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PROPORTION THEORY IN GREEK MATHEMATICS

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Abstract. The concepts of ratio, equal ratio, proportion played an important role in ancient Greek mathematics. The Book V of Euclid's Elements contains a proportion theory of (geometrical) magnitudes based on the equal ratio definition of Eudoxus (Definition V.5). In this paper we study both the pre-Eudoxian theory of proportion, and the genesis of Eudoxus's definition.

1. Proportion theory in Elements

First recall the most important definitions and propositions of the Elements for ratio, equal ratio and proportion. Most of them are formulated separately both for numbers and magnitudes. The citations are from Heath's translation [2].

Magnitude itself was not explicitly defined, so handled as a common notion, but the number was: "A number is a multitude composed of units" (Def. VII.2). Definition V.4 shows that not all kinds of magnitudes can have ratio only the so-called Archimedean ones: "Magnitudes are said to *have a ratio* to one another which are capable, when multiplied, of exceeding one another." Circle segment is one example for non-Archimedean magnitude.

In case of ratio, the situation is opposite. The ratio for numbers was not defined, only for magnitudes: "A ratio is a sort of relation in respect of size between two magnitudes of the same kind." (Def. V.3)

Eudoxus's celebrated equal ratio definition for magnitudes sounds like this: "Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order" (Def. V.5).

Thus, if a, b, c, d are magnitudes (of the same kind), then $a : b = c : d$ holds, if and only if for any positive integers ("numbers" in Greek usage) n, m

$$ma \gtreqless nb \Longrightarrow mc \gtreqless nd.$$

The arithmetical books of the Elements (Books VII-IX) are held the oldest ones, preserving the Pythagorean number theory for us. Euclid defines the proportionality (equal ratio) of numbers, and gives a criterion for it in Book VII:

"Numbers are proportional when the first is the same multiple, or the same part, or the same parts of the second that the third is of the fourth" (Def. VII.20).

"If four numbers be proportional, the number produced from the first and fourth will be equal to the number produced from the second and third; and, if the number

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produced from the first and fourth be equal to that produced from the second and third, the four numbers will be proportional" (Prop. VII.19).

In modern algebraic language, this theorem can be reformulated like this: Let a, b, c, d be positive integers. The equality $a : b = c : d$ holds if and only if $a \cdot d = b \cdot c$. In his proof Euclid uses propositions from Book V (Props 7, 9) on magnitudes being in the same ratio.

It is worth mentioning that in Prop. VI.16 a criterion is given for the proportionality of four straight lines: "If four straight lines be proportional, the rectangle contained by the extremes is equal to the rectangle contained by the means; and, if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines will be proportional." In other words: the criterion for the straight lines a, b, c, d to be proportional $a : b = c : d$ is that $a \cdot d = b \cdot c$. Production of straight lines (generally: magnitudes) was impossible for Greeks, in fact for all mathematicians before Descartes, so Euclid used the "equal area" language instead.

These equal ratio definitions are clearly older than that of Eudoxus's, whose secret of the success lies in the following facts. It does not use any disputed concepts of Greek mathematics, such as infinite divisibility, commensurability, number, ratio, etc., only those ones (greater, equal, and smaller) which are suitable to compare two equimultiples of magnitudes. Definition V.4 seemingly is that of ratio, but actually prescribes the existence of potential increasing infinity, which was generally accepted for both numbers and magnitudes. Eudoxus generalized the former equal ratio definitions for number and lines, so the theorems proved by them remained valid. It annuls only those theorems, which took all magnitudes commensurable. These theorems remain valid only for commensurable magnitudes, especially for numbers.

"Similar rectilineal figures are such as have their angles severally equal and the sides about the equal angles proportional." – says Definition VI.1. To prove all theorems in the book, Euclid could have used only Eudoxus's definition, so to give a special "proportional" definition for straight lines in Prop. VI.16 appears useless, unless Euclid wanted to preserve some former and simpler proofs. This also confirms that must have existed pre-Eudoxian theories of proportions.

