Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
19 (2003), 195–204
www.emis.de/journals

ON DERIVED AND INTEGRATED SETS OF BASIC SETS OF POLYNOMIALS OF SEVERAL COMPLEX VARIABLES

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ABSTRACT. In this paper, we study some derived and integrated basic sets of polynomials of several complex variables in complete Reinhardt domains and in hyperelliptical regions under some generalized differential and integral operators. Our new results extend and improve a lot of known works (see [3], [9], [10] and [17]).

1. Introduction

The problem of derived and integrated sets of basic sets of polynomials in one and two complex variables has been studied by many authors (see e.g. [3], [9], [10] and [17]). In all above studies the authors considered the unit disk $\Delta = \{z : |z| < 1\}$ in the complex plane $\mathbb{C}$, spherical regions and circles. In the present paper we will consider this problem for several complex variables in complete Reinhardt domains. A parallel study for this problem in hyperelliptical regions is also carried out.

Let $\mathbf{z} = (z_1, z_2, \ldots, z_n)$ be an element of $\mathbb{C}^n$; the space of several complex variables, an open hyperelliptical region of radii $r_s(> 0)$; $s \in I = \{1, 2, 3, \ldots, n\}$ is here denoted by $E_{[\mathbf{r}]}$ and its closure by $\overline{E}_{[\mathbf{r}]}$. Also, an open complete Reinhardt domain of radii $\rho_s(> 0)$; $s \in I$ is here denoted by $\Gamma_{[\rho]}$ and its closure by $\overline{\Gamma}_{[\rho]}$; we consider $D(\overline{\Gamma}_{[\rho]})$ and $D(\bar{E}_{[\mathbf{r}]})$ to denote unspecified domains containing the closed complete Reinhardt domain $\overline{\Gamma}_{[\rho]}$ and the closed hyperellipse $\overline{E}_{[\mathbf{r}]}$, respectively.

In terms of the introduced notations, these regions satisfy the inequalities:

$$\Gamma_{[\rho]} = \Gamma_{[\rho_1,\rho_2,\ldots,\rho_n]} = \big\{\mathbf{z} \in \mathbb{C}^n : |z_s| < \rho_s;\ \ s \in I\big\},$$
$$\overline{\Gamma}_{[\rho]} = \overline{\Gamma}_{[\rho_1,\rho_2,\ldots,\rho_n]} = \big\{\mathbf{z}_s \in \mathbb{C}^n : |z_s| \le \rho_s;\ \ s \in I\big\},$$

Consider unspecified domain containing the closed complete Reinhardt domain $\bar{\Gamma}_{[\rho]}$. This domain will be of radii ρ_s^*, $\rho_s^* > \rho_s$; $s \in I$, then making a contraction to this domain, we will get the domain $D([\rho^+]) = D([\rho_1^+, \rho_2^+, \ldots, \rho_n^+])$, where ρ_s^+ stand for the right-limits of ρ_s^* at ρ_s; $s \in I$. So,

$$D(\bar{\Gamma}_{[\rho]}) = D([\rho^+]) = D(\rho_1^+, \rho_2^+, \ldots, \rho_n^+) = \big\{\mathbf{z} \in \mathbb{C}^n : |z_s| \le \rho_s;\ \ s \in I\big\},$$
$$\mathbf{E}_{[\mathbf{r}]} = \{\mathbf{w} : |\mathbf{w}| < 1\}, \qquad \overline{\mathbf{E}}_{[\mathbf{r}]} = \{\mathbf{w} : |\mathbf{w}| \le 1\},$$

where $\mathbf{w} = (w_1, w_2, w_3, \ldots, w_n)$, $w_s = \frac{z_s}{r_s}$; $s \in I$.

Thus the function $f(\mathbf{z})$ of the complex variables z_s; $s \in I$ which is regular in $\overline{\mathbf{E}}_{[\mathbf{r}]}$ can be represented by the power series

$$f(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} a_{\mathbf{m}}\mathbf{z}^{\mathbf{m}} = \sum_{(m_1,m_2,\ldots,m_n)=\mathbf{0}}^{\infty} a_{m_1,m_2,\ldots,m_n} z_1^{m_1} z_2^{m_2} \ldots z_n^{m_n} \tag{1}$$

2000 *Mathematics Subject Classification.* 32A05, 32A15, 32A99.
Key words and phrases. Polynomials, Operators, Complete Reinhardt domains.

where $\mathbf{m} = (m_1, m_2, \ldots, m_n)$ be multi-indices of non-negative integers.

