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SOME INEQUALITIES FOR RICCI CURVATURE OF CERTAIN SUBMANIFOLDS IN SASAKIAN SPACE FORMS

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ABSTRACT. In the present paper, we obtain sharp inequalities between the Ricci curvature and the squared mean curvature for slant,semi-slant and bi-slant submanifolds in Sasakian space forms. Also, estimates of the scalar curvature and the k-Ricci curvature respectively, in terms of the squared mean curvature, are proved.

1. PRELIMINARIES

A $(2m+1)$-dimensional Riemannian manifold $(\tilde{M}, g)$ is said to be a *Sasakian manifold* if it admits an endomorphism ϕ of its tangent bundle $T\tilde{M}$, a vector field ξ and a 1-form η, satisfying:

$$\begin{cases} \phi^2 = -Id + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi), \\ (\tilde{\nabla}_X \phi)Y = -g(X, Y)\xi + \eta(Y)X,\ \tilde{\nabla}_X \xi = \phi X, \end{cases}$$

for any vector fields X, Y on $T\tilde{M}$, where $\tilde{\nabla}$ denotes the Riemannian connection with respect to g.

A plane section π in $T_p\tilde{M}$ is called a *ϕ-section* if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a *ϕ-sectional curvature*. A Sasakian manifold with constant ϕ-sectional curvature c is said to be a *Sasakian space form* and is denoted by $\tilde{M}(c)$.

The curvature tensor of $\tilde{M}(c)$ of a Sasakian space form $\tilde{M}(c)$ is given by [1]

$$\begin{aligned} \tilde{R}(X, Y)Z = \frac{c+3}{4}\{g(Y, Z)X - g(X, Z)Y\}+ \\ +\frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi+ \\ +g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\}, \end{aligned} \tag{1.1}$$

for any tangent vector fields X, Y, Z on $\tilde{M}(c)$.

As examples of Sasakian space forms we mention $\mathbb{R}^{2m+1}$ and S^{2m+1}, with standard Sasakian structures (see [1]).

In [9], A. Lotta has introduced the following notion of slant immersion in almost contact metric manifolds.

Definition. We call a differentiable distribution $\mathcal{D}$ on M a *slant distribution* if for each $x \in M$ and each nonzero vector $X \in \mathcal{D}_x$, the angle $\theta_{\mathcal{D}}(X)$ between ϕX and the vector subspace $\mathcal{D}_x$ is constant, which is independent of the choice of $x \in M$

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and $X \in \mathcal{D}_x$. In this case, the constant angle $\theta_{\mathcal{D}}$ is called the *slant angle* of the distribution $\mathcal{D}$.

Definition. A submanifold M tangent to ξ is said to be *slant* if for any $x \in M$ and any $X \in T_xM$, linearly independent of ξ, the angle between ϕX and T_xM is a constant $\theta \in [0, \frac{\pi}{2}]$, called the *slant angle* of M in $\tilde{M}$.

Examples of slant submanifolds. (see [2]).

Example 1. For any constant k,

$$x(u,v,t) = 2(e^{ku}\cos u\cos v, e^{ku}\sin u\cos v, e^{ku}\cos u\sin v, e^{ku}\sin u\sin v, t)$$

defines a slant submanifold of dimension 3 with slant angle $\theta = \arccos\frac{|k|}{\sqrt{1+k^2}}$, scalar curvature $\tau = \frac{-k^2}{3(1+k^2)}$ and non-constant mean curvature given by $\|H\| = \frac{2e^{-ku}}{3\sqrt{1+k^2}}$. Hence, the submanifold is not minimal.

Example 2. For any constant k,

$$x(u,v,t) = 2(u, k\cos v, v, k\sin v, t)$$

defines a slant submanifold M with slant angle $\theta = \arccos\frac{1}{\sqrt{1+k^2}}$, scalar curvature $\tau = \frac{-1}{3(1+k^2)}$, constant mean curvature given by $\|H\| = \frac{|k|}{3(1+k^2)}$. Moreover, the following statements are equivalent:

(a) $k = 0$;
(b) M is invariant;
(c) M is minimal;
(d) M has parallel mean curvature vector.