2. Development of Greek proportion theory

Since the only original source of Greek mathematics is Euclid's Elements, and even Eudoxus lived before Euclid, it is very hard to reconstruct the development. The other sources, such as Plato and Aristotle allusions, fragments of some mathematical works, and Proclus's commentaries allow different deductive reconstructions. In the following we also give a possible reconstruction. To support our train of thought, we refer mainly to Árpád Szabó's book, as well as to the Elements, and the following words of Proclus, a scholiast to Book X of the Elements, and Diogenes Laertius:

"The theory of commensurable magnitudes is developed primarily by arithmetic and then by geometry in imitation of it. This is why both sciences define commensurable magnitudes as those which have to one another the ratio of a number to a number, and this implies that commensurability exists primarily in numbers." [6, p. 49]

Euclid was "...Collecting many of the theorems of Eudoxus, perfecting many others by Theaetetus, and bringing *irrefragable* demonstrations the propositions which had only been somewhat loosely proved by his predecessors." [6, p. 8]

"The Pythagoreans were the first to make inquiry into commensurability, having first discovered it as a result of their observation of numbers; for though the unit is a common measure of all numbers, they could not find a common measure of all magnitudes.” [8, p. 215]

“He adds that Pythagoras specially applied himself to the arithmetical aspects of geometry, and he discovered the musical intervals on the monochord.” [8, p. 175]

We think that the so-called antanairesis method played a key role in the development of Greek mathematics, especially in proportion theory. Antanairesis was used for finding a common measure of two numbers or two magnitudes (e.g. two straight lines). Subtract in each step some multiple of the smaller from the greater, so that the remainder is not greater than the smaller. The procedure ends when the smaller measures the greater, i.e. some multiple of it is equal to the greater. This last remainder gives the greatest common measure of the two numbers or magnitudes. The unit (one) is a common measure of any two numbers, if they prime to one another it is the only common measure, as Def. VII.12 states. This means that any two numbers are commensurable; the antanairesis is a finite procedure.

In our opinion the development of proportion theory (generally: the development of Greek mathematics) could have happened in three phases. We also think that these phases are reflected in Euclid’s Elements.

2.1. **Pre-antanairesis stage.** Arithmetic and geometry developed separately. Arithmetic studied the properties of the discrete number, while geometry those of the continuous magnitude. Numbers were represented by rows or figures of pebbles, antanairesis was applied for numbers. To study their proportions they used Def. VII.20 or later its simpler equivalent Prop. VII.19.

Geometry was purely synthetic, so to define similarity with ratio of lines was impossible. Two circles were considered similar intuitively, while for their segments we find the following definition (Def. III.11): “Similar segments of circles are those which admit equal angles, or in which the angles are equal to one another.” This “equal angles” definition works for all regular polygons, for any triangles, but not for rectangles.

We know that the Greek word for equal (isos) had two meanings in this respect: congruent, or of equal area. The first meaning was used in case of regular polygons and triangles (see, for instance Prop. I.4), while the second in other cases, when ”equal and similar” expressed congruency (see e.g. Props. VI.29, XII.3). The central problem of geometry was the squaring (dynamis, tetragonism): to find a square equal to a given rectilinear figure. Theorems of this kind can be found in Books I-II of Elements. As Á. Szabó showed in [7], the word “dynamis” is a very old one, so these theorems must also be old.

Proposition II.11 is perhaps the most important in this respect: “To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment.” In strict sense, it is not a theorem but a constructional problem. The solution uses Props. I.46, I.47 (the Pythagorean theorem), and II.6. By II.11, the Pythagoreans could construct their symbolical regular pentagon without the concept of similarity (see Props. IV.10–14).

If the straight line is denoted by AB, the desired cutting point by C, then by II.11: $AB \cdot CB = AC^2$, or algebraically $ac = b^2$, where $AB = a$, $AC = b$, $CB = c$. Using the same notations, II.7 states for any point C on AB: $a^2 + c^2 = 2ac + b^2$. Thus, if especially fulfills the condition of II.11, then $a^2 + c^2 = 3b^2$.

There is another remarkable proposition from Book II, namely II.14: “To construct a square equal to a given rectilinear figure.” This is a general squaring problem solved by Props. I.45, I.47, and II.5. We note, that to recognize the close connection between II.11 and II.14 was not possible in this phase.

