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CLOSED NORMAL SUBGROUPS IN GROUPS OF UNITS OF COMPACT RINGS

ADELA TRIPE

Abstract. Normal subgroups of semiperfect rings were studied in [BS2]. We will study in this paper normal closed subgroups in groups of units of compact rings with identity.

1. Notation and convention

All topological rings are assumed to be associative, Hausdorff and with identity. The group of units (equal to invertible elements) of a ring R will be denoted by $U(R)$. If A, B are two subsets of a ring R, the put $A.B = \{ab : a \in A,\, b \in B\}$. The closure of a subset A of a topological space X will be denoted by $\overline{A}$. The subgroup of a group G generated by an element $g \in G$ will be denoted by $\langle g \rangle$. The Jacobson radical of a ring R is denoted by $J(R)$ or briefly by J. If I is a two-sided ideal of a ring R we will write $I \lhd R$. The set of all natural numbers will be denoted by $\mathbb{N}$, and $\mathbb{N}^{+}$stands for the set of positive integers. If $n \in \mathbb{N}$, and R is a ring, then $M\,(n, R)$ denotes the ring of $n \times n$ matrices over R. The theory of summable set of elements in topological Abelian groups is exposed in [B]. Accordingly, the sum of an arbitrary summable set $\{x_\alpha : \alpha \in \Omega\}$ is denoted by $\sum_{\alpha\in\Omega} x_\alpha$. If $\{R_\alpha : \alpha \in \Omega\}$ is a system of topological rings, then $\prod_{\alpha\in\Omega} R_\alpha$ stands for the topological product of these rings. If A is a subset of a ring R, then $\langle A \rangle$ denotes the subring of R generated by A, and $\langle A \rangle^{+}$ the subgroup of the additive group of R generated by A. If G is a group, $x, y \in G$, then $[x, y] = xyx^{-1}y^{-1}$ denotes the commutator of x and y. An *idempotent* $e \neq 0$ of a ring R is called *primitive* provided there are no non-zero orthogonal idempotents $e_1, e_2 \in R$ such that $e = e_1 + e_2$.

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2. Preliminaries

Definition 2.1. A compact ring Λ with identity is called a ring with a system of idempotents $\{e_\alpha : \alpha \in \Omega\}$ provided e_α are non-zero orthogonal idempotents and $1 = \sum_{\alpha\in\Omega} e_\alpha$.

Denote $\Lambda_{\alpha\beta} = e_\alpha \Lambda e_\beta$ and $\Lambda_{\alpha\alpha} = \Lambda_\alpha, (\alpha, \beta \in \Omega)$.

Definition 2.2. The subring $\Delta = \overline{\langle \Lambda_\alpha : \alpha \in \Omega \rangle}$ is called *the diagonal subring of the ring* Λ with respect to the system of idempotents $\{e_\alpha : \alpha \in \Omega\}$.

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Definition 2.3. The subgroup $D = U(\Delta) \subseteq U(\Lambda)$ is called a *torus* in $U(\Lambda)$ (or in Λ). A torus D is called *primitive* provided the system $\{e_\alpha : \alpha \in \Omega\}$ consists of primitive idempotents.

Evidently, each $\Lambda_{\alpha\beta}$ is a $(\Lambda_\alpha, \Lambda_\beta)$-bimodule.

Definition 2.4. A family $\sigma = (\sigma_{\alpha\beta})$ of closed sub-bimodules, is called a *net* in Λ if $\sigma_{\alpha\gamma}\sigma_{\gamma\beta} \subseteq \sigma_{\alpha\beta}$, for all $\alpha, \beta, \gamma \in \Omega$.

Definition 2.5. A net $\sigma = (\sigma_{\alpha\beta})$ is called a *D-net* in Λ provided $\sigma_{\alpha\alpha} = \Lambda_\alpha$ for all $\alpha \in \Omega$.

If Λ is a topological ring with identity then, in general, $U(\Lambda)$ with respect to the induced topology is not a topological group. When Λ is a compact ring with identity, $U(\Lambda)$ is closed and is a topological group with respect to the induced topology. The closedness of $U(\Lambda)$ was proved in [K]. The continuity of the mapping $x \mapsto x^{-1}$ in $U(\Lambda)$ follows from the boundedness of Λ (see [U]).

For any D-net σ, denote $M(\sigma) = \overline{\langle \sigma_{\alpha\beta} : \alpha, \beta \in \Omega \rangle}$. Evidently, $M(\sigma)$ is a closed subring of Λ, called *the subring of the net σ*.

Theorem 2.6. *Let Λ be a compact ring with a countable system of idempotents $\{e_i : i \in \mathbb{N}^+\}$. Then there exists a bijection between intermediate closed subgroups Σ ($\Delta \subseteq \Sigma \subseteq \Lambda$) and D-nets.*

Proof. It suffices to show that every intermediate closed subgroup Σ has the form $\Sigma = M(\sigma)$, where σ is a D-net. Put $\sigma_{ij} = e_i \Sigma e_j$. Since $e_i, e_j \in \Sigma$ ($i, j \in \mathbb{N}^+$), we obtain that $\sigma_{ij} \subseteq \Sigma$, hence $M(\sigma) \subseteq \Sigma$. Conversely, if $a = \sum_{i,j\in\mathbb{N}^+} a_{ij} \in \Sigma$, then $a_{ij} = e_i a e_j \in \sigma_{ij}$, hence $a \in M(\sigma)$. $\square$

We define on the set of all nets of a ring Λ, the relation $\leq$ as follows: if $\sigma = (\sigma_{\alpha\beta})$ and $\tau = (\tau_{\alpha\beta})$are two nets of Λ, we consider that $\sigma \leq \tau$ provided $\sigma_{\alpha\beta} \subseteq \tau_{\alpha\beta}$, for all $\alpha, \beta \in \Omega$.

For every D-net σ denote $G(\sigma) = U(M(\sigma))$. This group is called the *D-net subgroup.*

For $\alpha, \beta \in \Omega$, $\alpha \neq \beta$, and $\xi \in \sigma_{\alpha\beta}$, $t_{\alpha\beta}(\xi) = 1 + \xi$ is called an *elementary transvection* of $U(\Lambda)$. Note that $t_{\alpha\beta}(\xi)\ t_{\alpha\beta}(-\xi) = 1$, for every $\xi \in \sigma_{\alpha\beta}$. By $E(\sigma)$ is denoted the closure of the subgroup of $U(\Lambda)$ generated by all elementary transvections. $E(\sigma)$ is called the *elementary subgroup of the net σ.*

Definition 2.7. A fixed family $\sigma = (\sigma_{\alpha\beta})$ of closed subgroups of Λ is called an I-net (net ideal) provided $\Lambda_{\alpha\gamma}\sigma_{\gamma\beta} \subseteq \sigma_{\alpha\beta}$, $\sigma_{\alpha\gamma}\Lambda_{\gamma\beta} \subseteq \sigma_{\alpha\beta}$ for all $\alpha, \beta, \gamma \in \Omega$.