Invariant and *anti-invariant immersions* are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant immersion which is neither invariant nor anti-invariant is called a *proper slant immersion.*

Definition. We say that a submanifold M tangent to ξ is a *bi-slant* submanifold of $\tilde{M}$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that :

i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$.
ii) For any $i = 1, 2$, $\mathcal{D}_i$ is slant distribution with slant angle θ_i.

Let $2d_1 = \dim\mathcal{D}_1$ and $2d_2 = \dim\mathcal{D}_2$.

Remark. If either d_1 or d_2 vanishes, the bi-slant submanifold is a slant submanifold. Thus, slant submanifolds (and, therefore, invariant and anti-invariant submanifolds) are particular cases of bi-slant submanifolds.

Examples of bi-slant submanifolds. (see [2], [3])

Example 1. For any $\theta_1, \theta_2 \in [0, \frac{\pi}{2}]$,

$$x(u,v,w,s,t) = 2(u, 0, w, 0, v\cos\theta_1, v\sin\theta_1, s\cos\theta_2, s\sin\theta_2, t)$$

defines a five-dimensional bi-slant submanifold M, with slant angles θ_1 and θ_2, is $\mathbf{R}^9$ with its usual Sasakian structure (ϕ_0, ξ, η, g).

Furthermore, it is easy to see that

$$e_1 = 2(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}), \qquad e_2 = \cos\theta_1(2\frac{\partial}{\partial y^1}) + \sin\theta_1(2\frac{\partial}{\partial y^2}),$$
$$e_3 = 2(\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial z}), \qquad e_4 = \cos\theta_2(2\frac{\partial}{\partial y^3}) + \sin\theta_2(2\frac{\partial}{\partial y^4}),$$
$$e_5 = 2\frac{\partial}{\partial z} = \xi,$$

form a local orthonormal frame of TM. We define the distributions $\mathcal{D}_1 = \langle e_1, e_2\rangle$ and $\mathcal{D}_2 = \langle e_3, e_4\rangle$.

Then, it is clear that $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \langle\xi\rangle$ and it can be easily proved that $\mathcal{D}_i$ is a slant distribution with slant angle θ_i for any $i = 1, 2$. In particular, if we consider $\theta_1 = \theta_2 = \theta$ in the above, it results that M is a $\theta-$slant submanifold.

Example 2. For any $\theta_1 \in [0, \frac{\pi}{2}]$, we chose $\theta_2 \in (0, \frac{\pi}{2}]$, such that $\cos\theta_2 = \frac{\cos\theta_1}{\sqrt{2}}$. Then

$$x(u, v, w, s, t) = 2(u, 0, w, 0, v\cos\theta_1, v\sin\theta_1, s\cos\theta_2, s\sin\theta_2, t)$$

defines a five-dimensional bi-slant submanifold M in $(\mathbf{R}^9, \phi_0, \xi, \eta, g)$, with both slant angles equal to θ_2, but it is not slant submanifold. In fact we can chose a local orthonormal frame $\{e_1, \ldots, e_5\}$ of TM such that

$$e_1 = \frac{1}{\sqrt{2}}\{2(\frac{\partial}{\partial x^1} + y^1\frac{\partial}{\partial z}) + 2(\frac{\partial}{\partial x^4} + y^4\frac{\partial}{\partial z}), \qquad e_2 = \cos\theta_1(2\frac{\partial}{\partial y^1}) + \sin\theta_1(2\frac{\partial}{\partial y^2}),$$
$$e_3 = 2(\frac{\partial}{\partial x^3} + y^3\frac{\partial}{\partial z}), \qquad e_4 = \cos\theta_2(2\frac{\partial}{\partial y^3}) + \sin\theta_2(2\frac{\partial}{\partial y^4}),$$
$$e_5 = 2\frac{\partial}{\partial z} = \xi.$$

Now we define the distributions $\mathcal{D}_1 = \langle e_1, e_2\rangle$ and $\mathcal{D}_2 = \langle e_3, e_4\rangle$. It is easy to see that both $\mathcal{D}_1$ and $\mathcal{D}_2$ are slant distribution with the same slant angle θ_2. Nevertheless, we can obtain that M is not slant since $\theta_2 \neq 0$.