For the function $f(\mathbf{z})$, we have from [6] that

$$M\big[f;\mathbf{r}\big] = M\big[f; r_1, r_2, \ldots, r_n\big] = \sup_{\overline{\mathbf{E}}_{[\mathbf{r}]}} \big|f(\mathbf{z})\big|, \tag{2}$$

then it follows that

$$\lim_{<\mathbf{m}>\to\infty} \sup \left\{ \frac{|a_{\mathbf{m}}|}{\sigma_{\mathbf{m}} \prod_{s=1}^{n} \big[r_s\big]^{<\mathbf{m}>-m_s}} \right\}^{\frac{1}{<\mathbf{m}>}} \leq \frac{1}{\prod_{s=1}^{n} r_s},$$

where $<\mathbf{m}>= m_1 + m_2 + \cdots + m_n$ and

$$\sigma_{\mathbf{m}} = \inf_{|t|=1} \frac{1}{t^{\mathbf{m}}} = \frac{\big\{< \mathbf{m} >\big\}^{\frac{<\mathbf{m}>}{2}}}{\prod_{s=1}^{n} m_s^{\frac{m_s}{2}}} \quad \text{(see [6] and [15])},$$

$1 \leq \sigma_{\mathbf{m}} \leq (\sqrt{n})^{<\mathbf{m}>}$ on the assumption that $m_s^{\frac{m_s}{2}} = 1$, whenever $m_s = 0$; $s \in I$.

Also, as above if the function $f(\mathbf{z})$ of the complex variables z_s; $s \in I$ which is regular in $\overline{\Gamma}_{[\rho]}$ can be represented by the power series (1), then it follows that

$$|a_{\mathbf{m}}| \leq \frac{M(f, [\rho'])}{\rho'^{\mathbf{m}}}, \quad m_s \geq 0; \quad s \in I,$$

where

$$M(f, [\rho']) = \max_{\overline{\Gamma}_{[\rho']}} |f(\mathbf{z})|. \tag{3}$$

Hence, we get

$$\lim_{<\mathbf{m}>\to\infty} \sup \left\{ |a_{\mathbf{m}}| \prod_{s=1}^{n} \rho_s^{-<\mathbf{m}>+m_s} \right\}^{\frac{1}{<\mathbf{m}>}} \leq \frac{1}{\prod_{s=1}^{n} \rho'_s}.$$

Since, ρ'_s can be taken arbitrary near to ρ_s; $s \in I$, we conclude that

$$\lim_{<\mathbf{m}>\to\infty} \sup \left\{ |a_{\mathbf{m}}| \prod_{s=1}^{n} \rho_s^{-<\mathbf{m}>+m_s} \right\}^{\frac{1}{<\mathbf{m}>}} \leq \frac{1}{\prod_{s=1}^{n} \rho_s}.$$

Definition 1.1 ([13, 14])**.** A set of polynomials $\{P_{\mathbf{m}}[\mathbf{z}]\} = \big\{P_0, P_1, P_2, \ldots, P_n, \ldots\big\}$ is said to be basic when every polynomial in the complex variables z_s; $s \in I$; can be uniquely expressed as a finite linear combination of the elements of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$.

Thus according to [13 ; Th. 5] the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be basic if and only if there exists a unique row-finite matrix $\overline{P}$ such that

$$\overline{P}P = P\overline{P} = \mathbf{I}, \tag{4}$$

where $P = \big[P_{\mathbf{m},\mathbf{h}}\big]$ is the matrix of coefficients, ($\mathbf{h} = (h_1, h_2, \ldots, h_n)$ be multi-indices of non-negative integers), $\overline{P}$ is the matrix of operators of the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and $\mathbf{I}$ is the infinite unit matrix.

For the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ and its inverse $\big\{\overline{P}_{\mathbf{m}}[\mathbf{z}]\big\}$, we have

$$P_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}, \tag{5}$$

$$\mathbf{z}^{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P_{\mathbf{h}}[\mathbf{z}] = \sum_{\mathbf{h}} P_{\mathbf{m},\mathbf{h}} \overline{P}_{\mathbf{h}}[\mathbf{z}], \tag{6}$$

$$\overline{P}_{\mathbf{m}}[\mathbf{z}] = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} \mathbf{z}^{\mathbf{h}}, \tag{7}$$

Thus, for the function $f(\mathbf{z})$ given in (1) we get

$$f(\mathbf{z}) = \sum_{\mathbf{m}} \Pi_{\mathbf{m}} P_{\mathbf{m}}[\mathbf{z}],$$

where

$$\Pi_{\mathbf{m}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h},\mathbf{m}}\, a_{\mathbf{h}} = \sum_{\mathbf{h}} \overline{P}_{\mathbf{h},\mathbf{m}} \frac{f^{(\mathbf{h})}(\mathbf{0})}{h_s!},$$

where $h! = h(h-1)(h-2)\cdots 3\cdot 2\cdot 1$. The series $\sum_{\mathbf{m}=\mathbf{0}}^{\infty} \Pi_{\mathbf{m}} P_{\mathbf{m}}[\mathbf{z}]$ is the associated basic series of $f(\mathbf{z})$.

Let $N_{\mathbf{m}} = N_{m_1,m_2,\ldots,m_n}$ be the number of non-zero coefficients $\overline{P}_{\mathbf{m},\mathbf{h}}$ in the representation (6). A basic set satisfying the condition

$$\lim_{<\mathbf{m}>\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = 1 \tag{8}$$

is called, as in [13, 14] a Cannon set and if

$$\lim_{<\mathbf{m}>\to\infty} \{N_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = a > 1,$$

then the set is called a general basic set.