Remark. a) Let I be a two-sided closed ideal of Λ. Put $I_{\alpha\beta} = I \cap e_\alpha \Lambda e_\beta = e_\alpha I e_\beta$ for all $\alpha, \beta \in \Omega$. Then $(I_{\alpha\beta})$ is a I-net.

b) If σ is an I-net, then $M(\sigma) = \overline{\langle \sigma_{\alpha\beta} : \alpha, \beta \in \Omega \rangle}$ is a closed two-sided ideal of Λ.

The J-net of Λ is an I-net associated with $I = J = J(\Lambda)$, i.e. $\sigma = (\sigma_{\alpha\beta}), \sigma_{\alpha\beta} = e_\alpha J e_\beta$.

We consider below only compact rings with a countable system of idempotents $\{e_i : i \in \mathbb{N}^+\}$.

Definition 2.8. The subgroup

$$B = B(J) = \left\{ \sum_{i,j\in\mathbb{N}^+} a_{ij} \in U(\Lambda) : a_{ij} \in J_{ij} \text{ if } i > j \right\}$$

is called *the radical upper triangular subgroup of $U(\Lambda)$.*

Consider three conditions for a ring Λ with a countable system of idempotents:

(Σ) every idempotent of Λ_i ($i \in \mathbb{N}^+$) is a finite sum of invertible elements of Λ_i;
(θ) there exists an invertible element $\theta \in \Delta$, such that $1+\theta \in U(\Delta)$;
(θ_2) there exists an invertible element $\theta \in \Delta$, such that $1+\theta, 1+\theta+\theta^2 \in U(\Delta)$.

Definition 2.9. Let Λ be a compact ring with a countable system of idempotents $\{e_i : i \in \mathbb{N}^+\}$, satisfying the condition (Σ), and let H be a closed subgroup of $U(\Lambda)$, containing the torus D (i.e. $D \leq H \leq U(\Lambda)$). We will say that the D-net (σ_{ij}) *is associated to the subgroup* H if $\sigma_{ij} = \{\xi \in \Lambda_{ij} : t_{ij}(\xi) \in H\} = e_i \Lambda e_j \cap (H-1)$, for $i \neq j$, $i, j \in \mathbb{N}^+$.

Note that each σ_{ij} is closed.

If $\varepsilon \in U(\Lambda_i)$ and $\eta \in U(\Lambda_j)$, we have:

$$(*) \qquad d_i(\varepsilon)\, t_{ij}(\xi)\, d_i\left(\varepsilon^{-1}\right) = t_{ij}(\varepsilon\xi),$$

$$(**) \qquad d_i\left(\eta^{-1}\right) t_{ij}(\xi)\, d_i(\eta) = t_{ij}(\xi\eta).$$

Consider that the ring Λ satisfies the condition (Σ); then we will deduce that σ_{ij} is (Λ_i, Λ_j)-bimodule ($\Lambda_i \sigma_{ij=} \sigma_{ij}, \sigma_{ij}\Lambda_i = \sigma_{ij}$).

We denote $\sigma_= (\sigma_{ij})$. If $i \neq r$, $j \neq r$ and $\lambda \in \sigma_{ir}$, $\mu \in \sigma_{rj}$ we have:

$$[t_{ir}(\lambda), t_{rj}(\mu)] = t_{ij}(\lambda\mu) \in H,$$

hence if $\lambda \in \sigma_{ir}$, $\mu \in \sigma_{rj}$ we have $\lambda\mu \in \sigma_{ij}$, deci $\sigma_{ir}\sigma_{rj} \subseteq \sigma_{ij}$. We proved that σ is a D-net in Λ.

3. Main results

Lemma 3.1. *A finite simple ring $R = M(n, \mathbb{F}_q)$ satisfies the condition (θ_2) iff it is not isomorphic to one of the rings $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, M(2, \mathbb{F}_2)$.*

Proof. $\Rightarrow$: It is a routine to see that the rings $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, M(2, \mathbb{F}_2)$ do not satisfy the condition (θ_2).

$\Leftarrow$: Let $\mathbb{F}_q$ be a finite field, $q > 4$. Let S be the set of all solutions in $\mathbb{F}_q$ of the equation $x^2 + x + 1 = 0$. The set $\{-1\} \cup E$ has no more than three elements. But $|U(E)| > 3$, so there exists at least an element $\gamma \in U(E) - (\{-1\} \cup E)$. Evidently, γ verifies the condition (θ_2).

If $M(n, \mathbb{F}_q)$ is not isomorphic to $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, M(2, \mathbb{F}_2)$, then we have the following possible cases:

Case (1) $R = M(n, \mathbb{F}_2)$, $n > 2$. Then, the ring $M(n, \mathbb{F}_2)$ contains a subring with identity and is isomorphic to $\mathbb{F}_{2^n}$.

Case (2) $R = M(n, \mathbb{F}_3)$, $n > 3$. Then, the ring $M(n, \mathbb{F}_3)$ contains a subring with identity and is isomorphic to $\mathbb{F}_{3^n}$.

Case (3) $R = M(n, \mathbb{F}_4)$, $n > 1$. Then, the ring $M(n, \mathbb{F}_4)$ contains a subring with identity and is isomorphic to $\mathbb{F}_{2^{2n}}$. □

Corollary 3.2. *If R is a compact ring with identity, then it verifies the condition (θ_2) iff in the decomposition of $R/J(R)$ as topological product of finite simple discrete rings, the ringa $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, M(2, \mathbb{F}_2)$ do not appear.*

Lemma 3.3. *Let $\{e_\alpha : \alpha \in \Omega\}$ be a system of orthogonal idempotents. The ring $\overline{\langle e_\alpha \Lambda e_\alpha : \alpha \in \Omega \rangle}$ is topologically isomorphic to the topological product $\prod_{\alpha \in \Omega} e_\alpha \Lambda e_\alpha$.*

Proof. Consider the mapping:

$$\overline{\langle e_\alpha \Lambda e_\alpha : \alpha \in \Omega \rangle} \to \prod_{\alpha \in \Omega} e_\alpha \Lambda e_\alpha, \quad \sum_{\alpha \in \Omega} x_\alpha \mapsto \{x_\alpha\}_{\alpha \in \Omega}, x_\alpha \in e_\alpha \Lambda e_\alpha.$$

It is clear that this mapping is an algebraic isomorphism of $\sum_{\alpha\in\Omega} e_\alpha \Lambda e_\alpha$ on the subring A of $\prod_{\alpha\in\Omega} e_\alpha\Lambda e_\alpha$, consisting of elements $\{x_\alpha\}_{\alpha\in\Omega} \in \prod_{\alpha\in\Omega} e_\alpha\Lambda e_\alpha$, for which almost all coordinates x_α are zero.

