R)_{R \in \mathcal{R}}$ and $y = (y_R)_{R \in \mathcal{R}}$, and preordering $\mathcal{R}$ with the reverse set inclusion (the discrete preorder), we can easily see that (x, y) is a partially ordered (directed) net in $A \times (X \setminus A) \cup (X \setminus A) \times A$ such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$. Therefore, the assertion (2) also holds.

On the other hand, by using Corollary 12.5, we can also easily prove the following

Theorem 14.8. *If* $\mathcal{R}$ *is a properly connected relator on* X *and* $\operatorname{card}(X) > 1$, *then* $X = R(X) \cup R^{-1}(X)$ *for all* $R \in \mathcal{R}$.

Proof. In this case, by Corollary 12.5, the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is total. Therefore, we have $X = \big(R \cup R^{-1}\big)^{-1}(X) = \big(R \cup R^{-1}\big)(X) = R(X) \cup R^{-1}(X)$ for all $R \in \mathcal{R}$.

From the equality $\mathcal{R} \triangledown \mathcal{R}^{-1} = \mathcal{R}^{-1} \triangledown \big(\mathcal{R}^{-1}\big)^{-1}$, by Definition 14.1, it is clear that now we also have

Theorem 14.9. *If* $\mathcal{R}$ *is a relator on* X, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is properly connected;* *(2)* $\mathcal{R}^{-1}$ *is properly connected.*

Moreover, by using Theorem 14.5, we can also easily prove the following

Theorem 14.10. *If* $\mathcal{R}$ *and* $\mathcal{S}$ *are relators on* X *such that* $\mathcal{R}$ *is reflexive and properly connected and* $\mathcal{S}$ *is uniformly refined by* $\mathcal{R}$, *then* $\mathcal{S}$ *is also properly connected.*

Proof. If this is not the case, then by Theorem 14.5 there exists a function f of X onto $\{0, 1\}$ such that $f^{-1} \circ f \in \mathcal{S}^*$. Hence, since $\mathcal{S}^*$ is also uniformly refined by $\mathcal{R}$, we can infer that there exists a function g on X to X such that $f^{-1} \circ f \circ g \in \mathcal{R}^*$. Hence, since $\mathcal{R}^*$ is also reflexive, we can infer that $x \in f^{-1}\big(f\big(g(x)\big)\big)$, and thus $f(x) = f\big(g(x)\big)$ for all $x \in X$. Therefore, we have $f = f \circ g$, and thus $f^{-1} \circ f = f^{-1} \circ f \circ g \in \mathcal{R}^*$. Hence, by Theorem 14.5, it follows that $\mathcal{R}$ is also not properly connected, and this contradiction proves the theorem.

Remark 14.11. Later we shall see that the counterpart of Theorem 14.10 with 'connected' replaced by 'well-chained' is not true.

Now, as some immediate consequence of the corresponding definitions and Theorems 14.3 and 14.5, we can also state the following theorems.

Theorem 14.12. *If $\mathcal{R}$ is a relator on X and $\Box$ is an $*$–invariant operation for relators on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\Box$*–connected;*

(2) $S_A \notin \mathcal{R}^{\Box}$ *for every proper nonvoid subset* A *of* X*;*

(3) $S \notin \mathcal{R}^{\Box}$ *for every proper equivalence relation* S *on* X*;*

(4) $S \notin \mathcal{R}^{\Box}$ *for every proper nonvoid symmetric and transitive relation* S *on* X*.*

Theorem 14.13. *If $\mathcal{R}$ is a relator on X, $\Box$ is an $*$–invariant operation for relators on X and $\operatorname{card}(Y) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\Box$*–connected;*

(2) $f^{-1} \circ f \notin \mathcal{R}^{\Box}$ *for every non-constant function* f *of* X *into* Y*.*

Hence, by using Theorems 9.5, 9.6 and 9.7, we can also easily get the following two theorems.

Theorem 14.14. *If $\mathcal{R}$ is a relator on X, $\Box \in \{ *, \#, \wedge, \bullet \}$ and $\operatorname{card}(Y) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\Box$*–connected;*

(2) every $\Box$*–continuous function* f *of* $X(\mathcal{R})$ *into* $Y(\Delta_Y)$ *is constant.*

Theorem 14.15. *If $\mathcal{R}$ is a relator on X, $\Box \in \{ \triangle, \blacktriangle, \blacklozenge \}$ and $\operatorname{card}(Y) = 2$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\Box$*–connected;*

(2) every $\Box$*–continuous function* f *of* $X(\mathcal{R})$ *into* $Y(\Delta_Y)$ *is constant.*

Hint. If the assertion (1) does not hold, then by Theorem 14.13 there exists a function f of X onto Y such that $f^{-1} \circ \Delta_Y \circ f = f^{-1} \circ f \in \mathcal{R}^{\Box}$. Hence, by Theorem 9.7, it follows that f is a $\Box$–continous function of $X(\mathcal{R})$ onto $Y(\Delta_Y)$. Therefore, the assertion (2) does not also hold. Consequently, the implication $(2) \Longrightarrow (1)$ is true.

Remark 14.16. Note that if, for instance, $1 < \operatorname{card}(X) < \operatorname{card}(Y)$, then by Corollary 9.20 every $\triangle$–continuous function f of $X(\Delta_X)$ into $Y(\Delta_Y)$ is constant, but by the equivalence of the assertions (1) and (3) of Theorem 14.12 the relator $\{ \Delta_X \}$ is not $\triangle$–connected.

In this respect, it is also worth mentioning that, by using the corresponding results of Section 16, it can be easily shown that if $\Box \in \{ \triangle, \blacktriangle, \blacklozenge \}$ and f is a $\Box$–continuous function on a $\Box$–connected relator space $X(\mathcal{R})$ to a $\Box$–separated relator space $Y(\mathcal{S})$, then f is necessarily constant.

Finally, as some immediate consequence of the corresponding definitions and Theorem 14.9, we can also state

Theorem 14.17. *If $\mathcal{R}$ is a relator on X and $\Box$ is an inversion compatible operation for relators on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* $\Box$*–connected;* *(2)* $\mathcal{R}^{-1}$ *is* $\Box$*–connected.*

15. Connectedness of refinement relators

Definition 15.1. A relator $\mathcal{R}$ on X will be called uniformly, proximally, topologically, paratopologically, infinitesimally, ultrainfinitesimally, parainfinitesimally, and ultimately connected if it is $\square$–connected with $\square = \ast$, $\#$, $\wedge$, $\vartriangle$, $\bullet$, $\bigstar$, $\blacktriangle$, and $\blacklozenge$, respectively.

Remark 15.2. From the inclusions relations of the above operations, it is clear that 'paratopologically or infinitesimally connected' $\Longrightarrow$ 'topologically connected' $\Longrightarrow$ 'proximally connected' $\Longrightarrow$ 'uniformly connected' $\Longrightarrow$ 'properly connected'. And 'ultimately connected' $\Longrightarrow$ 'parainfinitesimally connected' $\Longrightarrow$ 'ultrainfinitesimally connected' $\Longrightarrow$ 'paratopologically and infinitesimally connected'.

Moreover, as an immediate consequence of Theorems 14.3 and 14.12, we can at once state the following

Theorem 15.3. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected; *(2) $\mathcal{R}$ is uniformly connected.*

Remark 15.4. Later we shall see that 'uniformly connected' $\not\Longrightarrow$ 'proximally connected' $\not\Longrightarrow$ 'topologically connected' $\not\Longrightarrow$ 'paratopologically or infinitesimally connected'. And 'paratopologically and infinitesimally connected' are independent notions. Moreover, 'paratopologically and infinitesimally connected' $\not\Longrightarrow$ 'ultrainfinitesimally connected' $\not\Longrightarrow$ 'parainfinitesimally connected' $\not\Longrightarrow$ 'ultimately connected'.