Now, let $\mathcal{D}_{\mathbf{m}} = \mathcal{D}_{m_1,m_2,m_3,\ldots,m_n}$ be the degree of the polynomial of the highest degree in the representation (6). That is to say, if $\mathcal{D}_{\mathbf{h}} = \mathcal{D}_{h_1,h_2,h_3,\ldots,h_n}$ is the degree of the polynomial $P_{\mathbf{h}}[\mathbf{z}]$, then $\mathcal{D}_{\mathbf{h}} \le \mathcal{D}_{\mathbf{m}}\ \forall h_s \le m_s$; $s \in I$, and since the elements of the basic set are linearly independent, then $N_{\mathbf{m}} \le 1+2+3+\cdots+\big(\mathcal{D}_{\mathbf{m}}+1\big) \le \lambda_1 \mathcal{D}_{\mathbf{m}}^2$, where λ_1 be a constant.

Therefore, the condition (8) for a basic set to be Cannon set implies the following condition:

$$\lim_{<\mathbf{m}>\to\infty} \{\mathcal{D}_{\mathbf{m}}\}^{\frac{1}{<\mathbf{m}>}} = 1 \quad \text{(see [17]).} \tag{9}$$

For any function $f(\mathbf{z})$ of several complex variables, there is formally an associated basic series $\sum_{\mathbf{h}=\mathbf{0}}^{\infty} P_{\mathbf{h}}[\mathbf{z}]$. When this associated series converges uniformally to $f(\mathbf{z})$ in some domain it is said to represent $f(\mathbf{z})$ in that domain; in other words, as in the classical terminology of Whittaker (see [19]), the basic set $P_{\mathbf{m}}[\mathbf{z}]$ will be effective in that domain. The convergence properties of basic sets of polynomials are classified according to the classes of functions represented by their associated basic series and also to the domain in which are represented.

To study the convergence properties of such basic sets of polynomials in complete Reinhardt domains and in hyperelliptical regions we consider the following notations for Cannon sums:

$$\mu(P_{\mathbf{m}},[\rho]) = \prod_{s=1}^{n} \{\rho_s\}^{<\mathbf{m}>-m_s} \sum_{\mathbf{h}} |\overline{P}_{\mathbf{m},\mathbf{h}}| M(P_{\mathbf{h}},[\rho]) \quad \text{(for Reinhardt domains, [8]);} \tag{10}$$

$$\Omega\big[P_{\mathbf{m}},\mathbf{r}\big] = \sigma_{\mathbf{m}} \prod_{s=1}^{n} \{r_s\}^{<\mathbf{m}>-m_s} \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} M[P_{\mathbf{h}},\mathbf{r}] \quad \text{(for hyperelliptical regions, [6]).} \tag{11}$$

Also, the Cannon function for the basic sets of polynomials in complete Reinhardt domains (see [8]) and in hyperelliptical regions (see [6]) were defined as follows:

$$\mu\big(P,[\rho]\big) = \lim_{<\mathbf{m}>\to\infty} \big\{\mu\big(P_{\mathbf{m}},[\rho]\big)\big\}^{\frac{1}{<\mathbf{m}>}} \quad \text{and} \quad \Omega\big[P,\mathbf{r}\big] = \lim_{<\mathbf{m}>\to\infty} \big\{\Omega\big[P_{\mathbf{m}},\mathbf{r}\big]\big\}^{\frac{1}{<\mathbf{m}>}}.$$

Concerning the effectiveness of the basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ in complete Reinhardt domains, we have the following results:

Theorem A ([16, 18]). *A necessary and sufficient condition for a Cannon set* $\{P_{\mathbf{m}}[\mathbf{z}]\}$ *to be*

(i) effective in $\overline{\Gamma}_{[\rho]}$ *is that* $\mu(P,[\rho]) = \prod_{s=1}^{n} \rho_s$,

(ii) effective in $D([\overline{\Gamma}_{\rho]})$ *is that* $\mu(P,[\rho^+]) = \prod_{s=1}^{n} \rho_s$.

Recently, Kishka and myself (see [6]) obtained the following results:

Theorem B ([6]). *The necessary and sufficient condition for the Cannon basic set* $\{P_{\mathbf{m}}[\mathbf{z}]\}$ *of polynomials of several complex variables to be effective in the closed hyperellipse* $\overline{\mathbf{E}}_{[\mathbf{r}]}$ *is that* $\Omega\big[P,\mathbf{r}\big] = \prod_{s=1}^{n} r_s$, *where* $\mathbf{r} = r_1, r_2, \ldots, r_n$.

Theorem C ([6]). *The Cannon basic set* $\{P_{\mathbf{m}}[\mathbf{z}]\}$ *of polynomials of several complex variables will be effective in* $D\big(\bar{E}_{[\mathbf{r}]}\big)$, *if and only if,* $\Omega\big[P,\mathbf{r}^+\big] = \prod_{s=1}^{n} r_s$.