We claim that this mapping is a topological isomorphism. Indeed, let V be an open ideal of Λ. There exists $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Omega$ such that $e_\alpha \in V$ for each $\alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_n$. Then $V \supseteq e_{\alpha_1} V e_{\alpha_1} + \cdots + e_{\alpha_n} V e_{\alpha_n} + \overline{\langle e_\alpha \Lambda e_\alpha : \alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_n\rangle}$. The image of V under the application described above coincides with:

$$\left(e_{\alpha_1} V e_{\alpha_1} \times \cdots \times e_{\alpha_n} V e_{\alpha_n} \times \overline{\langle e_\alpha V e_\alpha : \alpha \neq \alpha_1, \alpha_2, \ldots, \alpha_n\rangle}\right) \cap A.$$

It follows that $\langle e_\alpha\Lambda e_\alpha : \alpha \in \Omega\rangle$ and A are topologically isomorphic, therefore their completions $\overline{\langle e_\alpha\Lambda e_\alpha : \alpha \in \Omega\rangle}$ and $\prod \alpha \in \Omega e_\alpha \Lambda e_\alpha$ are topologically isomorphic too. $\square$

Corollary 3.4. $U\left(\overline{\langle e_\alpha\Lambda e_\alpha : \alpha \in \Omega\rangle}\right) \cong_{top} \prod_{\alpha\in\Omega} U\left(e_\alpha \Lambda e_\alpha\right)$.

Remark. It is well known that if Λ is a compact ring with identity and $\{e_\alpha : \alpha \in \Omega\}$ is a system of orthogonal idempotents, then for each $x_\alpha \in e_\alpha\Lambda e_\alpha$, the family $\{x_\alpha : \alpha \in \Omega\}$ is summable.

Lemma 3.5. *The topological groups* Λ *and* $\prod_{\alpha,\beta\in\Omega} e_\alpha\Lambda e_\beta$ *are isomorphic.*

Proof. Put

$$\rho \colon \Lambda \to \prod_{\alpha,\beta\in\Omega} e_\alpha \Lambda e_\beta, \quad \rho\left(x\right) = \left\{e_\alpha x e_\beta\right\}_{\alpha,\beta\in\Omega}.$$

It is obvious that ρ is a monomorphism, and a surjective homomorphism.

By Remark 3, any family $\left\{e_\alpha x_{\alpha\beta} e_\beta\right\}_{\alpha,\beta\in\Omega}$ is summable. Put $x = \sum e_\alpha x_{\alpha\beta} e_\beta$; then $\rho\left(x\right) = \{e_\alpha x_{\alpha\beta} e_\beta\}$.

It is a routine to prove that ρ is continuous. We will prove that ρ is open. Let V be an open ideal of Λ. There exists a finite subset Ω_0 of Ω such that $e_\alpha \in V$ if $\alpha \notin \Omega_0$ or $\beta \notin \Omega_0$. If follows from the definition of ρ that

$$\prod_{\alpha,\beta\in\Omega_0} e_\alpha V e_\beta \times \prod_{\substack{\alpha_1 \notin \Omega_0 \\ \text{or } \beta_1 \notin \Omega_0}} e_{\alpha_1} \Lambda e_{\beta_1} \subseteq \rho\left(V\right),$$

i.e. ρ is open. $\square$

We consider below only compact rings with a countable system of idempotents $\{e_i : i \in \mathbb{N}^+\}$.

Remark. B is a closed subgroup of $U\left(\Lambda\right)$.

Indeed,

$$B = \left\{ \sum_{i,j\in\mathbb{N}^+} a_{ij} \in U(\Lambda) : a_{ij} \in J_{ij} \text{ if } i > j \right\},$$

and by definition,

$$B = U(\Lambda) \cap \left\{ \sum_{i,j\in\mathbb{N}^+} a_{ij} \in \Lambda : a_{ij} \in J_{ij} \text{ if } i > j \right\}.$$

Since the set

$$\left\{ \sum_{i,j\in\mathbb{N}^+} a_{ij} \in \Lambda : a_{ij} \in J_{ij} \text{ if } i > j \right\}$$

is closed in Λ, B is closed in Λ.

Proposition 3.6. *The subring $S = \langle \Lambda_{ij} : i > j \rangle$ of the compact ring Λ is topologically nilpotent.*

Proof. Let V be an open ideal of Λ. There exists a finite subset K of $\mathbb{N}^+$ such that $e_i \in V$ for all $i \notin K$. Let $x = \sum_{i>j} x_{ij} \in S$; then $x = \sum_{i>j,i,j\in K} x_{ij} + v$, where $v \in V$. Consider for the simplicity that $K = \{1, \ldots, n\}$ and:

$$x = y + v;$$
$$x = \varepsilon_{21}x_{21} + \varepsilon_{31}x_{31} + \varepsilon_{32}x_{32} + \cdots + \varepsilon_{n\ n-1}x_{n\ n-1},$$
$$y = \varepsilon_{21}y_{21} + \varepsilon_{31}y_{31} + \varepsilon_{32}y_{32} + \cdots + \varepsilon_{n\ n-1}y_{n\ n-1},$$

where $\varepsilon_{ij} \in \{0, 1\}$, $\forall i > j$.

It is clear that there exists $k \in \mathbb{N}^+$such that $x^k \in V$. If follows that S is a topological nilring. Since Λ is compact, $\overline{S}$ is a topological nilring. □

Lemma 3.7 (Folklore)**.** *Let G be a topological group and K a compact subspace. Then for every neighborhood V of identity e there exists a neighborhood U of e such that $U.K \subseteq K.V$.*

Proof. For every $k \in K$ there exists a neighborhood W_k of k and a neighborhood $U_e^{(k)}$ of e such that $W_k^{-1}.U_e^{(k)}.W_k \subseteq V$ (the continuity of operations). Consider the cover $\{W_k : k \in K\}$ of K, and let $x_1, x_2, \ldots, x_n \in K$ such that $K \subseteq W_{x_1} \cup \ldots \cup W_{x_n}$.