Definition. We say that M tangent to ξ is a *semi-slant* submanifold of $\tilde{M}$ if there exist two orthogonal distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ on M such that :

i) TM admits the orthogonal direct decomposition $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$.
ii) The distribution $\mathcal{D}_1$ is an invariant distribution, i.e., $\phi(\mathcal{D}_1) = \mathcal{D}_1$.
ii) The distribution $\mathcal{D}_2$ is slant with angle $\theta \neq 0$.

Let $2d_1 = \dim\mathcal{D}_1$ and $2d_2 = \dim\mathcal{D}_2$.

In [3], the invariant distribution of a semi-slant submanifold is a slant distribution with zero angle. Thus, it is obvious that, in fact, semi-slant submanifolds are particular cases of bi-slant submanifolds. Moreover, it is clear that, if $\theta = \frac{\pi}{2}$, then the semi-slant submanifold is a semi-invariant submanifold.

(a) If $d_2 = 0$, then M is an invariant submanifold.
(b) If $d_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
(c) If $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, then M is a proper slant submanifold, with slant angle θ.

We say that a semi-slant submanifold is *proper* if $d_1 d_2 \neq 0$ and $\theta \neq \frac{\pi}{2}$.

Examples of semi-slant submanifolds. (see [3])

Example 1. Let $\mathbf{R}^6$ be the Euclidian space of dimension 6, with the standard metric and the almost complex structure given by $J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i}$, for any $i = 1, 2, 3$, where (x^i, y^i) denote the Cartesian coordinates.

Let $\mathbf{R}^5 \hookrightarrow \mathbf{R}^6$ be the usual immersion. Then, $C = \frac{\partial}{\partial y^3}$ is the unit normal to $\mathbf{R}^5$ and so, $\xi = -JC = \frac{\partial}{\partial x^3}$.

Now, for any $\theta \neq 0$, we can consider the immersions:

$$\varphi_1 : \mathbf{R}^4 \longrightarrow \mathbf{R}^6 : (u, v, t, s) \longmapsto (u\cos\theta, u\sin\theta, t, v, 0, s),$$
$$\varphi_2 : \mathbf{R}^3 \longrightarrow \mathbf{R}^5 : (u, v, t) \longmapsto (u\cos\theta, u\sin\theta, t, v, 0).$$

We can directly prove that φ_1 is a semi-slant immersion, with complex distribution $\mathcal{D}_1 = \left\langle \frac{\partial}{\partial x^3}, \frac{\partial}{\partial y^3} \right\rangle$ and slant distribution, with angle θ,

$$\mathcal{D}_2 = \left\langle \cos\theta \frac{\partial}{\partial x^1} + \sin\theta \frac{\partial}{\partial x^2}, \frac{\partial}{\partial y^1} \right\rangle.$$

On the other hand, φ_2 is a θ-slant immersion, where $\mathbf{R}^5$ has the almost contact metric structure induced by the described almost Hermitian structure on $\mathbf{R}^6$.

For the other properties and examples of slant, bi-slant and semi-slant submanifolds in Sasakian manifolds, we refer to [2], [3].

Let M be an n-dimensional submanifold of a Riemannian manifold $\tilde{M}$. We denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM, p \in M$, and ∇ the Riemannian connection of M. Also, let h be the second fundamental form and R the Riemann curvature tensor of M.

Then the equation of Gauss is given by

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = R(X,Y,Z,W)+ \\ +g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)), \end{aligned} \tag{1.2}$$

for any vectors X, Y, Z, W tangent to M.

Let $p \in M$ and $\{e_1, \ldots, e_n\}$ an orthonormal basis of the tangent space T_pM. We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{1.3}$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r) \tag{1.4}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)). \tag{1.5}$$

For any tangent vector field X to M, we put $\phi X = PX + FX$, where PX and FX are the tangential and normal components of ϕX, respectively. We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j). \tag{1.6}$$

Suppose L is a k-plane section of T_pM and X a unit vector in L. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of L such that $e_1 = X$.