Note that $D\big(\bar{E}_{[\mathbf{r}]}\big)$ can be defined similar to $D\big(\bar{\Gamma}_{[\rho]}\big)$.

For more information about the study of basic sets of polynomials, we refer to [1], [2], [4], [5], [6], [7], [12], [17] and [19].

Now we define the differential and integral operators J and Λ acting on the monomial $\mathbf{z}^{\mathbf{m}}$, such that

$$J(D) = D^n \mathbf{z}^{\mathbf{m}} = \begin{cases} \sum_{s=1}^{n} m_s(m_s-1)\ldots(m_s-n_s+1) z_s^{m_s}, & n_1 \le m_1 \text{ or } \ldots \text{ or } n_s \le m_s; \\ \mathbf{z}^{\mathbf{m}}, & n_s > m_s, \end{cases} \tag{12}$$

where $D^n = \sum_{s=1}^{n} z_s^{n_s} D_{z_s}^{n_s} = \sum_{s=1}^{n} z_s^{n_s} \frac{\partial}{\partial z_s}\frac{\partial}{\partial z_s}\cdots\frac{\partial}{\partial z_s}$, the derivatives is repeated n_s-times; $s \in I$.

The integral operator Λ will be

$$\Lambda(I) = I^n \mathbf{z}^{\mathbf{m}} = \sum_{s=1}^{n} \frac{1}{z_s^{n_s}} \int_0^{z_s}\int_0^{z_s}\int_0^{z_s}\cdots\int_0^{z_s} \mathbf{z}^{\mathbf{m}} dz_s dz_s \ldots dz_s$$

$$= \begin{cases} \mathbf{z}^{\mathbf{m}} \sum_{s=1}^{n}\left(\frac{1}{\prod_{j=1}^{n_s}(m_s+j)}\right), & \langle \mathbf{m}\rangle \ne \mathbf{0}; \\ 1, & \langle \mathbf{m}\rangle = \mathbf{0}, \end{cases} \tag{13}$$

where the integration is repeated $n_s -$ times.

Special cases of these operators $J(D)$ and $\Lambda(I)$ were introduced in [3], [9], [10] and [12].

2. Derived basic sets of polynomials in $\mathbb{C}^n$

Let $\{P_{\mathbf{m}}[\mathbf{z}]\}$ be a basic set of polynomials of the several complex variables z_s; $s \in I$ as given in (5). Consider the next questions: If we have a basic set of polynomials of several complex variables $\{P_{\mathbf{m}}[\mathbf{z}]\}$, and we effect on it by the operator $J(D)$, do the new set $\{J(D)P_{\mathbf{m}}[\mathbf{z}]\}$ still basic?. Also, if the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in specific regions (hyperelliptical regions or complete Reinhardt domains), do the set $\{J(D)P_{\mathbf{m}}[\mathbf{z}]\}$ still effective in the same regions?. In this paper we will give the answers of these questions. Similar results for the integral operator $\Lambda(I)$ can be obtained.

In this section, we shall study the differential set $\{P_k(J)P_{\mathbf{m}}[\mathbf{z}]\}$, where

$$J^k(D) = \left[\sum_{s=1}^{n} z_s^{n_s} D_{z_s}^{n_s}\right]^k \mathbf{z}^{\mathbf{m}} = J^{k-1} J \mathbf{z}^{\mathbf{m}}; \ k \text{ be a finite positive integer.} \tag{14}$$

Now, consider the differential operator $J^k(D)$ as given in (14) acting on the monomial $\mathbf{z}^{\mathbf{m}}$. suppose that the set

$$\{P_k(J)P_{\mathbf{m}}[\mathbf{z}]\} = \{P^*_{\mathbf{m}}[\mathbf{z}]\}$$

of polynomials is called the $P_k(J)-$set of polynomials of several complex variables z_s; $s \in I$, associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$. Thus, we get

$$\sum\nolimits_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P^*_{\mathbf{h}}[\mathbf{z}] = \sum_{j=1}^{k} \lambda_j \Big[\sum_{s=1}^{n} m_s(m_s-1)(m_s-2)\ldots(m_s-n_s+1) \Big]^j \mathbf{z}^{\mathbf{m}}, \tag{15}$$

where λ_j are constants $\neq 0$, $n_1 \leq m_1$ or $n_2 \leq m_2$ or ... or $n_s \leq m_s$ and

$$\sum_{\mathbf{h}} \overline{P}_{\mathbf{0},\mathbf{h}} P^*_{\mathbf{h}}[\mathbf{z}] = 1; \quad n_s > m_s \quad ; \quad s \in I.$$

Now, we shall prove that the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ is a basic set of polynomials. It is enough for this purpose to show that the polynomials $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ are linearly independent.