We claim that $t^{-1}.U.t \subseteq V$ for every $t \in K$, where $U = U_e(x_1) \cap \ldots \cap U_e(x_n)$. Indeed, let $t \in K$; there exists $i \in \{1, \ldots, n\}$ such that $t \in W_{x_i}$, hence $t^{-1}Ut \subseteq W_{x_i}^{-1}.U_e(x_i).W_{x_i} \subseteq V \Rightarrow U.t \subseteq t.V \Rightarrow U.K \subseteq K.V$. □

Remark. If $a \in U(\Lambda)$ and $b \in J(\Lambda)$ then $a + b \in U(\Lambda)$.

Indeed, $a^{-1}ba \in J(\Lambda)$ and $a + b = a\left(1 + a^{-1}b\right) \in U(\Lambda)$.

Lemma 3.8. *If R is a finite ring with identity and $x \in U(R)$, then $x^{-1} \in \langle x \rangle$.*

Proof. Since R is finite, there exist integers n, k, $n > k$, such that $x^n = x^k$. Then $x^{n-k} = 1$, hence $x^{-1} = x^{n-k} \in \langle x \rangle$. □

Lemma 3.9. *Let R' be a compact ring with identity and R a subring with identity. An element $x \in R$ is invertible in R if and only if it is invertible in R'.*

Proof. Since R' has identity, it has a local base consisting of two-sided ideals. Let V be an arbitrary open ideal of R'. Let x be an invertible element of R' and $xx^{-1} = x^{-1}x = 1$. By Lemma 3.8, $x^{-1} \in \langle x \rangle + V$, hence $x^{-1} \in \langle x \rangle \subseteq R$. We obtained that x is invertible in R. □

Proposition 3.10. *If σ is a D-net then $D.E(\sigma) = E(\sigma).D$.*

Proof. Let $a = \sum_{i\in\mathbb{N}^+} a_{ii} \in D$, where $a_{ii} \in U(\Lambda_i), i \in \mathbb{N}^+$. Let $\alpha \in \sigma_{ij}, i \neq j$; we have:

$$\begin{aligned} at_{ij}(\alpha)a^{-1} &= \left(\sum_{i\in\mathbb{N}^+} a_{ii}\right)(1+\alpha)\left(\sum_{i\in\mathbb{N}^+} a_{ii}^{-1}\right) \\ &= 1 + a_{ii}\alpha a_{jj}^{-1} \in E(\sigma). \end{aligned}$$

Since $a.E(\sigma) \subseteq E'(\sigma).a$, where $E'(\sigma)$ is a subgroup of $U(\Lambda)$ generated by transvections. By continuity, $a.E(\sigma) \subseteq E(\sigma).a$, hence $a.E(\sigma) = E(\sigma).a$. In analogous way, $E(\sigma).a \subseteq a.E(\sigma)$. We obtain that $D.E(\sigma) = E(\sigma).D$. □

Proposition 3.11. *If $a \in G(\sigma)$ then $a_{ii} \in U(\Lambda_i)$, for every $i \in \mathbb{N}^+$.*

Proof. The element $\sum_{i>j} a_{ij}$ belongs to J_{ij}, hence by the Remark 3,

$$a - \sum_{i>j} a_{ij} \in U(\Lambda),$$

hence $\sum_{i\le j} a_{ij} \in U(\Lambda)$.

Since $a \in U(\Lambda)$, by Lemma 3.9, $a \in U(L)$, where $L = \overline{\langle \Lambda_{ij} : i \le j\rangle}$. Since $\overline{\langle \Lambda_{ij} : i \le j\rangle} \subseteq J(L)$, we have that $a - \sum_{i<j} a_{ij} \in U(L)$, i.e. $\sum_{i\in\mathbb{N}^+} a_{ii} \in U(L)$. There exists $a' \in L$, such that $a'\left(\sum\limits_{i\in\mathbb{N}^+} a_{ii}\right) = \left(\sum_{i\in\mathbb{N}^+} a_{ii}\right) a' = 1$. Fix $i_0 \in \mathbb{N}^+$; then $e_{i_0}\left(\sum_{i\in\mathbb{N}^+} a_{ii}\right) a' = e_{i_0}$, or $a_{i_0 i_0} a' = e_{i_0}$, or $a_{i_0 i_0} e_{i_0} a' e_{i_0} = e_{i_0}$; analogously, $e_{i_0} a' e_{i_0} a_{i_0 i_0} = e_{i_0}$, hence $a_{i_0 i_0} \in U(\Lambda_{i_0})$. □

Theorem 3.12. *Let Λ be a ring with a countable system of idempotents and ρ a D-net, $B = G(\rho)$. If $\sigma \le \rho$ is a D-net, then $G(\sigma) = D.E(\sigma)$.*

Proof. By Lemma 3.7 there exists a neighborhood V of identity in $U(\Lambda)$ such that $V.E \subseteq E.W$. We may assume without loss of generality that:

$$V = V_{11} + V_{12} + \cdots + V_{nn} + \overline{\langle \Lambda_{ij} : i > n \text{ or } j > n\rangle},$$

where $n \in \mathbb{N}^+$, $V_{ii} = (1+Q)\cap e_i\Lambda e_i = e_i + e_iQe_i$, for $i \in \{1,\dots,n\}$, $V_{ij} = Q \cap e_i\Lambda e_j$, for $i \ne j$, $i,j \in \{1,\dots,n\}$, and that Q is an open ideal in Λ.

We will prove that $G(\sigma) \subseteq D.E(\sigma)$. Let $a \in G(\sigma)$, $a = \sum\limits_{i,j\in\mathbb{N}^+} a_{ij}$, $a_{ij} \in \Lambda_{ij}$.

Claim: For any $m \in \mathbb{N}^+$, there exist $y_m, x_m \in E(\sigma)$ and $d_m \in D$, such that: $d_m x_m a y_m \in e_1 + e_2 + \cdots + e_m + \overline{\langle \Lambda_{ij} : i > m \text{ or } j > m\rangle}$.