2.2. **Pro-antanairesis phase.** We know from Plato that the Greeks were convinced on the commensurability of any two geometric quantities: "Again, what of the relations of line and surface to volume, or of surface and line to another: do not all we Greeks imagine that they are commensurable in some way or other?" [5, vii 820d; 12, p. 25]

Applying the antanairesis to geometry the Pythagoreans could think they had found the proper method to verify this conviction. If any two magnitudes are commensurable, then any ratio is "expressible" as the ratio of two numbers. Consequently, any length, area, volume are expressible by numbers. Thus, number is the first principle, not the point or line, arithmetic surpasses geometry. They introduced arithmetic concepts in geometry trying to unite arithmetic and geometry.

According to Aristotle's Metaphysics, the Pythagoreans developed a number-based philosophy. Finding different means became interpretable for magnitudes, too, not only for numbers as before. Instead of squaring, finding different means was the basic problem in geometry. They recognized that (Aristotle: Met. 996b): "what squaring a rectangle is, viz. that is the finding of a mean."

The Pythagoreans used arithmetic, harmonic and geometrical means in music theory. Then studied those conditions in arithmetic under which these means are again numbers. That time the term "geometric mean" was not used, only the one: mean proportional. Á. Szabó is right saying in [7, p. 177] : "...to find a numerical geometric mean between the two numbers ... may have led, by way of arithmetic, first to the problem of geometrical similarity (for rectilinear figures) and then to the geometrical construction of a mean proportional ... "

Proportionality in arithmetic and similarity in geometry became gradually subtypes of the general equal ratio concept in the unified Pythagorean mathematics. Since every ratio was supposed to be expressible in numbers, so they could have stated a proportion like this: magnitudes have to one another a ratio which a number has to a number. The proof must have been based on Def. VII.20.

In this "similarity" phase, both II.11 and II.14 were reformulated, as well as their connection was noticed: "To cut a given finite straight line in extreme and mean ratio." (Prop. VI.30); "To two given straight lines to find a mean proportional." (Prop. VI.13) The figures of the proofs are the same as those of their "ancestors" in Book II. The difference is only in the use of similarity concept.

The equality $a^2 + c^2 = 3b^2$ obtained by applying II.11 to II.7 also has a counterpart in the Elements, namely Prop. XIII.4. The statements, the graphs of the proofs are the same in both propositions, except the use of the phrase "extreme and mean ratio". Hard to explain why Euclid did not use II.7 in his proof.

Prop. VI.13 offers an easy tool to make a construction required by VI.30. If our train of thought is correct, II.14 must antedate VI.13. Moreover, dynamis must be an earlier concept than mean proportional line, on the contrary of the generally accepted views.

Finding a mean proportional between two lines is always possible by Prop. VI.13, but it is not the case for numbers. Propositions VIII.18 and 20 give the criterion for the existence of a mean proportional number: the two numbers must be similar plane numbers. The square number clearly is a special plane number; nevertheless, Euclid formulates another proposition (VIII.11) for them. This reflects the former special role of squares in first phase similarity.

The phrase "similar plane numbers" sounds strange for us, but not for the Pythagoreans, as we can see from the definitions of Book VII. Since every length, area and volume are numbers, so a rectangle (its area), as a number is a product of two numbers, namely of its sides. The product itself was called plane number (Def.

VII.16). In case of a square the product's name was square number (Def. VII.18). Solid and cube numbers were defined similarly (Defs. VII.17 and 19).

"Similar plane and solid numbers are those which have their sides proportional" – says Def. VII.21. Believing in the commensurability of any two magnitudes, by proportionality the Pythagoreans might have meant the following. Let a, b, c, d be magnitudes of the same kind, and let n, m, q, r be numbers. Further, let $a : b = n : m$, $c : d = q : r$. The equality $a : b = c : d$ holds if and only if $n : m = q : r$, the criterion of which by Prop. VII.19: $nr = mq$. Let us call this proposition concept "equal product" one. To prove Prop. VI.31, the generalization of the Pythagorean theorem to similar rectangles, is also possible by this concept.