In fact suppose that there is a linear relation of the form:

$$\sum_{i=1}^{n} c_i P^*_{\mathbf{m};i}[\mathbf{z}] \equiv \mathbf{0}; \quad c_i \neq 0 \tag{16}$$

for at least one i, $i \in I$. Then

$$P_k(J)\Big[\sum_{i=1}^{n} c_i P_{\mathbf{m};i}[\mathbf{z}]\Big] = \mathbf{0}.$$

Hence, it follows that

$$\sum_{i=1}^{n} c_i P_{\mathbf{m},i}[\mathbf{z}] \equiv \mathbf{0}.$$

This means that $\{P_{\mathbf{m}}[\mathbf{z}]\}$ would not be linearly independent, then the set would not be basic, consequently (16) is impossible. Thus the relation (15) is unique and hence the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ is a basic set.

Hence a representation of the monomial $\mathbf{z}^{\mathbf{m}}$ by the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials is possible.

To obtain a fundamental inequality concerning the Cannon sum $\mu(P^*_{\mathbf{m}}[\mathbf{z}], [\rho])$ of the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ in complete Reinhardt domains, write

$$\mu(P^*_{\mathbf{m}}, [\rho]) = \frac{\big[\prod_{s=1}^{n} \{\rho_s\}^{\langle \mathbf{m}\rangle - m_s}\big] \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} M(P^*_{\mathbf{h}}, [\rho])}{\sum_{j=1}^{k} \lambda_j \big[\sum_{s=1}^{n} m_s(m_s-1)\ldots(m_s-n_s+1)\big]^j}.$$

Since,

$$P^*_{\mathbf{h}}[\mathbf{z}] = \Big[\sum_{s=1}^{n} z_s^{n_s} D^{n_s}_{z_s} P_{\mathbf{h}}[\mathbf{z}]\Big]^k, \quad \text{then}$$

$$M\big(P^*_{\mathbf{h}}, [\rho]\big) = \max_{\overline{\Gamma}[\rho]} |P^*_{\mathbf{h}}| \leq \zeta \, N_{\mathbf{m}} \; M\big(P_{\mathbf{h}}, [\rho]\big).$$

where ζ be a constant. Therefore,

$$\mu\big(P^*_{\mathbf{h}}, [\rho]\big) \leq \zeta \, N_{\mathbf{m}} \; \frac{\big[\prod_{s=1}^{n} \{\rho_s\}^{<\mathbf{m}>-m_s}\big] \sum_{\mathbf{h}} |\overline{P}_{\mathbf{m},\mathbf{h}}| M(P_{\mathbf{h}}, [\rho])}{\sum_{j=1}^{k} \lambda_j \big[\sum_{s=1}^{n} m_s(m_s-1)\ldots(m_s-n_s+1)\big]^j},$$

which implies that,

$$\mu\big(P^*_{\mathbf{m}}, [\rho]\big) \leq \zeta \frac{N_{\mathbf{m}} \; \mu\big(P_{\mathbf{m}}, [\rho]\big)}{\sum_{j=0}^{k} \lambda_j \big[\sum_{s=1}^{n} m_s(m_s-1)\ldots(m_s-n_s+1)\big]^j}.$$

Thus,

$$\begin{aligned}\mu\big(P^*,[\rho]\big) &= \lim_{<\mathbf{m}>\to\infty}\sup\Big\{\mu\big(P^*_{\mathbf{m}},[\rho]\big)\Big\}^{\frac{1}{<\mathbf{m}>}}\\ &\le \lim_{<\mathbf{m}>\to\infty}\sup\Big\{\mu\big(P_{\mathbf{m}},[\rho]\big)\Big\}^{\frac{1}{<\mathbf{m}>}} = \mu(P,[\rho]) = \prod_{s=1}^{n}\rho_s.\end{aligned}$$

But

$$\mu\big(P^*,[\rho]\big) \ge \prod_{s=1}^{n}\rho_s.$$

Then, we deduce that

$$\mu\big(P^*,[\rho]\big) = \prod_{s=1}^{n}\rho_s. \tag{17}$$

Therefore, by using Theorem A, we obtain the following theorem:

Theorem 2.1. *If the Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables z_s; $s\in I$, possess one or another of the properties*

1. Effectiveness in the closed complete Reinhardt domain $\overline{\Gamma}_{[\rho]}$.

2. Effectiveness in the closed region $D(\overline{\Gamma}_{[\rho]})$.

Then the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will possess the same property as well as.

To get the relation concerning the Cannon sum $\Omega[P^*_{\mathbf{m}},\mathbf{r}]$ of the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ in $\overline{E}_{[\mathbf{r}]}$, suppose that $f(\mathbf{z})$ be a regular function in $\overline{E}_{[\mathbf{r}]}$, then

$$\begin{aligned} f^*(\mathbf{z}) &= P_k(J)f(\mathbf{z})\\ &= \eta a_{\mathbf{0}} + \sum_{\mathbf{m}} a_{\mathbf{m}}\sum_{j=0}^{k}\lambda_j\Big[\sum_{s=1}^{n} m_s(m_s-1)(m_s-2)\ldots(m_s-n_s+1)\Big]^j\mathbf{z}^{\mathbf{m}},\end{aligned} \tag{18}$$

where η be a constant.