Induction on m:

For $m = 1$, put $y_1 = x_1 = 1, d_1 = a_{11}^{-1} + 1 - e_1 \in a_{11}^{-1} + \overline{\langle \Lambda_{ij} : i > 1 \text{ or } j > 1\rangle} \in D$. We have:

$$\begin{aligned} d_1x_1ay_1 = d_1a &= \left(a_{11}^{-1} + 1 - e_1\right) a \\ &\in \left(a_{11}^{-1} + \overline{\langle \Lambda_{ij} : i > 1 \text{ or } j > 1\rangle}\right)\left(\sum_{p,q\in\mathbb{N}^+} a_{pq}\right) \\ &\subseteq \sum_{q\in\mathbb{N}^+} a_{11}^{-1}a_{1q} + \overline{\langle \Lambda_{ij} : i > 1 \text{ or } j > 1\rangle} \\ &\subseteq e_1 + \sum_{q\ge 2} a_{11}^{-1}a_{1q} + \overline{\langle \Lambda_{ij} : i > 1 \text{ or } j > 1\rangle} \\ &\subseteq e_1 + \overline{\langle \Lambda_{ij} : i > 1 \text{ or } j > 1\rangle}. \end{aligned}$$

Assume that the claim was proved for m, and we will prove it for $m+1$. By induction, there exist $y_m, x_m \in E(\sigma)$ and $d_m \in D$, such that:

$$\begin{aligned} d_mx_may_m &\in e_1 + e_2 + \cdots + e_m + \overline{\langle \Lambda_{ij} : i > m \text{ or } j > m\rangle} \\ &\subseteq e_1 + e_2 + \cdots + e_m + \sum_{k=1}^{m} \lambda_{m+1\ k} + \sum_{s=1}^{m} \lambda_{s\ m+1} + \lambda_{m+1\ m+1} \\ &+ \overline{\langle \Lambda_{ij} : i > m+1 \text{ or } j > m+1\rangle}. \end{aligned}$$

Consider the elements $x_{m+1}^{(i)}, y_{m+1}^{(i)} \in E(\sigma)$, defined as follows:

$$\begin{aligned} x_{m+1}^{(i)} &= t_{m+1\ i}(-\lambda_{m+1\ i}), \\ y_{m+1}^{(i)} &= t_{i\ m+1}(-\lambda_{i\ m+1}), \end{aligned}$$

where $i \in \{1, \ldots, m\}$. Then,

$$\begin{aligned}
&x_{m+1}^{(1)}\left(d_m x_m a y_m\right)\\
&= \left(1 - \lambda_{m+1\ i}\right)\left(d_m x_m a y_m\right)\\
&\in \left(1 - \lambda_{m+1\ i}\right)\left(e_1 + e_2 + \vdots + e_m + \sum_{k=1}^{m} \lambda_{m+1\ k} + \sum_{s=1}^{m} \lambda_{s\ m+1} + \lambda_{m+1\ m+1}\right.\\
&\left. + \overline{\langle \Lambda_{ij} : i > m+1 \text{ or } j > m+1 \rangle}\right)\\
&\subseteq e_1 + e_2 + \cdots + e_m + \sum_{k=1}^{m} \lambda_{m+1\ k} + \sum_{s=1}^{m} \lambda_{s\ m+1} + \lambda_{m+1\ m+1}\\
&+ \overline{\langle \Lambda_{ij} : i > m+1 \text{ or } j > m+1 \rangle} - \lambda_{m+1\ 1} - \lambda_{m+1\ 1}\lambda_{1\ m+1}\\
&\subseteq e_1 + e_2 + \cdots + e_m + \sum_{k=2}^{m} \lambda_{m+1\ k} + \sum_{s=1}^{m} \lambda_{s\ m+1} + \widetilde{\lambda}_{m+1\ m+1}\\
&+ \overline{\langle \Lambda_{ij} : i > m+1 \text{ or } j > m+1 \rangle};
\end{aligned}$$

Furthermore, we have:

$$\begin{aligned}
&x_{m+1}^{(1)}\left(d_m x_m a y_m\right) y_{m+1}^{(1)} = x_{m+1}^{(1)}\left(d_m x_m a y_m\right)\left(1 - \lambda_{i\ m+1}\right)\\
&\in \left(e_1 + e_2 + \cdots + e_m + \sum_{k=2}^{m} \lambda_{m+1\ k} + \sum_{s=1}^{m} \lambda_{s\ m+1} + \widetilde{\lambda}_{m+1\ m+1}\right.\\
&\left. + \overline{\langle \Lambda_{ij} : i > m+1 \text{ or } j > m+1 \rangle}\right) \cdot \left(1 - \lambda_{i\ m+1}\right)\\
&\subseteq e_1 + e_2 + \cdots + e_m + \sum_{k=2}^{m} \lambda_{m+1\ k} + \sum_{s=2}^{m} \lambda_{s\ m+1} + \widetilde{\lambda}_{m+1\ m+1}\\
&+ \overline{\langle \Lambda_{ij} : i > m+1 \text{ or } j > m+1 \rangle}.
\end{aligned}$$

Continuing, we obtain:

$$\begin{aligned}
x_{m+1}^{(m)} x_{m+1}^{(m-1)} \ldots x_{m+1}^{(1)}\left(d_m x_m a y_m\right) y_{m+1}^{(1)} \ldots y_{m+1}^{(m-1)} y_{m+1}^{(m)}&\\
\in e_1 + e_2 + \cdots + e_m + \widetilde{\lambda}_{m+1\ m+1}^{-1}&\\
+ \overline{\langle \Lambda_{ij} : i > m+1 \text{ or } j > m+1 \rangle};&
\end{aligned}$$

By Lemma 3.11, $\widetilde{\lambda}_{m+1\ m+1} \in U\left(\Lambda_{m+1}\right)$. Put

$$\begin{aligned}
d'_{m+1} &= 1 - e_{m+1} + \widetilde{\lambda}_{m+1\ m+1}^{-1}\\
&= e_1 + e_2 + \cdots + e_m + \widetilde{\lambda}_{m+1\ m+1}^{-1} + e_{m+2} + \cdots .
\end{aligned}$$

We have:

$$\begin{aligned}
&d'_{m+1} x_{m+1}^{(m)} x_{m+1}^{(m-1)} \ldots x_{m+1}^{(1)}\left(d_m x_m a y_m\right) y_{m+1}^{(1)} \ldots y_{m+1}^{(m-1)} y_{m+1}^{(m)}\\
&\in (e_1 + e_2 + \cdots + e_m + \widetilde{\lambda}_{m+1\ m+1}^{-1})(e_1 + e_2 + \cdots + e_m + \widetilde{\lambda}_{m+1\ m+1}^{-1}\\
&+ \overline{\langle \Lambda_{ij} : i > m+1 \text{ or } j > m+1 \rangle})\\
&\subseteq e_1 + e_2 + \cdots + e_m + e_{\ m+1} + \overline{\langle \Lambda_{ij} : i > m+1 \text{ or } j > m+1 \rangle}.
\end{aligned}$$