However, as an immediate consequence of Theorems 14.3 and 14.12 and Corollary 11.14, we still have the following counterpart of Theorem 13.3.

Theorem 15.5. *If $\mathcal{R}$ is a uniformly filtered relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected; *(2) $\mathcal{R}$ is proximally connected.*

Moreover, as an immediate consequence of Theorem 14.12 and 11.12, we can also at once state

Theorem 15.6. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally connected; *(2) $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} = \{ \emptyset , X \}$.*

In addition to this theorem, it is also worth proving the following

Theorem 15.7. *If $\mathcal{R}$ is a uniformly filtered relator on X and $\operatorname{card}(X) > 1$, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally connected;

(2) $B \in \operatorname{Int}_{\mathcal{R}}(A)$ and $X \setminus A \in \operatorname{Int}_{\mathcal{R}}(X \setminus B)$ imply $A \not\subset B$ for all $A, B \subset X$ with $A \neq X$ and $B \neq \emptyset$;

(3) $X = A \cup B$ implies $B \in \operatorname{Cl}_{\mathcal{R}}(A)$ or $A \in \operatorname{Cl}_{\mathcal{R}}(B)$ for all $A, B \subset X$ with $A \neq \emptyset$ and $B \neq \emptyset$.

Proof. From Theorem 15.5 we know that $\mathcal{R}$ is proximally connected if and only if $\mathcal{R}$ is properly connected. Moreover, from Corollary 11.6, by using Theorems 14.3 and 12.3, we can see $\mathcal{R}$ is properly connected if and only if $\mathcal{R} \vee \mathcal{R}^{-1}$ is properly well-chained.

On the other hand, from Theorem 12.10 we know that $\mathcal{R} \vee \mathcal{R}^{-1}$ is properly well-chained if and only if $B \in \operatorname{Int}_{\mathcal{R} \vee \mathcal{R}^{-1}}(A)$ implies $A \not\subset B$ for all $A, B \subset X$ with $A \neq X$ and $B \neq \emptyset$. Moreover, from Theorem 6.12 we know that $B \in \operatorname{Int}_{\mathcal{R} \vee \mathcal{R}^{-1}}(A)$ if and only if $B \in \operatorname{Int}_{\mathcal{R}}(A)$ and $B \in \operatorname{Int}_{\mathcal{R}^{-1}}(A)$. Furthermore, from Theorem 3.2, we know that $B \in \operatorname{Int}_{\mathcal{R}^{-1}}(A)$ if and only if $X \setminus A \in \operatorname{Int}_{\mathcal{R}}(X \setminus B)$. Therefore, the assertions (1) and (2) are equivalent. The equivalence of the assertions (2) and (3) is again immediate from Theorem 3.1.

From Theorems 15.6 and 15.7, by Theorem 3.9, it is clear that we also have the following

Theorem 15.8. *If $\mathcal{R}$ is a uniformly filtered relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally connected;

(2) for each proper nonvoid subset A of X, there exists a net (x, y) in $A \times (X \setminus A)$ or $(X \setminus A) \times A$ such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$;

(3) for any two nonvoid subsets A and B of X, with $X = A \cup B$, there exists a net (x, y) in $A \times B$ or $B \times A$ such that $y \in \operatorname{Lim}_{\mathcal{R}}(x)$ $\big(y \in \operatorname{Adh}_{\mathcal{R}}(x)\big)$.

From Theorem 15.6, by Theorem 2.8 and Corollary 5.15, it is clear that we also have the following

Theorem 15.9. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically connected; *(2) $\mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}} = \{\emptyset, X\}$.*

In addition to Theorem 15.9, we can also easily prove

Theorem 15.10. *If $\mathcal{R}$ is a topologically filtered relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically connected;

(2) $B \subset \operatorname{int}_{\mathcal{R}}(A)$ and $X \setminus A \subset \operatorname{int}_{\mathcal{R}}(X \setminus B)$ imply $A \not\subset B$ for all $A, B \subset X$ with $A \neq X$ and $B \neq \emptyset$;

(3) $X = A \cup B$ implies $B \cap \operatorname{cl}_{\mathcal{R}}(A) \neq \emptyset$ or $A \cap \operatorname{cl}_{\mathcal{R}}(B) \neq \emptyset$ for all $A, B \subset X$ with $A \neq \emptyset$ and $B \neq \emptyset$.

Proof. By Theorem 2.8, it is clear that $\mathcal{R}$ is topologically connected if and only if $\mathcal{R}^{\wedge}$ is proximally connected. Moreover, since $\mathcal{R}^{\wedge}$ is now uniformly filtered, from Theorem 15.7 we know that $\mathcal{R}^{\wedge}$ is proximally connected if and only if $B \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(A)$ and $X \setminus A \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(X \setminus B)$ imply $A \not\subset B$ for all $A, B \subset X$ with $A \neq X$ and $B \neq \emptyset$. Moreover, from Theorem 5.14, we know that $B \in \operatorname{Int}_{\mathcal{R}^{\wedge}}(A)$ if and only if $B \subset \operatorname{int}_{\mathcal{R}}(A)$. Therefore, the assertions (1) and (2) are equivalent. Moreover, by Theorem 3.1, it is clear that the assertions (2) and (3) are also equivalent.

Remark 15.11. The assertion (2) of Theorem 15.9 can be briefly verbalized by saying that no proper nonvoid subset of X is topologically clopen.

While, the assertion (3) of Theorem 15.10 can be briefly verbalized by saying that X cannot be decomposed into the union of two nonvoid separated sets.

From Theorems 15.9 and 15.10, by using Theorem 3.12, we can also get the following

Theorem 15.12. *If $\mathcal{R}$ is a topologically filtered relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically connected;

(2) for each proper nonvoid subset A of X there exist a net x in A and a point y in $X \setminus A$, or a net x in $X \setminus A$ and a point y in A such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$;

(3) for any two nonvoid subsets A and B of X, with $X = A \cup B$, there exist a net x in A and a point y in B, or a net x in B and a point y in A such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$.

Hence, analogously to Ward [73, Theorem 82, p. 66], we can also state

Corollary 15.13. *If $\mathcal{R}$ is a topologically filtered relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is topologically connected;

(2) for each proper nonvoid topologically open subset A of $X(\mathcal{R})$ there exist a net x in A and a point y in $X \setminus A$ such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$;

(3) for each proper nonvoid topologically closed subset A of $X(\mathcal{R})$ there exist a net x in $X \setminus A$ and a point y in A such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$.

Hint. To prove the implication (2) $\Longrightarrow$ (1), suppose on the contrary that the assertion (2) holds, but the assertion (1) does not hold. Then, by Theorem 15.9 there exists a poper nonvoid subset A of X such that A is both topologically open and closed in $X(\mathcal{R})$. Therefore, by the assertion (2), there exists a net x in A and point y in $X \setminus A$ such that $y \in \lim_{\mathcal{R}}(x)$ $\big(y \in \operatorname{adh}_{\mathcal{R}}(x)\big)$. Hence, by Theorem 3.12, it follows that $y \in \operatorname{cl}_{\mathcal{R}}(A) \subset A$, and this contradicts to the fact that $y \in X \setminus A$.

In addition to Theorems 15.3 and 15.5, it is also worth proving the following

Theorem 15.14. *If $\mathcal{R}$ is a reflexive Lebesgue relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected;

(2) $\mathcal{R}$ is uniformly connected;

(3) $\mathcal{R}$ is proximally connected;

(4) $\mathcal{R}$ is topologically connected.