It is clear that $f^*(\mathbf{z})$ is also regular in $\overline{E}_{[\mathbf{r}]}$. Hence from (1) and (18), we get

$$f^*(\mathbf{z}) \sim \sum_{\mathbf{h}} \overline{P}_{\mathbf{h}} f^*_{\mathbf{0}} P^*_{\mathbf{h}}[\mathbf{z}],$$

where

$$\overline{P}_{\mathbf{h}} f^*_{\mathbf{0}} = \eta a_{\mathbf{0}} + \sum_{\mathbf{m}=1} a_{\mathbf{m}}\sum_{j=0}^{k}\lambda_j\Big[\sum_{s=1}^{n} m_s(m_s-1)(m_s-2)\ldots(m_s-n_s+1)\Big]^j\overline{P}_{\mathbf{m};\mathbf{h}}.$$

If the basic series associated with every regular function in $\overline{E}_{[\mathbf{r}]}$ converges uniformly in $\overline{E}_{[\mathbf{r}]}$ to $f^*(\mathbf{z})$, then we can say that the basic $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ is a Cannon set of polynomials. Then, the Cannon sum $\Omega\big[P^*_{\mathbf{m}},\mathbf{r}\big]$ of the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ in $\overline{E}_{[\mathbf{r}]}$, will has the form

$$\Omega\big[P^*_{\mathbf{m}},\mathbf{r}\big] = \frac{\sigma_{\mathbf{m}}\prod_{s=1}^{n}\{r_s\}^{<\mathbf{m}>-m_s}\sum_{\mathbf{h}}|\overline{P}_{\mathbf{m},\mathbf{h}}|M\big[P^*_{\mathbf{h}},\mathbf{r}\big]}{\sum_{j=1}^{k}\lambda_j\big[\sum_{s=1}^{n} m_s(m_s-1)(m_s-2)\ldots(m_s-n_s+1)\big]^j}$$

where,

$$M\big[P^*_{\mathbf{h}},\mathbf{r}\big] = \max_{\overline{E}_{[\mathbf{r}]}}\big|P^*_{\mathbf{h}}[\mathbf{z}]\big|.$$

Now we let, $\mathcal{D}_{\mathbf{m}}$ be the degree of the polynomial of the highest degree in the representation (6). Hence by Cauchy's inequality we see that

$$
\begin{aligned}
(19)\quad M\big[P^*_{\mathbf{m}},\mathbf{r}\big] &= \max_{\overline{E}_{[\mathbf{r}]}}\big|P^*_{\mathbf{h}}[\mathbf{z}]\big| \le |P_{\mathbf{h};\mathbf{0}}| + \sum_{\mathbf{h}\ge 1}|P^*_{\mathbf{m};\mathbf{h}}|\frac{\prod_{s=1}^{n} r_s^{h_s}}{\sigma_{\mathbf{h}}} = |P_{\mathbf{h};\mathbf{0}}| \\
&+ \sum_{\mathbf{h}\ge 1}\sum_{j=1}^{k}\lambda_j\left[\sum_{s=1}^{n} m_s(m_s-1)(m_s-2)\dots(m_s-n_s+1)\right]^j \sigma_{\mathbf{h}}\frac{M\big[P_{\mathbf{h}},\mathbf{r}\big]}{\prod_{s=1}^{n} r_s^{h_s}}\frac{\prod_{s=1}^{n} r_s^{h_s}}{\sigma_{\mathbf{h}}} \\
&\le K_1 N_{\mathbf{m}}\mathcal{D}^n_{\mathbf{m}} M\big[P_{\mathbf{h}},\mathbf{r}\big] \le K_1\mathcal{D}^{n+2}_{\mathbf{m}} M\big[P_{\mathbf{h}},\mathbf{r}\big],
\end{aligned}
$$

where K_1 be a constant and the power n here because we differentiated n_s-times. Then,

$$
\begin{aligned}
(20)\quad \Omega\big[P^*_{\mathbf{m}},\mathbf{r}\big] &\le K_1\frac{\sigma_{\mathbf{m}}\mathcal{D}^{n+2}_{\mathbf{m}}\prod_{s=1}^{n}\{r_s\}^{<\mathbf{m}>-m_s}\sum_{\mathbf{h}}|\overline{P}_{\mathbf{m},\mathbf{h}}|M\big[P_{\mathbf{h}},\mathbf{r}\big]\big]}{\sum_{j=0}^{k}\lambda_j\left[\sum_{s=1}^{n} m_s(m_s-1)(m_s-2)\dots(m_s-n_s+1)\right]^j} \\
&= K_2\,\Omega[P_{\mathbf{m}},\mathbf{r}],
\end{aligned}
$$

where

$$
K_2 = \frac{K_1\sigma_{\mathbf{m}}\mathcal{D}^{n+2}_{\mathbf{m}}}{\sum_{j=1}^{k}\lambda_j\left[\sum_{s=1}^{n} m_s(m_s-1)(m_s-2)\dots(m_s-n_s+1)\right]^j}
$$