Define the *Ricci curvature* Ric_L of L at X by

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k}, \tag{1.7}$$

where K_{ij} denotes the *sectional curvature of the 2-plane section spanned by* e_i, e_j. We simply called such a curvature a k-Ricci curvature.

The *scalar curvature* τ *of the* k*-plane section* L is given by

$$\tau(L) = \sum_{1\le i<j\le k} K_{ij}. \tag{1.8}$$

For each integer k, $2 \le k \le n$, the Riemannian invariant Θ_k on an n-dimensional Riemannian manifold M is defined by

$$\Theta_k(p) = \frac{1}{k-1} \inf_{L,X} \mathrm{Ric}_L(X), \quad p \in M, \tag{1.9}$$

where L runs over all k-plane sections in T_pM and X runs over all unit vectors in L.

Recall that for a submanifold M in a Riemannian manifold, the *relative null space of* M *at a point* $p \in M$ is defined by

$$\mathcal{N}_p = \{X \in T_pM \,|\, h(X,Y) = 0, \text{ for all } Y \in T_pM\}. \tag{1.10}$$

2. Ricci curvature and squared mean curvature

Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for submanifolds in real space forms (see [6]).

We prove similar inequalities for slant, bi-slant and semi-slant submanifolds in a Sasakian space form.

We consider submanifolds M tangent to the Reeb vector field ξ.

Theorem 2.1. *Let* M *be an* $(n = 2k+1)-$*dimensional* $\theta-$ *slant submanifold tangent to* ξ *in a* $(2m+1)$*-dimensional Sasakian space form* $\tilde{M}(c)$*. Then:*

(i) *For each unit vector* $X \in T_pM$ *orthogonal to* ξ*, we have*

$$\mathrm{Ric}(X) \le \frac{1}{4}\{(n-1)(c+3) + \frac{1}{2}(3\cos^2\theta - 2)(c-1) + n^2\|H\|^2\}. \tag{2.1}$$

(ii) *If* $H(p) = 0$*, then a unit tangent vector* $X \in T_pM$ *orthogonal to* ξ *satisfies the equality case of (2.1) if and only if* $X \in \mathcal{N}_p$*.*

(iii) *The equality case of (2.1) holds identically for all unit tangent vectors orthogonal to* ξ *at* p *if and only if* p *is a totally geodesic point.*

Proof. Let $X \in T_pM$ be a unit tangent vector X at p, orthogonal to ξ. We choose an orthonormal basis $e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}$ such that $e_1, \ldots, e_n$ are tangent to M at p, with $e_1 = X$.

Then, from the equation of Gauss, we have

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - n(n-1)\frac{c+3}{4} - [3(n-1)\cos^2\theta - 2n + 2]\frac{c-1}{4}. \tag{2.2}$$

From (2.2), we get

$$n^2\|H\|^2 = 2\tau + \sum_{r=n+1}^{2m+1}[(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2\sum_{i<j}(h_{ij}^r)^2] \tag{2.3}$$

$$- 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - n(n-1)\frac{c+3}{4} - [3(n-1)\cos^2\theta - 2n + 2]\frac{c-1}{4}$$

$$= 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m+1}[(h_{11}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2] + 2\sum_{r=n+1}^{2m+1}\sum_{i<j}(h_{ij}^r)^2$$

$$- 2\sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n} h_{ii}^r h_{jj}^r - n(n-1)\frac{c+3}{4} - [3(n-1)\cos^2\theta - 2n + 2]\frac{c-1}{4}.$$

From the equation of Gauss, we find

$$K_{ij} = \sum_{r=n+1}^{2m+1}[h_{11}^r h_{22}^r - (h_{12}^r)^2] + 3\cos^2\theta \cdot \frac{c-1}{4} + \frac{c+3}{4}$$

and consequently

$$\sum_{2\le i<j\le n} K_{ij} = \sum_{r=n+1}^{2m+1}\sum_{2\le i<j\le n}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)}{2}\frac{c+3}{4} + \tag{2.4}$$
$$+ [3(n-1)\cos^2\theta - 3\cos^2\theta - 2n + 4]\frac{c-1}{8}.$$

Substituting (2.4) in (2.3), one gets

$$\frac{1}{2}n^2\|H\|^2 \ge 2\,\mathrm{Ric}(X) - 2(n-1)\frac{c+3}{4} - (3\cos^2\theta - 2)\frac{c-1}{4},$$

which is equivalent to (2.1).