Proof. Suppose that the assertion (1) holds. Since $\mathcal{R}$ is a Lebesgue relator, $\mathcal{R}^\wedge$ is uniformly refined by $\mathcal{R}$. Hence, since $\mathcal{R}^\wedge = \big(\mathcal{R}^\wedge\big)^*$, by using Theorem 14.10 we can infer that $\mathcal{R}^\wedge$ is also properly connected. Therefore, the assertion (4) also holds. Moreover, from Remark 15.2, we know that the implications (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) are always true.

From Theorem 15.14, by Theorem 9.0, it is clear that in particular we also have

Corollary 15.15. *If $\mathcal{R}$ is a reflexive, uniformly filtered, strongly topologically transitive and topologically compact relator on X, then $\mathcal{R}$ is properly connected if and only if it is topologically connected.*

16. Some further results on the connectedness of refinement relators

Theorem 16.1. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected; *(2) $\mathcal{E}_{\mathcal{R}} \setminus \{\emptyset\} \subset \mathcal{D}_{\mathcal{R}} \cup \{X\}$.*

Proof. By Theorem 14.12, the assertion (1) holds if and only if $S_A \notin \mathcal{R}^{\Delta}$ for all proper nonvoid subset A of X. Moreover, by Theorem 11.18, for any proper nonvoid subset A of X we have $S_A \notin \mathcal{R}^{\Delta}$ if and only if $A \notin \mathcal{E}_{\mathcal{R}} \setminus \mathcal{D}_{\mathcal{R}}$. Therefore, the assertion (1) is equivalent to the condition that $A \notin \mathcal{E}_{\mathcal{R}} \setminus \mathcal{D}_{\mathcal{R}}$ for all proper nonvoid subset A of X. The latter condition means only that $\mathcal{E}_{\mathcal{R}} \setminus \{\emptyset, X\} \subset \mathcal{D}_{\mathcal{R}}$, that is, $\mathcal{E}_{\mathcal{R}} \setminus \{\emptyset\} \subset \mathcal{D}_{\mathcal{R}} \cup \{X\}$. Therefore, the assertions (1) and (2) are also equivalent.

Corollary 16.2. *If $\mathcal{R}$ is a paratopologically connected relator on X such that* $\operatorname{card}(X) > 1$, *then $\mathcal{R}$ is a total relator on X.*

Proof. If this is not the case, then by Theorem 5.3 we have $\mathcal{E}_{\mathcal{R}} = \mathcal{P}(X)$ and $\mathcal{D}_{\mathcal{R}} = \emptyset$. Hence, using that $\operatorname{card}(X) > 1$, we can see that the assertion (2) of Theorem 16.1 fails to hold. And this contradicts the paratopological connectedness of $\mathcal{R}$.

Remark 16.3. Later, we shall see that even an infinitesimally connected relator $\mathcal{R}$ on X need not be total.

Theorem 16.4. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected;

(2) $\mathcal{E}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}}$; *(3) $\mathcal{R}(x) \subset \mathcal{D}_{\mathcal{R}}$ for all $x \in X$.*

Proof. If the assertion (1) holds, then by Corollary 16.2 and Theorem 5.3 we have $\emptyset \notin \mathcal{E}_R$ and $X \in \mathcal{D}_{\mathcal{R}}$. And hence, by Theorems 16.1, it is clear that the assertion (2) also holds. Moreover, by Theorem 16.1, it is clear that the converse implication is true even if X is a singleton.

Finally, to complete the proof, we note that the equivalence of the assertions (2) and (3) is immediate from the facts that $\mathcal{E}_{\mathcal{R}}$ is the smallest ascending family in X containing $\mathcal{R}(x) = \{R(x) : R \in \mathcal{R}\}$ for all $x \in X$, and moreover $\mathcal{D}_{\mathcal{R}}$ is also an ascending family in X.

Theorem 16.5. *If $\mathcal{R}$ is a relator on X and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ is paratopologically connected;

(2) $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{E}_{\mathcal{R}}$;

(3) $R(x) \cap S(y) \neq \emptyset$ *for all* $x, y \in X$ *and* $R, S \in \mathcal{R}$.

Proof. From Theorem 16.4 we know that the assertion (1) is equivalent to the inclusion $\mathcal{E}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}}$. Moreover, from Theorem 3.7 we can see that the inclusion $\mathcal{E}_{\mathcal{R}} \subset \mathcal{D}_{\mathcal{R}}$ equivalent to the assertion (2).

On the other hand, by the corresponding definitions, it is clear that the assertions (2) and (3) are also equivalent.

Corollary 16.6. *If* $\mathcal{R}$ *is a topological relator on* X, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically connected;*

(2) $\mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\} \subset \mathcal{D}_{\mathcal{R}}$; *(3)* $U \cap V \neq \emptyset$ *for all* $U, V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$.

Proof. Now, by Theorem 5.8, for any $A \subset X$, we have $A \in \mathcal{E}_{\mathcal{R}}$ if and only if there exists a $V \in \mathcal{T}_{\mathcal{R}} \setminus \{\emptyset\}$ such that $V \subset A$. Therefore, by Theorems 16.4 and 16.5, the required assertions are equivalent whenever $\operatorname{card}(X) > 1$. However, since a topological relator is in particular total, the above cardinality condition can be omitted.

Remark 16.7. Hence, it is clear that the hyperconnectedness of Steen and Seebach [54, p. 29], studied also by Levine [25] and several other people (see [52], [42] and [1]), is a particular case of our paratopological connectedness.

Moreover, from Theorem 16.5 we can also at once see that the semi-directedness of Száz [60] coincides with our paratopological connectedness. Therefore, according to the corresponding results of [60], we also have the following theorems.

Theorem 16.8. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically connected;*

(2) $X \setminus A \notin \mathcal{E}_{\mathcal{R}}$ *for all* $A \in \mathcal{E}_{\mathcal{R}}$;

(3) $A \in \mathcal{D}_{\mathcal{R}}$ *whenever* $A \subset X$ *such that* $A \cap B \neq \emptyset$ *for all* $B \in \mathcal{D}_{\mathcal{R}}$.

Theorem 16.9. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically connected;*

(2) $A \in \mathcal{D}_{\mathcal{R}}$ *or* $X \setminus A \in \mathcal{D}_{\mathcal{R}}$ *for all* $A \subset X$;

(3) $A \in \mathcal{D}_{\mathcal{R}}$ *or* $B \in \mathcal{D}_{\mathcal{R}}$ *whenever* $X = A \cup B$.

Theorem 16.10. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is paratopologically connected;* *(2)* $\mathcal{R}^{-1} \circ \mathcal{R} = \{X^2\}$;

(3) $\rho_{\mathcal{R}^{-1} \circ \mathcal{R}} = X^2$; *(4)* $cl_{\mathcal{R} \square \mathcal{R}}(\Delta_X) = X^2$.

Proof. If the assertion (1) holds, then by Theorem 16.5 for any $x, y \in X$ and $R, S \in \mathcal{R}$ we have $R(x) \cap S(y) \neq \emptyset$. Hence, it follows that

$$y \in S^{-1}\big(R(x)\big) = \big(S^{-1} \circ R\big)(x).$$

Therefore, we have $\left(S^{-1}\circ R\right)(x) = X = X^2(x)$ for all $x\in X$. And hence, it follows that $S^{-1}\circ R = X^2$. Therefore, $\mathcal{R}^{-1}\circ\mathcal{R} = \left\{X^2\right\}$. That is, the assertion (2) also holds. The converse implication $(2)\Longrightarrow(1)$ can be proved quite similarly by reversing the above argument.