Put

$$\begin{aligned}
x'_{m+1} &= x_{m+1}^{(m)} x_{m+1}^{(m-1)} \ldots x_{m+1}^{(1)} \in E\left(\sigma\right),\\
y'_{m+1} &= y_{m+1}^{(1)} \ldots y_{m+1}^{(m-1)} y_{m+1}^{(m)} \in E\left(\sigma\right).
\end{aligned}$$

We obtain:

$$\begin{aligned}
&d'_{m+1}x^{(m)}_{m+1}x^{(m-1)}_{m+1}\dots x^{(1)}_{m+1}\left(d_m x_m a y_m\right) y^{(1)}_{m+1}\dots y^{(m-1)}_{m+1}y^{(m)}_{m+1}\\
&= d'_{m+1}x'_{m+1}\left(d_m x_m a y_m\right) y'_{m+1}\\
&= d'_{m+1}x'_{m+1}d_m x_m a y_m y'_{m+1}\\
&\in e_1+e_2+\dots+e_m+e_{\ m+1}+\overline{\langle\Lambda_{ij}: i>m+1 \text{ or } j>m+1\rangle}.
\end{aligned}$$

By Proposition 3.10, there exist $d'\in D$ and $x'\in E(\sigma)$, such that: $x'_{m+1}d_m = d'x'$. We obtain:

$$\begin{aligned}
d'_{m+1}x'_{m+1}d_m x_m a y_m y'_{m+1} &= d'_{m+1}\left(x'_{m+1}d_m\right) x_m a y_m y'_{m+1}\\
&= d'_{m+1}\left(d'x'\right) x_m a y_m y'_{m+1}\\
&= d'_{m+1}d'x'x_m a y_m y'_{m+1}.
\end{aligned}$$

Denote: $d_{m+1}=d'_{m+1}d'$, $y_{m+1}=y_m y'_{m+1}$, $x_{m+1}=x'x_m$, where $d_{m+1}\in D$, x_{m+1}, $y_{m+1}\in E(\sigma)$; we have:

$$d'_{m+1}d'x'x_m a y_m y'_{m+1} = d_{m+1}x_{m+1}a y_{m+1}$$

We obtained:

$$\begin{aligned}
&d'_{m+1}x'_{m+1}d_m x_m a y_m y'_{m+1}\\
&= d_{m+1}x_{m+1}a y_{m+1}\\
&\in e_1+e_2+\dots+e_{\ m+1}+\overline{\langle\Lambda_{ij}: i>m+1 \text{ or } j>m+1\rangle},
\end{aligned}$$

so the affirmation is true for every $m\in\mathbb{N}^+$. In particular, there exist $y_n, x_n\in E(\sigma)$, $d_n\in D$, such that:

$$d_n x_n a y_n \in e_1+e_2+\dots+e_n+\overline{\langle\Lambda_{ij}: i>n \text{ or } j>n\rangle}\subseteq V.$$

It follows that:

$$a = x_n^{-1}d_n^{-1}V y_n^{-1}\subseteq E(\sigma).D.V.E(\sigma)\subseteq D.E(\sigma).V.E(\sigma)\subseteq D.E(\sigma).W.$$

We obtain:

$G(\sigma)\subseteq\cap\{D.E(\sigma).W : W \text{ is a neighborhood of identity in } U(\Lambda)\} = D.E(\sigma)$.

$\square$

Lemma 3.13. *Let Λ be a compact ring with identity, with a countable system of idempotents, which satisfies the condition (θ). Let H be a closed subgroup of $U(\Lambda)$, $D\subseteq U(\Lambda)$. If $a=\sum_{i,j\in\mathbb{N}^+}a_{ij}\in H$, and r is a fixed natural number for which $a_{rj}=0$, $(\forall)\ j\neq r$, then $t_{ir}(a_{ir})\in H$ for all $i\neq r$.*

Proof. Using condition (θ) we find elements $\varepsilon\in\Lambda_r$, $\eta\in\Lambda_i$, such that $\varepsilon, e_r+\varepsilon\in U(\Lambda_r)$, and $\eta, e_i+\eta\in U(\Lambda_i)$.

Denote $a^{-1}=\sum_{i,j\in\mathbb{N}^+}a'_{ij}$, where a^{-1} is the inverse of a. We have,

$$1=\left(\sum_{i,j\in\mathbb{N}^+}a_{ij}\right)\left(\sum_{s,k\in\mathbb{N}^+}a'_{sk}\right).$$

Multiplying on the left by e_r we obtain $a_{rr}\left(\sum_{k\in\mathbb{N}^+}a'_{rk}\right)=e_r$. Multiplying on the right by e_j we obtain $a_{rr}a'_{rj}=0$, hence $a'_{rj}=0$ for all $j\neq r$.

Put $b=ad_r(e_r+\varepsilon)a^{-1}\in H$. We have:

$$b=1+a\varepsilon a^{-1}=1+\sum_{k,j\in\mathbb{N}^+}a_{kr}\varepsilon a'_{rj}=1+\sum_{k\in\mathbb{N}^+}a_{kr}\varepsilon a'_{rr}.$$

It is easy to prove that $b^{-1}=1+\sum\limits_{k\in\mathbb{N}^+}a_{kr}\overline{\varepsilon}a'_{rr}$, where $\overline{\varepsilon}=-\ \varepsilon\left(e_r+\varepsilon\right)^{-1}$, is the inverse of b.