Consider condition (9), we obtain that

$$
\begin{aligned}
\Omega\big[P^*,\mathbf{r}\big] &= \lim_{<\mathbf{m}>\to\infty}\sup\Big\{\Omega\big[P^*_{\mathbf{m}},\mathbf{r}\big]\Big\}^{\frac{1}{<\mathbf{m}>}} \\
&\le \lim_{<\mathbf{m}>\to\infty}\sup\Big\{K_2\,\Omega\big[P_{\mathbf{m}},\mathbf{r}\big]\Big\}^{\frac{1}{<\mathbf{m}>}} \le \Omega\big[P,\mathbf{r}\big] = \prod_{s=1}^{n} r_s.
\end{aligned}
$$

But

$$
\Omega\big[P^*,\mathbf{r}\big] \ge \prod_{s=1}^{n} r_s \qquad \text{(see [6]).}
$$

Then,

$$
(21)\qquad \Omega\big[P^*,\mathbf{r}\big] = \prod_{s=1}^{n} r_s.
$$

Now, suppose that the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective in $\overline{E}_{[\mathbf{r}]}$, then according to (21) and using Theorems B and C, we obtain the following theorem:

Theorem 2.2. *If the Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables z_s; $s \in I$ for which the condition (9) is satisfied, is effective in $\overline{E}_{[\mathbf{r}]}$, then the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will be effective there.*

If, condition (9) not satisfied then the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$ can not be effective in $\overline{E}_{[\mathbf{r}]}$, where the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ is effective there. To ensure this, we give the following example.

Example 2.1. Suppose that the set $\{P_{m_1,m_2}[z_1,z_2]\}$ of polynomials of the two complex variables z_1, z_2, given by

$$P_{m_1,m_2}[z_1,z_2] = \begin{cases} \sigma_{m_1,m_2} z_1^{m_1} z_2^{m_2} + \sigma_{am_1,am_2} z_1^{am_1} z_2^{am_2}, & n_1 \leq m_1 \text{ or } n_2 \leq m_2, \\ \sigma_{m_1,m_2} z_1^{m_1} z_2^{m_2}; & \text{otherwise.} \end{cases}$$

where, $a = b^{m_1+m_2}$; $\quad b > 1$.

It is easy to see that $\Omega\big[P,1\big] = 1$, then the set $\{P_{m_1,m_2}[z_1,z_2]\}$ is effective in $\overline{E}_{[r_1,r_2]}$; $r_1 = r_2 = 1$. Also,

$$\Omega\big[P^*_{m_1,m_2}, r_1, r_2\big] = \zeta(a) \prod_{s=1}^{2} r_s^{a(m_1+m_2)} + \prod_{s=1}^{2} r_s^{m_1+m_2},$$

where $\zeta(a) > 0$ is a constant depending only on a and

$$\Omega\big[P^*, 1\big] = \zeta(a) + 1 > 1,$$

that is to say that the set $\{P^*_{m_1,m_2}[z_1,z_2]\}$ is not effective in $\overline{E}_{[r_1,r_2]}$; $r_1 = r_2 = 1$ although the set $\{P_{m_1,m_2}[z_1,z_2]\}$ is effective there. The reason for this obviously that condition (21) is not satisfied by the set $\{P_{m_1,m_2}[z_1,z_2]\}$ as required.

Remark 2.1. It should be remarked that in Theorem 2.2, we have restricted ourselves by condition (9) for the degree of the derived basic set of polynomials in the hyperellitical regions and not in complete Reinhardt domains because after taking the maximum modulus in the region $\overline{E}_{[\mathbf{r}]}$, the existence of $\sigma_{\mathbf{m}}$ will make the Cannon function for this set tends to infinity, where $\mathcal{D}_{\mathbf{m}} = \mathcal{O}(<\mathbf{m}>)$. We do not need condition (9) for the integrated basic set of polynomials.

3. Integrated basic sets of polynomials in $\mathbb{C}^n$

Now, we consider the integral operator $\Lambda(I)$ as given in (13) acting on the monomial $\mathbf{z}^{\mathbf{m}}$, on the assumption that the integration is carried out with respect to each variable while the others are constants. Suppose that the set $\{\Lambda(I)P_{\mathbf{m}}[\mathbf{z}]\} = \{P^{**}_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials and is called the $\Lambda-$set of polynomials of several complex variables z_s; $s \in I$, associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$. Thus, we get that

$$\begin{cases} \mathbf{z}^{\mathbf{m}}\alpha(\mathbf{m},n) = \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} P^{**}_{\mathbf{h}}[\mathbf{z}]; & n_s \geq k \text{ and} \\ 1 = \sum_{\mathbf{h}} \overline{P}_{\mathbf{0},\mathbf{h}} P^{**}_{\mathbf{h}}[\mathbf{z}]; & \text{otherwise} \end{cases} \tag{22}$$

where,

$$\alpha(\mathbf{m},n) = \sum_{s=1}^{n} \bigg(\frac{1}{\prod_{j=1}^{n_s}(m_s+j)} \bigg).$$

Similarly as in the set $\{P^*_{\mathbf{m}}[\mathbf{z}]\}$, it is easy to prove that the set $\{P^{**}_{\mathbf{m}}[\mathbf{z}]\}$ is a basic set of polynomials.