While, to prove the equivalences $(2)\Longleftrightarrow(3)$ and $(3)\Longleftrightarrow(4)$, it is enough to note that

$$\rho_{\mathcal{R}^{-1}\circ\mathcal{R}} = \bigcap \mathcal{R}^{-1}\circ\mathcal{R} \qquad \text{and} \qquad \rho_{\mathcal{R}^{-1}\circ\mathcal{R}} = \mathrm{cl}_{\mathcal{R}\square\mathcal{R}}(\Delta_X).$$

Remark 16.11. The above theorem is a counterpart of the more familiar statement that a relator $\mathcal{R}$ on X is reflexive and properly separating if and only if $\rho_{\mathcal{R}^{-1}\circ\mathcal{R}} = \Delta_X$, or equivalently $cl_{\mathcal{R}\square\mathcal{R}}(\Delta_X) = \Delta_X$.

Theorem 16.12. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is infinitesimally connected;

(2) $A\cap\rho_{\mathcal{R}}(X\setminus A)\neq\emptyset$ or $(X\setminus A)\cap\rho_{\mathcal{R}}(A)\neq\emptyset$ for every proper nonvoid subset A of X;

(3) $A\cap\rho_{\mathcal{R}}(B)\neq\emptyset$ or $B\cap\rho_{\mathcal{R}}(A)\neq\emptyset$ for any two nonvoid subsets A and B of X with $X=A\cup B$.

Proof. By Theorem 14.12, the assertion (1) holds if and only if $S_A\notin\mathcal{R}^{\bullet}$ for all proper nonvoid subset A of X. Moreover, by Theorem 11.19, for any $A\subset X$ we have $S_A\notin\mathcal{R}^{\bullet}$ if and only if $A\cap\rho_{\mathcal{R}}(X\setminus A)\neq\emptyset$ or $(X\setminus A)\cap\rho_{\mathcal{R}}(A)\neq\emptyset$. Therefore, the assertions (1) and (2) are also equivalent. Now, since the equivalence of the assertions (2) and (3) is quite obvious, the proof is complete.

Theorem 16.13. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is ultrainfinitesimally connected;

(2) $X=\rho_{\mathcal{R}}(A)$ or $X=\rho_{\mathcal{R}}(X\setminus A)$ for every proper nonvoid subset A of X;

(3) $X=\rho_{\mathcal{R}}(A)$ or $X=\rho_{\mathcal{R}}(B)$ for any two nonvoid subset A and B of X with $X=A\cup B$.

Proof. By Theorem 14.12, the assertion (1) holds if and only if $S_A\notin\mathcal{R}^{\star}$ for all proper nonvoid subset A of X. Moreover, by Theorem 11.20, for any $A\subset X$ we have $S_A\notin\mathcal{R}^{\star}$ if and only if $X=\rho_{\mathcal{R}}(A)$ or $X=\rho_{\mathcal{R}}(X\setminus A)$. Therefore, the assertions (1) and (2) are equivalent.

Theorem 16.14. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is parainfinitesimally connected; (2) $E_{\mathcal{R}}\neq\emptyset$; (3) $D_{\mathcal{R}}\neq X$.

Proof. By Theorem 14.12, the assertion (1) holds if and only if $S_A\notin\mathcal{R}^{\blacktriangle}$ for all proper nonvoid subset A of X. Moreover, by Theorem 11.21, for any proper nonvoid subset A of X, we have $S_A\notin\mathcal{R}^{\blacktriangle}$ if and only if $E_{\mathcal{R}}\neq\emptyset$. Therefore, the assertions (1) and (2) are equivalent. Moreover, by Theorem 3.8, it is clear that the assertions (2) and (3) are also equivalent.

Remark 16.15. Later, we shall see that the inverse of a topologically, paratopologically, or parainfinitesimally connected relator need not have the same connectedness property.

However, the following counterpart of Theorem 13.10 shows that the inverse of an ultimately connected relator is still ultimately connected.

Theorem 16.16. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is ultimately connected;* *(2)* $\mathcal{R} = \{X^2\}$.

Proof. By Theorem 14.12, the assertion (1) holds if and only if $S_A \notin \mathcal{R}^{\blacklozenge}$ for all proper nonvoid subset A of X. Moreover, by Theorem 11.22, for a proper nonvoid subset A of X, we have $S_A \notin \mathcal{R}^{\blacklozenge}$ if and only if $\mathcal{R} = \{X^2\}$. Therefore, the assertions (1) and (2) are also equivalent.

Finally, by calling a relator σ–infinitesimally connected if it is $\star$–connected, we can also prove the following

Theorem 16.17. *If* $\mathcal{R}$ *is a relator on* X *and* $\operatorname{card}(X) > 1$, *then the following assertions are equivalent:*

(1) $\mathcal{R}$ *is* σ*–infinitesimally connected;*

(2) $\operatorname{card}(X) = 1$ *or* $\big(\operatorname{card}(X) = 2$ *and* $\{x\} \notin \mathcal{T}_{\mathcal{R}} \cap \mathcal{F}_{\mathcal{R}}$ *for all* $x \in X\big)$ *or* $\big(\operatorname{card}(X) = 3$ *and* $\{x\} \notin \mathcal{T}_{\mathcal{R}}$ *for all* $x \in X\big)$.

Proof. By Theorem 14.12, the assertion (1) holds if and only if $S_A \notin \mathcal{R}^{\star}$ for all proper nonvoid subset A of X. Moreover, by Theorem 11.23, for any $A \subset X$, we have $S_A \notin \mathcal{R}^{\star}$ if and only if $\big(\operatorname{card}(A) = 1$ and $A \notin \mathcal{T}_{\mathcal{R}}\big)$ or $\big(\operatorname{card}(X \setminus A) = 1$ and $A \notin \mathcal{F}_{\mathcal{R}}\big)$. Therefore, the assertions (1) and (2) are also equivalent.

Remark 16.18. From Theorem 14.3, by Remark 10.18, we can at once see that a relator $\mathcal{R}$ on X is ∞–connected if and only if it is properly connected. Therefore, the 'quasi-connectedness properties' of relators need not also be studied.

Moreover, from Theorem 14.12, by Remark 10.18, we can at once see that the 'almost uniform (almost proximal) and the superproximal (supertopological) connectedness properties' of relators need not also be studied.

However, by Remark 13.19, a relator $\mathcal{R}$ on X may naturally be called properly connected at a point $x \in X$ if the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is properly well-chained at x. Therefore, localized forms of the corresponding connectedness properties of relators may also be investigated.

17. RELATIONSHIPS BETWEEN WELL-CHAINEDNESS AND CONNECTEDNESS PROPERTIES

As an immediate consequence of the corresponding definitions, we can at once state the following

Theorem 17.1. *If $\mathcal{R}$ is a properly well-chained relator on X, then $\mathcal{R}$ is, in particular, properly connected.*

Proof. If $\mathcal{R}$ is properly well-chained, then $R^\infty = X^2$ for all $R \in \mathcal{R}$. Hence, since $R \subset R \cup R^{-1}$, it is clear that $\left(R \cup R^{-1} \right)^\infty = X^2$ for all $R \in \mathcal{R}$. Therefore, the relator $\mathcal{R} \triangledown \mathcal{R}^{-1}$ is also properly well-chained, and thus $\mathcal{R}$ is properly connected.

Remark 17.2. Later we shall see that even a parainfinitesimally connected relator need not be properly well-chained.

However, as an immediate consequence of the corresponding definitions, we still have the following

Theorem 17.3. *If $\mathcal{R}$ is a strongly symmetric relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is properly connected:* *(2)* *$\mathcal{R}$ is properly well-chained.*

Proof. If $\mathcal{R}$ is strongly symmetric, then $R^{-1} = R$, and hence $R \cup R^{-1} = R$ for all $R \in \mathcal{R}$. Therefore, $\mathcal{R} \triangledown \mathcal{R}^{-1} = \mathcal{R}$, and hence in particular $\left(\mathcal{R} \triangledown \mathcal{R}^{-1} \right)^\infty = \mathcal{R}^\infty$. Thus, the assertions (1) and (2) equivalent.

Concerning properly connected relators, we can also easily prove the following theorems.