Put $c = [d_i (e_i + \eta), b] \in H$ and $\overline{\eta} = -(e_i + \eta)^{-1} \in U(\Lambda_i)$. Then:

$$c = (e_i + \eta)\, b\, (e_i + \eta)^{-1} b^{-1} = (e_i + \eta)^{-1} b \overline{\eta} b^{-1}.$$

But

$$b\overline{\eta}b^{-1} = \left(1 + \sum_{k\in\mathbb{N}^+} a_{kr} \varepsilon a'_{rr}\right) \overline{\eta} \left(1 + \sum_{k\in\mathbb{N}^+} a_{kr} \overline{\varepsilon} a'_{rr}\right) = \overline{\eta} \left(1 + \sum_{k\in\mathbb{N}^+} a_{kr} \overline{\varepsilon} a'_{rr}\right),$$

and

$$c = (e_i + \eta)\, \overline{\eta} \left(1 + \sum_{k\in\mathbb{N}^+} a_{kr} \overline{\varepsilon} a'_{rr}\right) = 1 + a_{ir} \overline{\varepsilon} a'_{rr}.$$

We proved that $t_{ir} (a_{ir} \overline{\varepsilon} a'_{rr}) \in H$. But $\overline{\varepsilon} a'_{rr} \in U(\Lambda_r)$, so applying the formula $(**)$ we obtain that $t_{ir} (a_{ir}) \in H$. □

Note. An analogous lemma is true in the case of a matrix $a \in H$, for which $a_{jr} = 0$, $\forall j \neq r$.

Theorem 3.14. *If C is the subring of Λ defined as follows:*

$$C = \left\{ \sum_{i,j\in\mathbb{N}^+} a_{ij} \in \Lambda : a_{ij} \in J_{ij} \text{ if } i > j \right\},$$

then

$$J(C) = \left\{ \sum_{i,j\in\mathbb{N}^+} a_{ij} \in \Lambda : a_{ij} \in J_{ij} \text{ if } i \geq j \right\}.$$

Proof. Let $I = \left\{ \sum_{i,j\in\mathbb{N}^+} a_{ij} \in \Lambda : a_{ij} \in J_{ij} \text{ if } i \geq j \right\}$.

The subset $X = \left\{ \sum_{i,j\in\mathbb{N}^+} x_{ij} \in I : \text{almost all } x_{ij} \text{ are } 0 \right\}$ is dense in I. Indeed, let

$$V = \sum_{\substack{i\leq n,\\ j\leq n}} V_{ij} + \overline{\langle \Lambda_{ij} : i > n \text{ or } j > n \rangle}$$

be a neighborhood of identity in $U(\Lambda)$, where $n \in \mathbb{N}^+$,

$$V_{ii} = (1 + Q) \cap e_i \Lambda e_i = e_i + e_i Q e_i$$

for $i \in \{1, \ldots, n\}$ and $V_{ij} = Q \cap e_i \Lambda e_j$ for $i \neq j, i, j \in \{1, \ldots, n\}$, and Q is an open ideal in Λ.

Every element $x = \sum_{i,j\in\mathbb{N}^+} x_{ij}$ of I can be written in the form

$$x = x_{11} + x_{12} + \cdots + x_{nn} + v,$$

where $v \in V$. Therefore, X is dense in I. It is a routine to prove that I is a two-sided ideal in C.

Consider the following application:

$$\rho \colon C \to S, \quad \sum_{i,j} c_{ij} \mapsto \{c_{ii} + J(\Lambda_{ii})\}_{i\in\mathbb{N}^+},$$

where $R = \prod_{i\geq 1} \Lambda_i / J(\Lambda_i)$.

The application defined above is a morphism. Let $a, c \in C$; we have:

$$\begin{aligned}\rho(a + c) &= \rho\left(\sum_{i,j\in\mathbb{N}^+} (a_{ij} + c_{ij})\right) = \{(a_{ii} + c_{ii}) + J(\Lambda_i)\}_{i\in\mathbb{N}^+} \\ &= \{a_{ii} + J(\Lambda_i)\}_{i\in\mathbb{N}^+} + \{c_{ii} + J(\Lambda_i)\}_{i\in\mathbb{N}^+} = \rho(a) + \rho(c)\end{aligned}$$

If $ac = d$, then:

$$\rho(ac) = \{d_{ii} + J(\Lambda_i)\}_{i\in\mathbb{N}^+}, \quad \rho(a)\rho(c) = \{a_{ii}c_{ii} + J(\Lambda_i)\}_{i\in\mathbb{N}^+}$$

Fix $i \in \mathbb{N}^+$, then

$$d_{ii} = \sum_{s\in\mathbb{N}^+} a_{is}c_{si} = a_{ii}c_{ii} + J(\Lambda_i),$$

where $\rho(ac) = \rho(a)\rho(c)$.

We affirm that ρ is continuous. Let U be a neighborhood of identity in R,

$$U = \prod_{1\le i\le n} [U_i + J(\Lambda_i)] \times \prod_{k>n} [\Lambda_k / J(\Lambda_k)];$$

where U_i are neighborhoods of identity in $\Lambda_i (1 \le i \le n$.

Let V be a neighborhood of identity in C,

$$V = \Big(U_1 + U_2 + \cdots + U_n + \langle U_{ij} : 1 \le i \le n, 1 \le j \le n, i \ne j\rangle + \overline{\langle \Lambda_{ij} : i < j\rangle}\Big) \cap C,$$

where U_{ij} are neighborhoods of 0 in Λ_{ij}, with $1 \le i \le n, 1 \le j \le n$, $i \ne j$.

Then, $\rho(V) = U$. Hence ρ is an open continuous morphism.

Note that ρ is surjective and $\ker\rho = I$. By continuity of ρ it follows that I is closed in C. Since R is semisimple, $J(C) \subseteq I$.

We affirm that $I \subseteq J(C)$. Since X is a dense subset of I, if suffices to show that every element of X is quasiregular (because I is a compact ideal). We will prove that for any $x \in X$, the element $x + J(C)$ is quasiregular. Let $x = \sum_{i,j\in\mathbb{N}^+} x_{ij} \in X$; then, by the definition of I, $x_{ij} \in J_{ij}$ for $i \ge j$. Fix $s \in \{1, \ldots, n\}$. Then $K = \langle J_{si} : i \in \mathbb{N}^+\rangle$ is a right ideal of C. Note that the ideal K is topologically nilpotent. We will prove by induction on t, that $K^t \subseteq J_{ss}^{t-1}J_{s1} + J_{ss}^{t-1}J_{s2} + \cdots$. If $t = 1$ the inclusion is obvious (put $J_{ss}^0 = C$). If the inclusion is true for t, then

$$K^{t+1} \subseteq \big(\langle J_{si} : i \in \mathbb{N}^+\rangle\big)\big(J_{ss}^{t-1}J_{s1} + J_{ss}^{t-1}J_{s2} + \cdots\big).$$

Since

$$\big(\langle J_{si} : i \in \mathbb{N}^+\rangle\big) J_{ss}^{t-1} \subseteq J_{ss}^t,$$

the inclusion is true for every t. Since Λ has a local base consisting of two-sided ideals, and $J_{ss} \subseteq J(\Lambda)$ is topologically nilpotent, hence K is topologically nilpotent.