(ii) Assume $H(p) = 0$. Equality holds in (2.1) if and only if

$$\begin{cases} h_{12}^r = \ldots = h_{1n}^r = 0, \\ h_{11}^r = h_{22}^r + \cdots + h_{nn}^r, r \in \{n+1, \ldots, 2m\}. \end{cases} \tag{2.5}$$

Then $h_{1j}^r = 0$, for every $j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}$, that is $X \in \mathcal{N}_p$.

(iii) The equality case of (2.1) holds for all unit tangent vectors orthogonal to ξ at p if and only if

$$\begin{cases} h_{ij}^r = 0, i \ne j, r \in \{n+1, \ldots, 2m\}, \\ h_{11}^r + \cdots + h_{nn}^r - 2h_{ii}^r = 0, \quad i \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}. \end{cases} \tag{2.6}$$

In this case, since ξ is tangent to M, it follows that a totally umbilical point is totally geodesic. □

Theorem 2.2. *Let M be an $(n = 2d_1 + 2d_2 + 1)-$dimensional bi-slant submanifold satisfying $g(X, \phi Y) = 0$, for any $X \in \mathcal{D}_1$ and any $X \in \mathcal{D}_2$, tangent to ξ in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then:*

(i) *For each unit vector $X \in T_pM$ orthogonal to ξ and if*

a) *X is tangent to $\mathcal{D}_1$ we have*

$$\mathrm{Ric}(X) \le \frac{1}{4}\{(n-1)(c+3) + \frac{1}{2}(3\cos^2\theta_1 - 2)(c-1) + n^2\|H\|^2\} \tag{2.7}$$

and if

b) *X is tangent to $\mathcal{D}_2$ we have*

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{(n-1)(c+3) + \frac{1}{2}(3\cos^2\theta_2 - 2)(c-1) + n^2\|H\|^2\}. \tag{2.7$'$}$$

(ii) *If $H(p) = 0$, then a unit tangent vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of (2.7) and (2.7$'$) if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of (2.7) and (2.7$'$) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Proof. Let $X \in T_pM$ be a unit tangent vector X at p, orthogonal to ξ. We choose an orthonormal basis $e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}$ such that $e_1, \ldots, e_n$ are tangent to M at p, with $e_1 = X$.

Then, from the equation of Gauss, we have

$$n^2\|H\|^2 = 2\tau + \|h\|^2 - n(n-1)\frac{c+3}{4} - [6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 2n + 2]\frac{c-1}{4}. \tag{2.8}$$

From (2.8), we get

$$\begin{aligned} n^2\|H\|^2 = {}& 2\tau + \sum_{r=n+1}^{2m+1}[(h_{11}^r)^2 + (h_{22}^r + \cdots + h_{nn}^r)^2 + 2\sum_{i<j}(h_{ij}^r)^2] - \\ & -2\sum_{r=n+1}^{2m+1}\sum_{2\leq i<j\leq n} h_{ii}^r h_{jj}^r - n(n-1)\frac{c+3}{4} - [6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 2n + 2]\frac{c-1}{4} = \\ & = 2\tau + \frac{1}{2}\sum_{r=n+1}^{2m+1}[(h_{11}^r + \cdots + h_{nn}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{nn}^r)^2] + \\ & + 2\sum_{r=n+1}^{2m+1}\sum_{i<j}(h_{ij}^r)^2 - 2\sum_{r=n+1}^{2m+1}\sum_{2\leq i<j\leq n} h_{ii}^r h_{jj}^r - n(n-1)\frac{c+3}{4} - \\ & - [6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 2n + 2]\frac{c-1}{4}. \end{aligned} \tag{2.9}$$