Theorem 17.4. *If $\mathcal{R}$ is a uniformly filtered relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is properly connected;* *(2)* *$\mathcal{R} \vee \mathcal{R}^{-1}$ is properly well-chained.*

Proof. In this case, by Corollary 11.6, for any $A \subset X$ we have $S_A \in \mathcal{R}^*$ if and only if $R_A \in \left(\mathcal{R} \vee \mathcal{R}^{-1} \right)^*$. Therefore, by Theorems 14.3 and 12.3, the required assertions are also equivalent.

Theorem 17.5. *If $\mathcal{R}$ is a reflexive relator on X, then the following assertions are equivalent:*

(1) *$\mathcal{R}$ is properly connected;*

(2) *$\mathcal{R} \circledcirc \mathcal{R}^{-1}$ is properly well-chained;*

(3) *$\mathcal{R}^{-1} \circledcirc \mathcal{R}$ is properly well-chained.*

Proof. In this case, by Theorem 11.7, for any $A \subset X$ we have $S_A \in \mathcal{R}^*$ if and only if $R_A \in \left(\mathcal{R} \circledcirc \mathcal{R}^{-1} \right)^*$, or equivalently $R_A \in \left(\mathcal{R}^{-1} \circledcirc \mathcal{R} \right)^*$. Therefore, by Theorems 14.3 and 12.3, the required assertions are also equivalent.

Remark 17.6. The proper well-chainedness of the relator $\mathcal{R}^{-1} \circledcirc \mathcal{R}$, that is, the condition $\left(\mathcal{R}^{-1} \circledcirc \mathcal{R} \right)^\infty = \left\{ X^2 \right\}$, by Remark 12.2, means only that for any $x, y \in X$, with $x \neq y$, and any $R \in \mathcal{R}$ there exists a finite family $(x_i)_{i=0}^n$ in X such that $x_o = x$, $x_n = y$ and $(x_{i-1}, x_i) \in R^{-1} \circ R$, that is, $R(x_{i-1}) \cap R(x_i) \neq \emptyset$ for all $i = 1, \ldots, n$.

Theorem 17.7. *If $\mathcal{R}$ is a uniformly filtered reflexive relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly connected;

(2) $\mathcal{R}\circ\mathcal{R}^{-1}$ is properly well-chained;

(3) $\mathcal{R}^{-1}\circ\mathcal{R}$ is properly well-chained.

Proof. In this case, by Corollary 11.9, for any $A\subset X$ we have $S_A\in\mathcal{R}^{*}$ if and only if $R_A\in\big(\mathcal{R}\circ\mathcal{R}^{-1}\big)^{*}$, or equivalently $R_A\in\big(\mathcal{R}^{-1}\circ\mathcal{R}\big)^{*}$. Therefore, by Theorems 14.3 and 12.3, the required assertions are also equivalent.

As an immediate consequence of Theorem 17.1, we can at once state

Theorem 17.8. *If $\Box$ is a unary operation for relators on X and $\mathcal{R}$ is a $\Box$–well-chained relator on X, then $\mathcal{R}$ is, in particular, $\Box$–connected.*

Moreover, in addition to Theorem 17.3, we can also easily prove the following

Theorem 17.9. *If $\mathcal{R}$ is a quasi-proximally symmetric relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is proximally connected; *(2) $\mathcal{R}$ is proximally well-chained.*

Proof. In this case, by Theorem 8.5, we have $\bar{\tau}_{\mathcal{R}}=\tau_{\mathcal{R}}$, and hence $\tau_{\mathcal{R}}\cap\bar{\tau}_{\mathcal{R}}=\tau_{\mathcal{R}}$. Therefore, by Theorems 15.6, 12.8 and 13.3, the required assertions are equivalent.

From Theorem 17.9, by Theorems 15.3, 15.5 and 13.3, it is clear that in particular we also have the following

Corollary 17.10. *If $\mathcal{R}$ is a uniformly filtered and quasi-proximally symmetric relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is properly (uniformly) connected;

(2) $\mathcal{R}$ is properly (uniformly) well-chained.

Moreover, from Theorem 17.9, we can also easily get the following

Theorem 17.11. *If $\Box$ is a $\#$–invariant operation for relators on X and $\mathcal{R}$ is a quasi–$\Box$–symmetric relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is $\Box$–connected: *(2) $\mathcal{R}$ is $\Box$–well-chained.*

Proof. Since $\mathcal{R}$ is a quasi–$\Box$–symmetric, we have $\big(\mathcal{R}^{\Box\infty}\big)^{-1}=\mathcal{R}^{\Box\infty}$. Hence, since $\mathcal{R}^{\Box\#}=\mathcal{R}^{\Box}$, we can infer that $\Big(\big(\mathcal{R}^{\Box}\big)^{\#\infty}\Big)^{-1}=\big(\mathcal{R}^{\Box}\big)^{\#\infty}$. Therefore, the relator $\mathcal{R}^{\Box}$ is quasi-proximally symmetric.

Now, by Theorem 17.9, it is clear that the relator $\mathcal{R}^{\Box}$ is proximally connected if and only if it is proximally well-chained. That is, the relator $\mathcal{R}^{\Box\#}$ is properly connected if and only if it is properly well-chained. And hence, since $\mathcal{R}^{\Box\#}=\mathcal{R}^{\Box}$, it is clear that the required assertions are also equivalent.

Remark 17.12. From Theorem 17.11, in particular, it is clear that a quasi-topologically symmetric relator is topologically connected if and only if it is topologically well-chained.

On the other hand, from Theorems 17.1 and 15.14, we can at once see that a properly well-chained reflexive Lebesque relator is already topologically connected, despite that it need not be topologically well-chained by the following

Example 17.13. If $X = [\,0\,,\,1\,]$ and

$$R_n = \{\,(x,\,y) \in X^2 : \quad |\,x - y\,| < 1/n\,\}$$

for all $n \in \mathbb{N}$, then $\mathcal{R} = \{\,R_n\}_{n=1}^{\infty}$ is a properly well-chained reflexive Lebesgue relator on X such that $\mathcal{R}$ is not topologically well-chained.

Clearly, $\mathcal{R}$ is a properly filtered and strictly uniformly transitive tolerance relator on X. Moreover, by Theorem 8.24, it is clear that $\mathcal{R}$ is topologically compact. Therefore, by Theorem 8.27, $\mathcal{R}$ is a Lebesgue relator. Moreover, by Remark 12.2 and Theorem 13.5, it is clear that $\mathcal{R}$ is properly, but not topologically well-chained.

Finally, we note that, by Theorems 16.16, 13.10 and 13.14 and Proposition 12.6, we also have the following

Theorem 17.14. *If $\mathcal{R}$ is a relator on X, then the following assertions are equivalent:*

(1) $\mathcal{R}$ is ultimately connected; *(2) $\mathcal{R}$ is ultimately well-chained.*

Remark 17.15. The latter theorem can also be easily derived from Theorem 17.11 since the operation ♦ is invariant under the operations $\#$ and -1.

18. A FEW ILLUSTRATING EXAMPLES

The following example shows that even a parainfinitesimally connected preorder relator need not be properly well-chained.

Example 18.1. If $X = \{\,1\,,\,2\,\}$ and $R \subset X^2$ such that $R(1) = \{1\}$ and $R(2) = X$, then $\mathcal{R} = \{\,R\,\}$ is a parainfinitesimally connected preorder relator on X such that $\mathcal{R}$ is neither properly well-chained nor ultimately connected.

Note that $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} = R(1) \cap R(2) = \{1\} \neq \emptyset$. Therefore, by Theorem 16.14, $\mathcal{R}$ is parainfinitesimally connected. But, $R^{\infty} = R \neq X^2$. Therefore, by Definition 12.1, $\mathcal{R}$ is not properly well–chained. Moreover, by Theorem 16.16, $\mathcal{R}$ is not ultimately connected.