Then we have:

$$x + J(C) = \sum_{i,j\in\mathbb{N}^+} x_{ij} + J(C).$$

But $\sum_{i,j\in\mathbb{N}^+} x_{ij}$ is a nilpotent element, so $x + J(C)$ is nilpotent. Furthermore I is a quasiregular ideal, hence $I \subseteq J(C)$. □

Theorem 3.15. *Let Λ be a compact ring with identity and a countable system of idempotents, satisfying conditions $(\theta_2), (\Sigma)$. Let D be a torus and B the radical upper triangular subgroup of $U(\Lambda)$. Then for every closed subgroup H, $D \subseteq H \subseteq B$, there exists a D-net σ such that $H = G(\sigma)$.*

Proof. Let $\sigma = (\sigma_{ij}), i, j \in \mathbb{N}^+$, be the D-net associated to H (see the Definition 2.9). Since $D \le H$, $E(\sigma) \le H$, and by Theorem 3.12, we obtain that $G(\sigma) = D.E(\sigma) \le H$.

We will prove that $H \le G(\sigma)$. It suffices to show that $a_{ij} \in \sigma_{ij}, i \ne j$, for all $a = \sum_{i,j\in\mathbb{N}^+} a_{ij} \in H$, or equivalently, $t_{ij}(a_{ij}) \in H$, for $i \ne j$.

By condition (θ_2), there exists an element $\varepsilon \in \Lambda_r$, for any $r \in \mathbb{N}^+$, such that $\varepsilon, e_r + \varepsilon, e_r + \varepsilon + \varepsilon^2 \in U(\Lambda_r)$.

Let $a = \sum_{i,j\in\mathbb{N}^+} a_{ij} \in H$; denote $a^{-1} = \sum\limits_{i,j\in\mathbb{N}^+} a'_{ij}$. Put $b = ad_r(e_r+\varepsilon)a^{-1} \in H$; we obtain:

$$b = 1 + a\varepsilon a^{-1} = 1 + \sum_{i,j\in\mathbb{N}^+} a_{ir}\varepsilon a'_{rj}.$$

Since $H \leq G(\rho) = B$, by the Proposition 3.11, we obtain that

$$a_{rr}, a'_{rr}, b_{rr} = e_r + a_{rr}\varepsilon a'_{rr} \in U(\Lambda_r).$$

Denote

$$\eta = b_{rr}^{-1} a_{rr}\varepsilon a'_{rr} = (e_r + a_{rr}\varepsilon a'_{rr})^{-1} a_{rr}\varepsilon a'_{rr} \in U(\Lambda_r).$$

ut $c = bd_r(\eta)a \in H$. We have:

$$\begin{aligned} c &= \left(1 + \sum_{i,j\in\mathbb{N}^+} a_{ir}\varepsilon a'_{rj}\right)(1 - e_r + \eta) \sum_{i,j\in\mathbb{N}^+} a_{ij} \\ &= \left[1 + \sum_{i,j\in\mathbb{N}^+} a_{ir}\varepsilon a'_{rj} - e_r + \eta + \sum_{i,j\in\mathbb{N}^+} a_{ir}\varepsilon a'_{rr}(\eta - e_r)\right] \sum_{i,j\in\mathbb{N}^+} a_{ij} \\ &= \sum_{i,j\in\mathbb{N}^+} a_{ij} + \sum_{i\in\mathbb{N}^+} a_{ir}\varepsilon + (\eta - e_r)\sum_{j\in\mathbb{N}^+} a_{rj} + \sum_{i,j\in\mathbb{N}^+} a_{ir}\varepsilon a'_{rr}(\eta - e_r)a_{rj}. \end{aligned}$$

Then $c_{rr} = a_{rr}\varepsilon$ and $c_{ri} = 0$, for $i \neq r$. Applying Lemma 3.13 for $c \in H$, we obtain that $t_{ir}(c_{ir}) \in H$, for every $i \neq r$.

We have:

$$c_{ir} = a_{ir} + a_{ir}\varepsilon + a_{ir}\varepsilon a'_{rr}(\eta - e_r)a_{rr} = a_{ir}\left[e_r + \varepsilon + \varepsilon a'_{rr}(\eta - e_r)a_{rr}\right].$$

Denote $\mu = e_r + \varepsilon + \varepsilon a'_{rr}(\eta - e_r)a_{rr}$; then, $c_{ir} = a_{ir}\mu$. Since

$$\eta - e_r = -(e_r + a_{rr}\varepsilon a'_{rr})^{-1},$$

we obtain that:

$$\begin{aligned} \mu &= e_r + \varepsilon - \left[a_{rr}^{-1}\left(e_r + a_{rr}\varepsilon a'^{-1}_{rr}\varepsilon^{-1}\right)\right]^{-1} \\ &= e_r + \varepsilon - \left(a_{rr}^{-1}a'^{-1}_{rr}\varepsilon^{-1} + 1\right)^{-1} \\ &= e_r + \varepsilon - \left[(e_r + \varepsilon a'_{rr}a_{rr})\,a_{rr}^{-1}a'^{-1}_{rr}\varepsilon^{-1}\right]^{-1} \\ &= e_r + \varepsilon - \varepsilon a'_{rr}a_{rr}(e_r + a_{rr}\varepsilon a'_{rr})^{-1}. \end{aligned}$$

Hence, $\mu(e_r + \varepsilon a'_{rr}a_{rr}) = e_r + \varepsilon + \varepsilon^2 a'_{rr}a_{rr}$.

Using Theorem 3.14, we have that $e_r+\varepsilon+\varepsilon^2 a'_{rr}a_{rr} = e_r+\varepsilon+\varepsilon^2 \pmod{J(C)}$, and since $e_r + \varepsilon + \varepsilon^2 \in U(\Lambda_r)$, we obtain $e_r + \varepsilon + \varepsilon^2 a'_{rr}a_{rr} \in U(\Lambda_r)$, hence $\mu \in U(\Lambda_r)$.

Since $t_{ir}(c_{ir}) = t_{ir}(a_{ir}\mu) \in H$, and using the relation $(**)$, we obtain that $t_{ir}(a_{ir}) \in H$ if $i \neq j$. We proved that $H \leq G(\sigma)$, hence $H = G(\sigma)$. □

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Department of Mathematics,
University of Oradea,
Oradea, ROMANIA
E-mail address: tadela@uoredea.ro