The arithmetic version of the Pythagorean theorem is also can be found in the Elements, strangely enough in Book X: To find two square numbers such that their sum is also square (Lemma 1 after Prop. 28). The proof is based on three theorems from Book IX, and Prop. II.6.

2.3. **Incommensurability phase.** Some Pythagorean (Hippasus?) discovered that the antanairesis procedure is endless when applied to the diagonal and side of the square. The existence of lines whose lengths or ratio is "inexpressible" by numbers terminated the dominant role of number in mathematics. The concept of magnitude took over its role. The line, plane figure, solid body exist, but their length, area, volume are numbers only in special cases. Their ratios can only be expressed in numbers, when they are commensurable (see Props. X.5–6). It is remarkable that Euclid used Def. VII. 20 to show the proportionality (equal ratio) of two magnitudes and two numbers.

The discovery of incommensurable lines invalidated the geometrical proofs involving the concept of proportion (similarity). They remained valid only for commensurable magnitudes, especially for numbers together the equal product conception of proportionality. The need for a new proportion concept, valid for both commensurable and incommensurable magnitudes, has arisen.

Á. Szabó quotes in [7, p. 101] a same ratio definition from Aristotle Topics 158b: "Those magnitudes are in the same ratio whose anthyphairesis is the same". This anthyphairesis (antanairesis) definition is allegedly attributed to Theaetetus, a contemporary of Eudoxus.

The antanairesis definition of equal ratio was born probably to avoid the difficulties with incommensurable straight lines. Some historians share this view, some others, including Á. Szabó, reject it. Anyway, with this definition, many properties of proportions found in Books V and VI can be proved for straight lines, but not all, for instance Prop. VI.12 and VI.13. Omar Khajjam's attempt to prove Prop. 13 by this definition is discussed in [3]. With Eudoxus's equal ratio definition, Euclid could give "irrefragable" proofs for all properties of proportions, and all former theorems containing similarity concept.

In Books I–IV the word "similar" occurs only in connection with circle segments, since being non-Archimedean magnitudes the problem of commensurability is meaningless for them. As mentioned before, Euclid gave two proofs for many theorems: one in Book I–IV without proportion, and one later using proportion theory. Euclid's idea behind this strange doubling probably was: giving first a proof (found by his predecessors, or by him) using not debated concepts helps to accept and understand the new proofs including new, debated concepts.

Putting the earlier Pythagorean arithmetic after Book V (Books VII–IX) seems not to be logical, since Euclid does not use any results from the previous books except some properties of proportions of magnitudes. He applied them for numbers without any proof. On the other hand, to some other theorems on proportions he gave two proofs: one in Book V for magnitudes by Def. V.5, and one in Book VII for numbers by Def. VII.20. This shows that Euclid was not consequent in taking number (and straight line, too) as special magnitude.

Euclid's Elements reflects the influence of Aristotle's philosophy, so let us quote him on the relation between number and magnitude [1, An. Post. I.6-7 75a]:

"You cannot therefore when proving a thing pass form one genus to another; e.g. you cannot prove a geometrical proposition by arithmetic. ... Now the things from which the proof starts (the axioms) may be the same (whatever the subject); but where the genus is different, say in arithmetic and geometry, it is not possible to apply the arithmetical demonstrations to the properties of magnitudes *unless the magnitudes are numbers*."

For us any geometrical magnitude is simply a real number, positive integers are special real numbers, so all statements for real numbers remain valid for positive integers, but the converse is not true. In the Elements, both magnitudes and numbers were symbolised by straight lines, and Euclid handled straight lines mainly also as special magnitudes.

Many theorems in Book VII concerning antanairesis have a correspondent in Book X. Comparing them one can recognize that in some cases the latter simply a translation of the former using the following vocabulary: number – magnitude, prime to one another - incommensurable, not prime to one another – commensurable. Take for example propositions VII.1 and X.2, as well as VII.2 and X.3 (their numbering also suggests parallelism).

Prop. VII.1: Two unequal numbers being set out, and the less being continually subtracted from the greater, if the number which is left never measures the one before it until an unit is left, the original numbers will be prime to one another.