The following example shows that even a uniformly connected preorder relator need not be proximally connected. Moreover, the hypotheses of Theorems 15.5 and 17.3 cannot be significantly weakened.

Example 18.2. If $X = \{\,1\,,\,2\,\}$ and $R \subset X^2$ such that $R(1) = \{\,1\,\}$ and $R(2) = X$, then then $\mathcal{R} = \big\{\,R\,,\,R^{-1}\big\}$ is a uniformly connected, proximally filtered and properly symmetric preorder relator on X such that $\mathcal{R}$ is neither properly well-chained nor proximally connected.

Note that $S_{\{1\}} = S_{\{2\}} = \Delta_X$ is not contained in $\mathcal{R}^*$. Therefore, by Theorem 14.12, $\mathcal{R}$ is uniformly connected. But, $\{1\} \in \tau_{\mathcal{R}}$, and moreover $\{2\} \in \tau_{\mathcal{R}}$, and hence $\{1\} = X \setminus \{2\} \in \bar{\tau}_{\mathcal{R}}$. Therefore, by Theorem 15.6, $\mathcal{R}$ is not proximally connected. The required filteredness property of $\mathcal{R}$ is immediate from the fact that $\Delta_X \in \mathcal{R}^{\#}$.

The following two examples show that even a paratopologically connected tolerance or preorder relator need not be infinitesimally connected. Therefore, the counterpart of Corollary 13.15 with 'well-chained' replaced by 'connected' is not true.

Example 18.3. If $X = \{1, 2, 3\}$ and $R_i \subset X^2$ for $i = 1, 2$ such that

$$R_1(1) = X, \qquad R_1(2) = \{1, 2\}, \qquad R_1(3) = \{1, 3\},$$

$$R_2(1) = \{1, 2\}, \qquad R_2(2) = X, \qquad R_2(3) = \{2, 3\},$$

then $\mathcal{R} = \{R_1, R_2\}$ is a proximally well-chained and paratopologically connected tolerance relator on X such that $\mathcal{R}$ is neither topologically well-chained nor infinitesimally connected.

Note that $\tau_{\mathcal{R}} = \{\emptyset, X\}$. Therefore, by Theorems 12.8 and 13.3, $\mathcal{R}$ is proximally well-chained. But, $\{1, 2\} \in \mathcal{T}_{\mathcal{R}}$. Therefore, by Theorem 13.5, $\mathcal{R}$ is not topologically well-chained.

Moreover, we evidently have $R_i(x) \cap R_j(y) \neq \emptyset$ for all $i, j \in \{1, 2\}$ and $x, y \in X$. Therefore, by Theorem 16.5, $\mathcal{R}$ is paratopologically connected. But, $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = S_{\{3\}}$, and hence $S_{\{3\}} \in \{\rho_{\mathcal{R}}^{-1}\}^* = \mathcal{R}^\bullet$. Therefore, by Theorem 14.12, $\mathcal{R}$ is not infinitesimally connected.

Example 18.4. If $X = \{1, 2, 3\}$ and $R_i \subset X^2$ for all $i \in X$ such that

$$R_1(1) = X, \qquad R_1(2) = \{2, 3\}, \qquad R_1(3) = \{2, 3\},$$

$$R_2(1) = \{1, 3\}, \qquad R_2(2) = X, \qquad R_2(3) = \{1, 3\},$$

$$R_3(1) = \{1, 2\}, \qquad R_3(2) = \{1, 2\}, \qquad R_3(3) = X,$$

then $\mathcal{R} = \{R_1, R_2, R_3\}$ is a paratopologically connected preorder relator on X such that $\mathcal{R}$ is neither properly well-chained nor infinitesimally connected.

Note that $R_i(x) \cap R_j(y) \neq \emptyset$ for all $i, j \in X$ and $x, y \in X$. Therefore, by Theorem 16.5, $\mathcal{R}$ is paratopologically connected. But, $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = \Delta_X$, and hence $\Delta_X \in \{\rho_{\mathcal{R}}^{-1}\}^* = \mathcal{R}^\bullet$. Therefore, by Theorem 14.12, $\mathcal{R}$ is not infinitesimally connected.

The following example shows that even an infinitesimally well-chained and ultrainfinitesimally connected reflexive relator need not be parainfinitesimally connected.

Example 18.5. If $X = \{1, 2, 3\}$ and $R \subset X^2$ such that

$$R(1) = \{1, 2\}, \qquad R(2) = \{2, 3\}, \qquad R(3) = \{1, 3\},$$

then $\mathcal{R} = \{R\}$ is an infinitesimally well-chained and ultrainfinitesimally connected reflexive relator on X such that $\mathcal{R}$ is neither paratopologically well-chained nor parainfinitesimally connected.

Note that $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = R^{-1}$. Moreover, $R^2 = X^2$, and hence $R^{\infty} = X^2$. Therefore, $\rho_{\mathcal{R}}^{\infty} = \big(R^{-1}\big)^{\infty} = \big(R^{\infty}\big)^{-1} = X^2$, and thus by Theorem 13.16 $\mathcal{R}$ is infinitesimally well-chained. But, by Theorem 13.10, $\mathcal{R}$ is not paratopologically well-chained.

Moreover, $R^{-1}(1) = \{1, 3\}$, $R^{-1}(2) = \{1, 2\}$ and $R^{-1}(3) = \{2, 3\}$. Therefore, $\rho_{\mathcal{R}}(A) = R^{-1}(A) = X$ for all $A \subset X$ with $\operatorname{card}(A) = 2$. Thus, by Theorem 16.13, $\mathcal{R}$ is ultrainfinitesimally connected. But, now we also have $E_{\mathcal{R}} = \bigcap \mathcal{E}_{\mathcal{R}} = \bigcap_{i=1}^{3} R(i) = \emptyset$. Therefore, by Theorem 16.14, $\mathcal{R}$ is not parainfinitesimally connected.

The following example shows that even an infinitesimally well-chained tolerance relator need not be paratopologically connected.

Example 18.6. If $X = \{i\}_{i=1}^{4}$ and $R \subset X^2$ such that

$$R(1) = \{1, 2\}, \qquad R(2) = \{1, 2, 3\}, \qquad R(3) = \{2, 3, 4\}, \qquad R(4) = \{3, 4\},$$

then $\mathcal{R} = \{R\}$ is an infinitesimally well-chained and infinitesimally connected tolerance relator on X such that $\mathcal{R}$ is neither paratopologically well-chained nor paratopologically connected.

Note that $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = R^{-1} = R$. Moreover, $R^3 = X^2$, and hence $R^{\infty} = X^2$. Therefore, $\rho_{\mathcal{R}}^{\infty} = X^2$, and thus by Theorem 13.16 $\mathcal{R}$ is infinitesimally well-chained. And hence, by Theorem 17.8, $\mathcal{R}$ is infinitesimally connected. But, $R(1) \cap R(4) = \emptyset$, and thus by Theorem 16.5 $\mathcal{R}$ is not paratopologically connected. Moreover, by Theorem 13.10, $\mathcal{R}$ is not paratopologically well-chained.

The following example shows that even a topologically well-chained tolerance relator need not be infinitesimally connected.

Example 18.7. If $X = \{i\}_{i=1}^{4}$ and $R_i \subset X^2$ for $i = 1, 2$ such that

$$R_1(1) = \{1,2,3\}, \quad R_1(2) = \{1,2,4\}, \quad R_1(3) = \{1,3,4\}, \quad R_1(4) = \{2,3,4\},$$

$$R_2(1) = \{1,2,4\}, \quad R_2(2) = \{1,2,3\}, \quad R_2(3) = \{2,3,4\}, \quad R_2(4) = \{1,3,4\},$$

then $\mathcal{R} = \{R_1, R_2\}$ is a topologically well-chained and topologically connected tolerance relator on X such that $\mathcal{R}$ is neither infinitesimally well-chained nor infinitesimally connected.