From the equation of Gauss, we find:

a) if X is tangent to $\mathcal{D}_1$

$$K_{ij} = \sum_{r=n+1}^{2m+1}[h_{11}^r h_{22}^r - (h_{12}^r)^2] + 3\cos^2\theta_1 \cdot \frac{c-1}{4} + \frac{c+3}{4}$$

and consequently

$$\begin{aligned} \sum_{2\leq i<j\leq n} K_{ij} = {}& \sum_{r=n+1}^{2m+1}\sum_{2\leq i<j\leq n}[h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)}{2}\frac{c+3}{4} + \\ & + [6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 3\cos^2\theta_1 - 2n + 4]\frac{c-1}{8}. \end{aligned} \tag{2.10}$$

Substituting (2.10) in (2.9), one gets

$$\frac{1}{2}n^2\|H\|^2 \geq 2\,\mathrm{Ric}(X) - 2(n-1)\frac{c+3}{4} - (3\cos^2\theta_1 - 2)\frac{c-1}{4},$$

which is equivalent to (2.7).

b) Similar if X is tangent to $\mathcal{D}_2$, we have

$$K_{ij} = \sum_{r=n+1}^{2m+1}[h_{11}^r h_{22}^r - (h_{12}^r)^2] + 3\cos^2\theta_2 \cdot \frac{c-1}{4} + \frac{c+3}{4}$$

and consequently

$$\sum_{2\leq i<j\leq n} K_{ij} = \sum_{r=n+1}^{2m+1} \sum_{2\leq i<j\leq n} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] + \frac{(n-1)(n-2)}{2}\frac{c+3}{4} + \tag{2.11}$$
$$+[6(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - 3\cos^2\theta_2 - 2n + 4]\frac{c-1}{8}.$$

Substituting (2.11) in (2.9), one gets

$$\frac{1}{2}n^2\|H\|^2 \geq 2\operatorname{Ric}(X) - 2(n-1)\frac{c+3}{4} - (3\cos^2\theta_2 - 2)\frac{c-1}{4},$$

which is equivalent to $(2.7^{'})$.

(ii) Assume $H(p) = 0$. Equality holds in (2.7) and $(2.7^{'})$ if and only if

$$\begin{cases} h_{12}^r = \ldots = h_{1n}^r = 0, \\ h_{11}^r = h_{22}^r + \cdots + h_{nn}^r, r \in \{n+1, \ldots, 2m\}. \end{cases} \tag{2.12}$$

Then $h_{1j}^r = 0$, for every $j \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}$, that is $X \in \mathcal{N}_p$.

(iii) The equality case of (2.7) and $(2.7^{'})$ holds for all unit tangent vectors orthogonal to ξ at p if and only if

$$\begin{cases} h_{ij}^r = 0, i \neq j, r \in \{n+1, \ldots, 2m\}, \\ h_{11}^r + \cdots + h_{nn}^r - 2h_{ii}^r = 0, \quad i \in \{1, \ldots, n\}, r \in \{n+1, \ldots, 2m\}. \end{cases} \tag{2.13}$$

In this case, since ξ is tangent to M, it follows that a totally umbilical point is totally geodesic. □

Corollary 2.3. *Let M be an $(n = 2d_1 + 2d_2 + 1)-$dimensional semi-slant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then:*

(i) For each unit vector $X \in T_pM$ orthogonal to ξ and if
a X is tangent to $\mathcal{D}_1$ we have

$$\operatorname{Ric}(X) \leq \frac{1}{4}\{(n-1)(c+3) - (c-1) + n^2\|H\|^2\} \tag{2.14}$$

and if
b X is tangent to $\mathcal{D}_2$ we have

$$\operatorname{Ric}(X) \leq \frac{1}{4}\{(n-1)(c+3) + \frac{1}{2}(3\cos^2\theta - 2)(c-1) + n^2\|H\|^2\}. \tag{$2.14^{'}$}$$

(ii) If $H(p) = 0$, then a unit tangent vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of (2.14) and $(2.14^{'})$ if and only if $X \in \mathcal{N}_p$.