Now the Cannon sums $\mu\big(\Lambda(I)P_{\mathbf{m}}, [\rho]\big)$ and $\Omega\big[\Lambda(I)P_{\mathbf{m}}, \mathbf{r}\big]$ for the complete Reinhardt domain $\overline{\Gamma}_{[\rho]}$ and the hyperellipse $\overline{E}_{[\mathbf{r}]}$, respectively will be such that

$$\mu\big(\Lambda(I)P_{\mathbf{m}}, [\rho]\big) = \frac{\prod_{s=1}^{n}\{\rho_s\}^{<\mathbf{m}>-m_s}}{\alpha(\mathbf{m},\mathbf{n})} \sum_{\mathbf{h}} \overline{P}_{\mathbf{m},\mathbf{h}} M\big(P^{**}_{\mathbf{h}}, [\rho]\big),$$

where

$$M\big(P^{**}_{\mathbf{h}}, [\rho]\big) = \max_{\overline{\Gamma}_{[\rho]}} |P^{**}_{\mathbf{h}}[\mathbf{z}]| \leq \zeta_1\, N_{\mathbf{m}}\, M\big(P_{\mathbf{h}}[\mathbf{z}], [\rho]\big),$$

where ζ_1 be a constant. Hence,

$$\begin{aligned}(23)\quad &\mu\big(P^{**}_{\mathbf{m}},[\rho]\big)\\ &\le \zeta_1\, N_{\mathbf{m}}\,\frac{\prod_{s=1}^{n}\{\rho_s\}^{<\mathbf{m}>-m_s}}{\alpha(\mathbf{m},\mathbf{n})}\sum_{\mathbf{h}}|\overline{P}_{\mathbf{m},\mathbf{h}}|M\big(P_{\mathbf{h}},[\rho]\big)=\frac{\zeta_1\, N_{\mathbf{m}}}{\alpha(\mathbf{m},n)}\mu\big(P_{\mathbf{m}},[\rho]\big).\end{aligned}$$

Thus,

$$\begin{aligned}(24)\quad \mu\big(P^{**},[\rho]\big)&=\lim_{<\mathbf{m}>\to\infty}\sup\Big\{\mu\big(P^{**}_{\mathbf{m}},[\rho]\big)\Big\}^{\frac{1}{<\langle\mathbf{m}>}}\\ &\le \lim_{<\mathbf{m}>\to\infty}\sup\Big\{\mu\big(P_{\mathbf{m}},[\rho]\big)\Big\}^{\frac{1}{<\mathbf{m}>}}=\mu\big(P,[\rho]\big)=\prod_{s=1}^{n}\rho_s.\end{aligned}$$

In the same manner, we can proceed as in (24) to get the following inequality concerning the hyperelliptical regions

$$\Omega\big[P^{**}_{\mathbf{m}},\mathbf{r}\big]=\frac{N_{\mathbf{m}}\,\sigma_{\mathbf{m}}\prod_{s=1}^{n}\{r_s\}^{<\mathbf{m}>-m_s}}{\alpha(\mathbf{m},n)}\sum_{\mathbf{h}}\bar{P}_{\mathbf{m},\mathbf{h}}M\big[P^{**}_{\mathbf{h}},\mathbf{r}\big]\le\frac{N_{\mathbf{m}}\,\zeta^*}{\alpha(\mathbf{m},n)}\Omega\big[P_{\mathbf{m}},\mathbf{r}\big].$$

where ζ^* be a constant. Thus, we can obtain that

$$(25)\qquad \Omega\big[P^{**},\mathbf{r}\big]\le\Omega\big[P,\mathbf{r}\big]=\prod_{s=1}^{n}r_s.$$

Hence, we have the following theorem:

Theorem 3.1. *If the Cannon basic set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials of the several complex variables z_s; $s\in I$, possess one or another of the properties*

1. *Effectiveness in the closed complete Reinhardt domain $\overline{\Gamma}_{[\rho]}$.*
2. *Effectiveness in the closed domain $D(\overline{\Gamma}_{[\rho]})$.*
3. *Effectiveness in the closed hyperellipse $\overline{E}_{[\mathbf{r}]}$.*
4. *Effectiveness in the closed region $D(\bar{E}_{[\mathbf{r}]})$.*

Then the set $\{P^{**}_{\mathbf{m}}[\mathbf{z}]\}$ of polynomials associated with the set $\{P_{\mathbf{m}}[\mathbf{z}]\}$ will possess the same property as well as.

Remark 3.1. Similar results for derived and integrated general basic sets of polynomials of several complex variables in complete Reinhardt domains and in hyperelliptical regions can be obtained.

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Received October 04, 2002.

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