Note that $\mathcal{T}_{\mathcal{R}} = \{\emptyset, X\}$. Therefore, by Theorem 13.5, $\mathcal{R}$ is topologically well-chained. And hence, by Theorem 17.8, $\mathcal{R}$ is topologically connected. But, if $A = \{1, 2\}$, then $\rho_{\mathcal{R}}^{-1} = \bigcap \mathcal{R} = S_A$, and hence $S_A \in \big\{\rho_{\mathcal{R}}^{-1}\big\}^{*} = \mathcal{R}^{\bullet}$. Therefore, by Theorem 14.12, $\mathcal{R}$ is not infinitesimally connected. And thus, by Theorem 17.8, $\mathcal{R}$ is not infinitesimally well-chained.

The following example shows that even a proximally well-chained tolerance relator need not be topologically connected.

Example 18.8. If $X = \{i\}_{i=1}^{4}$ and $R_i \subset X^2$ for all $i \in X$ such that

$$R_1(1) = \{1, 2\}, \qquad R_1(2) = X, \qquad R_1(3) = R_1(4) = \{2, 3\ 4\},$$

$$R_2(1) = X, \qquad R_2(2) = \{1, 2\}, \qquad R_2(3) = R_2(4) = \{1, 3, 4\},$$

$$R_3(1) = R_3(2) = \{1, 2, 4\}, \qquad R_3(3) = \{3, 4\}, \qquad R_3(4) = X,$$

$$R_4(1) = R_4(2) = \{1, 2, 3\}, \qquad R_4(3) = X, \qquad R_4(4) = \{3, 4\},$$

then $\mathcal{R} = \{ R_i \}_{i=1}^{4}$ is a proximally well-chained and proximally connected tolerance relator on X such that $\mathcal{R}$ is neither topologically well-chained nor topologically connected.

Note that $\tau_{\mathcal{R}} = \{ \emptyset , X \}$. Therefore, by Theorems 12.8 and 13.3, $\mathcal{R}$ is proximally well-chained. And hence, by Theorem 17.8, $\mathcal{R}$ is proximally connnected. But, $\{1, 2\} \in \mathcal{T}_{\mathcal{R}}$, and moreover $\{3, 4\} \in \mathcal{T}_{\mathcal{R}}$, and hence $\{1, 2\} \in \mathcal{F}_{\mathcal{R}}$. Therefore, by Theorem 15.9, $\mathcal{R}$ is not topologically connected. And hence, by Theorem 17.8, $\mathcal{R}$ is not topologically well-chained.

The following example shows that the inverse of even a topologically well-chained, paratopologically and infinitesimally connected reflexive relator need not be topologically well-chained.

Example 18.9. If $X = \{ 1 , 2 , 3 \}$ and $R_i \subset X^2$ for $i = 1 , 2$ such that

$$R_1(1) = \{ 1 , 2 \}, \qquad R_1(2) = \{ 2 , 3 \}, \qquad R_1(3) = X ,$$

$$R_2(1) = \{ 1 , 3 \}, \qquad R_2(2) = \{ 2 , 3 \}, \qquad R_2(3) = X ,$$

then $\mathcal{R} = \{ R_1, R_2 \}$ is a topologically well-chained, paratopologically and infinitesimally connected reflexive relator on X such that $\mathcal{R}$ is neither infinitesimally well-chained nor ultrainfinitesimally connected. Moreover, $\mathcal{R}^{-1}$ is a proximally well-chained and parainfinitesimally connected reflexive relator on X such that $\mathcal{R}^{-1}$ is neither topologically well-chained nor ultimately connected.

Note that $\mathcal{T}_{\mathcal{R}} = \{ \emptyset , X \}$. Therefore, by Theorem 13.5, $\mathcal{R}$ is topologically well-chained. Hence, by Remark 13.2 and Theorem 12.17, $\mathcal{R}^{-1}$ is proximally well-chained. Moreover, $R_i(x) \cap R_j(y) \neq \emptyset$ for all $i , j \in \{ 1 , 2 \}$ and $x , y \in X$. Therefore, by Theorem 16.5, $\mathcal{R}$ is paratopologically connected.

Moreover, it can be easily seen that

$$R_1^{-1}(1) = \{ 1 , 3 \}, \qquad R_1^{-1}(2) = X , \qquad R_1^{-1}(3) = \{ 2 , 3 \};$$

$$R_2^{-1}(1) = \{ 1 , 3 \}, \qquad R_2^{-1}(2) = \{ 2 , 3 \}, \qquad R_2^{-1}(3) = X .$$

Hence, since $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = R_1^{-1} \cap R_2^{-1}$, it is clear that

$$\rho_{\mathcal{R}}(1) = \{ 1 , 3 \} \qquad \text{and} \qquad \rho_{\mathcal{R}}(2) = \rho_{\mathcal{R}}(3) = \{ 2 , 3 \}.$$

Therefore, $\{ 2 , 3 \} \cap \rho_{\mathcal{R}}(1) \neq \emptyset$, $\{ 1 , 3 \} \cap \rho_{\mathcal{R}}(2) \neq \emptyset$ and $\{ 1 , 2 \} \cap \rho_{\mathcal{R}}(3) \neq \emptyset$. Thus, by Theorem 16.12, $\mathcal{R}$ is infinitesimally connected.

Moreover, we can also easily see that

$$E_{\mathcal{R}^{-1}} = \bigcap \mathcal{E}_{\mathcal{R}^{-1}} = \bigcap \{ R_i^{-1}(x) : \quad x \in X , \quad i \in \{ 1 , 2 \} \} = \{ 3 \} \neq \emptyset .$$

Therefore, by Theorem 16.14, $\mathcal{R}^{-1}$ is parainfinitesimally connected.

On the other hand, if $A = \{ 1 \}$, then we can at once see that $\rho_{\mathcal{R}}(A) = \{ 1 , 3 \}$ and $\rho_{\mathcal{R}}(X \setminus A) = \{ 2 , 3 \}$. Thus, by Theorem 16.13, $\mathcal{R}$ is not ultrainfinitesimally connected. Moreover, we can also at once see that $\{ 2 , 3 \} \in \mathcal{T}_{\mathcal{R}^{-1}}$. Therefore, by Theorem 13.5, $\mathcal{R}^{-1}$ is not topologically well-chained. Thus, by Remark 13.2 and Theorem 12.17, $\mathcal{R}$ is not infinitesimally well-chained. Finally, by Theorem 16.16, it is clear that $\mathcal{R}^{-1}$ is not ultimately connected.

Remark 18.10. From Example 18.9 we can at once see that the inverse of even a proximally well-chained and parainfinitesimally connected reflexive relator need not be either infinitesimally well-chained or ultrainfinitesimally connected.

Moreover, by using Example 18.4, we can easily show that the inverse of even a paratopologically connected preorder relator need not be topologically connected. Therefore, neither the topological nor the paratopological connectedness is inverse invariant.

Example 18.11. If X and $\mathcal{R}$ are as in Example 18.4, then $\mathcal{R}$ is a paratopologically connected preorder relator on X such that $\mathcal{R}^{-1}$ is a preorder relator on X such that $\mathcal{R}^{-1}$ is not even topologically connected.