Prop. X.2: If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.

Prop. VII.2: Given two numbers not prime to one another, to find their greatest common measure.

Prop. X.3: Given two commensurable magnitudes to find their greatest common measure.

The argumentation and even the wording of the proofs of these counterparts are nearly the same. In case of Props. VII.2 and X.3 the porisms are also the same: From this it is manifest that if a number measures two numbers, it will also measure their greatest common measure (VII.2). From this it is manifest that if a magnitude measures two magnitudes, it will also measure their greatest common measure. (X.2) The proofs of Book VII theorems use only generally accepted concepts, but those of Book X require the debated exhaustion principle, which is the mathematical form of infinite divisibility. Euclid proved this principle in Prop. X.1 by Def. V.4, which is also behind Def V.5, Eudoxus's equal ratio definition.

2.4. **Genesis of Eudoxus's equal ratio definition.** To the best of my knowledge, nobody has studied the deduction of Eudoxus's, except V.J. Katz, who writes in his excellent book [4, p. 80]: "It is not known what inspired Eudoxus to his new definition, but a reasonable guess can be made." Katz derives Eudoxus's definition from Theaetetus's one, in spite his negative judgment on the latter: "for specific magnitudes this definition may be awkward to use." In my opinion, the argumentation of his deduction is not convincing enough.

We will show that Eudoxus's new equal ratio concept originated a former one: namely from Prop. VII.19. Eudoxus could not simply rewrite it for magnitudes, since to multiply magnitudes was unimaginable for him. Nevertheless, the idea might have come from it in the following manner. If n, m, q, r are arbitrary numbers, there are three cases for their ratios $n : m$ and $q : r$, namely: $n : m > q : r$, $n : m = q : r$, $n : m < q : r$, or equivalently: $nr > mq$, $nr = mq$, $nr < mq$. Combining the cases we have:

$$n : m \gtreqless q : r, \text{ or } nr \gtreqless mq.$$

Now replace the numbers q and r by the magnitudes a and b, respectively, and substitute in the above relations:

$$n : m > a : b, \quad n : m = a : b, \quad n : m < a : b, \quad nb > ma, \quad nb = ma, \quad nb < ma.$$

If a and b are commensurable, then there exists exactly one n and m for which the equality holds, while if they are incommensurable, then for all m and n one and only one of these inequalities is valid. Thus for arbitrary a and b exactly one holds from the above relations, which can be written in the following unified forms:

$$a : b \gtreqless n : m, \text{ or } ma \gtreqless nb.$$

We can write similar relations for the magnitudes c and d, and may suppose that the condition for the equal ratio is the same relation for the same numbers n and m, i.e. $a : b = c : d$, if for all numbers n, m

$$ma \gtreqless nb \implies mc \gtreqless nd.$$

This definition (i.e. Def. V.5) seems to be logical, but the Greeks surely needed an "irrefragable" proof for it, which must have been an indirect one regarding the problems with the lemmas in a direct proof. We remark that the first generally accepted proof for the incommensurability of the diagonal and side of the square was also an indirect one.

Suppose that $a : b = c : d$, but the above implication is not true for all numbers n, m. Then there exist at least one n and m for which one of the following three relations holds:

(1) $ma > nb$ and $mc \leq nd$, (or $a : b > n : m \geq c : d$),
(2) $ma = nb$ and $mc \neq nd$, (or $a : b = n : m \neq c : d$),
(3) $ma < nb$ and $mc \geq nd$, (or $a : b < n : m \leq c : d$).

Since $a : b$ can be equal to $c : d$ in neither case, therefore numbers n, m with these properties cannot exist. Thus we have proved the correctness of Eudoxus's definition, which is in fact a statement. We note that Case 1 expresses the definition of greater ratio by Def. V.7: "When, of the equimultiples, the multiple of the first magnitude exceeds the multiple of the second, but the multiple of the third does not exceed the multiple of the fourth, then the first is said to have a *greater ratio* to the second than the third has to the fourth."

This coincidence strengthens our conviction that we correctly reconstructed Eudoxus's deduction of his equal ratio definition.

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