Note that, under the notation of Example 18.4, we have

$$\begin{array}{lll} R_1^{-1}(1)=\{1\}, & R_1^{-1}(2)=X, & R_1^{-1}(3)=X, \\ R_2^{-1}(1)=X, & R_2^{-1}(2)=\{2\}, & R_2^{-1}(3)=X, \\ R_3^{-1}(1)=X, & R_3^{-1}(2)=X, & R_3^{-1}(3)=\{3\}. \end{array}$$

Hence, since $\mathcal{R}^{-1}=\{R_1^{-1}, R_2^{-1}, R_3^{-1}\}$, we can see that $\{1\}\in\mathcal{T}_{\mathcal{R}^{-1}}$, and moreover $\{2,3\}\in\mathcal{T}_{\mathcal{R}^{-1}}$, and hence $\{1\}\in\mathcal{F}_{\mathcal{R}^{-1}}$. Therefore, by Theorem 15.9, $\mathcal{R}^{-1}$ is not topologically connected.

The following example shows that the hypothesis of the reflexivity of the relator $\mathcal{R}$ in Theorem 14.10 is essential.

Example 18.12. If $X=\{1,2\}$ and $R=X\times\{1\}$, then $\mathcal{R}=\{R\}$ is a parainfinitesimally connected, strongly transitive relator and $\mathcal{S}=\{\Delta_X\}$ is an equivalence relator on X such that $\mathcal{S}$ is properly refined by $\mathcal{R}$, but $\mathcal{S}$ is not properly connected.

Note that $E_{\mathcal{R}}=\bigcap\mathcal{E}_{\mathcal{R}}=R(1)\cap R(2)=\{1\}\neq\emptyset$. Therefore, by Theorem 16.14, $\mathcal{R}$ is parainfinitesimally connected. Moreover, $f=R$ is a function of X into itself such that $\Delta_X\circ f=R\in\mathcal{R}$. Therefore, $\mathcal{S}$ is properly refined by $\mathcal{R}$. But, for instance, $S_{\{1\}}=\Delta_X\in\mathcal{S}\subset\mathcal{S}^*$, and thus by Theorem 14.3 $\mathcal{S}$ is not properly connected.

The following example shows that the reflexivity of the relator $\mathcal{S}$ in Theorem 14.10 cannot be stated.

Remark 18.13. If $X=\{1,2\}$ and $S\subset X^2$ such that $S(1)=\{2\}$ and $S(2)=X$, then $\mathcal{R}=\{X^2\}$ is a paratopologically well-chained and ultimately connected equivalence relator and $\mathcal{S}=\{S\}$ is an infinitesimally well-chained and parainfinitesimally connected strongly symmetric relator on X such that $\mathcal{S}$ is properly refined by $\mathcal{R}$, but $\mathcal{S}$ is not reflexive.

By Theorems 13.10 and 16.16, it is clear that the relator $\mathcal{R}$ is paratopologically well-chained and ultimately connected. On the other hand, we can easily see that $\rho_{\mathcal{S}}=\bigcap\mathcal{S}^{-1}=S^{-1}=S$. Moreover, $S^2=X^2$, and hence $S^\infty=X^2$. Therefore, $\rho_{\mathcal{S}}^\infty=X^2$, and thus by Theorem 13.16 $\mathcal{S}$ is infinitesimally well-chained. Moreover, it is clear that $E_{\mathcal{R}}=\bigcap\mathcal{E}_{\mathcal{S}}=S(1)\cap S(2)=\{2\}\neq\emptyset$. Therefore, by Theorem 16.14, $\mathcal{S}$ is parainfinitesimally connected. Finally, we can observe that $f=X\times\{2\}$ is a function of X into itself such that $S\circ f=X^2\in\mathcal{R}$. Therefore, $\mathcal{S}$ is properly refined by $\mathcal{R}$. But, despite this, $\mathcal{S}$ is not reflexive.

The following similar example shows that the counterpart of Theorem 14.10 with ‘connected’ replaced by ‘well-chained’ is not true.

Example 18.14. If $X = \{1, 2\}$ and $S \subset X^2$ such that $S(1) = \{1\}$ and $S(2) = X$, then $\mathcal{R} = \{X^2\}$ is a paratopologically well-chained and ultimately connected equivalence relator and $\mathcal{S} = \{S\}$ is a parainfinitesimally connected preorder relator on X such that $\mathcal{S}$ is properly refined by $\mathcal{R}$, but $\mathcal{S}$ is not properly well-chained.

As a more delicate example of the above types, we can also at once state

Example 18.15. If $X = \{1, 2\}$ and $S_i \subset X^2$ for all $i \in X$ such that

$$S_1(1) = \{1\}, \qquad S_1(2) = X, \qquad \text{and} \qquad S_2(1) = X, \qquad S_2(2) = \{2\},$$

then $\mathcal{R} = \{X^2\}$ is a paratopologically well-chained and ultimately connected equivalence relator and $\mathcal{S} = \{S_1, S_2\}$ is a uniformly connected, properly filtered and properly symmetric preorder relator on X such that $\mathcal{S}$ is properly refined by $\mathcal{R}$, but $\mathcal{S}$ is neither properly well-chained nor proximally connected. Thus, in particular, the relator $\mathcal{S}^{\#}$ cannot be refined by $\mathcal{R}^{\blacklozenge}$.

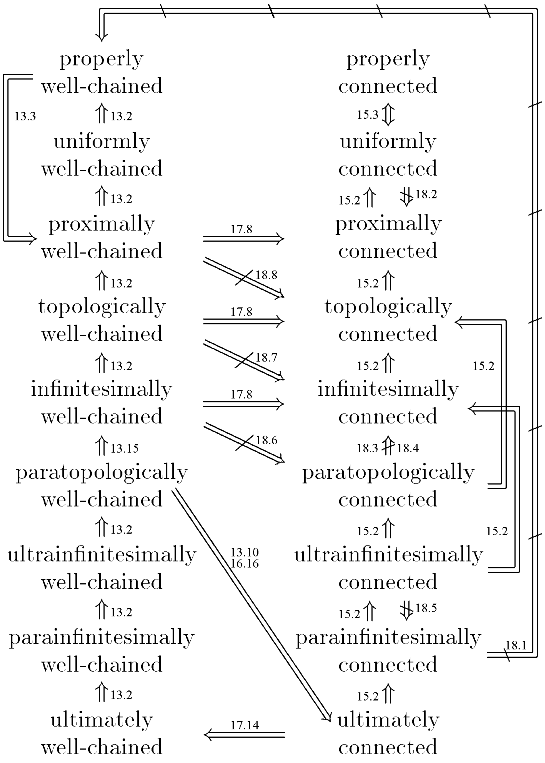
The following example shows that, in contrast to Corollaries 12.5 and 16.2, an infinitesimally connected relator need not be total.

Example 18.16. If $X = \{1, 2\}$ and $R = \{2\} \times X$, then $\mathcal{R} = \{R\}$ is an infinitesimally connected relator on X such that $\mathcal{R}$ is not total. Thus, in particular, $\mathcal{R}$ cannot be properly well-chained and paratopologically connected.

Note that $\rho_{\mathcal{R}} = \bigcap \mathcal{R}^{-1} = R^{-1} = X \times \{2\}$, and hence $\{2\} \cap \rho_{\mathcal{R}}(1) \neq \emptyset$. Therefore, by Theorem 16.12, $\mathcal{R}$ is infinitesimally connected. But, despite this $\mathcal{R}$ is not total.

19. A SUMMARY OF IMPLICATIONS

The relations between the various well-chainedness and connectedness properties of relators may be summarized by the following set of implications.


properly well-chained
uniformly well-chained
proximally well-chained
topologically well-chained
infinitesimally well-chained
paratopologically well-chained
ultrainfinitesimally well-chained
parainfinitesimally well-chained
ultimately well-chained
properly connected
uniformly connected
proximally connected
topologically connected
infinitesimally connected
paratopologically connected
ultrainfinitesimally connected
parainfinitesimally connected
ultimately connected
13.3
13.2
13.15
13.10
16.16
15.3
15.2
18.2
18.3
18.4
18.5
18.1
17.8
18.8
18.7
18.6
17.14

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