(iii) The equality case of (2.14) and $(2.14^{'})$ holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.

Corollary 2.4. *Let M be an $(n = 2k+1)$-dimensional invariant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then:*

(i) *For each unit vector $X \in T_pM$ orthogonal to ξ, we have*

$$\operatorname{Ric}(X) \leq \frac{1}{4}\{(n-1)(c+3) + \frac{1}{2}(c-1)\}. \tag{2.15}$$

(ii) *A unit tangent vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of (2.15) if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case of (2.15) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.*

Corollary 2.5. *Let M be an $(n = 2k+1)$-dimensional anti-invariant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$. Then:*

(i) For each unit vector $X \in T_pM$ orthogonal to ξ, we have

$$\mathrm{Ric}(X) \leq \frac{1}{4}\{(n-1)(c+3) - (c-1) + n^2\|H\|^2\}. \tag{2.16}$$

(ii) If $H(p) = 0$, then a unit tangent vector $X \in T_pM$ orthogonal to ξ satisfies the equality case of (2.16) if and only if $X \in \mathcal{N}_p$.

(iii) The equality case of (2.16) holds identically for all unit tangent vectors orthogonal to ξ at p if and only if p is a totally geodesic point.

3. k-Ricci curvature

In this section, we prove a relationship between the k-Ricci curvature and the squared mean curvature for slant, bi-slant and semi-slant submanifolds in a Sasakian space form.

We state an inequality between the scalar curvature and the squared mean curvature for submanifolds tangent to ξ.

Theorem 3.1. *Let M be an $(n = 2k+1)-$dimensional $\theta-$slant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3}{4} - \frac{[3(n-1)\cos^2\theta - 2n + 2](c-1)}{4n(n-1)}. \tag{3.1}$$

Proof. We choose an orthonormal basis $\{e_1, \ldots, e_n = \xi, e_{n+1}, \ldots, e_{2m+1}\}$ at p such that e_{n+1} is parallel to the mean curvature vector $H(p)$ and $e_1, \ldots, e_n$ diagonalize the shape operator A_{n+1}. Then the shape operators take the forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & . & . & . & 0 \\ 0 & a_2 & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & a_n \end{pmatrix} \tag{3.2}$$

$$A_r = (h_{ij}^r), i, j = 1, \ldots, n, r = n+2, \ldots, 2m+1, trace\, A_r = \sum_{i=1}^{n} h_{ii}^r = 0. \tag{3.3}$$

From (2.2), we get

$$\begin{aligned} n^2\|H\|^2 = 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{2m+1}\sum_{i,j=1}^{n}(h_{ij}^r)^2 - n(n-1)\frac{c+3}{4} - \\ -[3(n-1)\cos^2\theta - 2n + 2]\frac{c-1}{4}. \end{aligned} \tag{3.4}$$

On the other hand, since

$$0 \leq \sum_{i<j}(a_i - a_j)^2 = (n-1)\sum_i a_i^2 - 2\sum_{i<j} a_i a_j,$$

we obtain

$$n^2\|H\|^2 = (\sum_{i=1}^{n} a_i)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{i<j} a_i a_j \leq n\sum_{i=1}^{n} a_i^2,$$

which implies

$$\sum_{i=1}^{n} a_i^2 \geq n\|H\|^2.$$

Since we have that

$$n^2\|H\|^2 \geq 2\tau + n\|H\|^2 - n(n-1)\frac{c+3}{4} - [3(n-1)\cos^2\theta - 2n + 2]\frac{c-1}{4}, \tag{3.5}$$

which is equivalent to (3.1).

Let $\{e_1, \dots e_n\}$ be an orthonormal basis of T_pM. Denote by $L_{i_1\dots i_k}$ the k-plane section spanned by $e_{i_1}, \dots, e_{i_k}$. It follows from (1.7) and (1.8) that

$$\tau(L_{i_1\dots i_k}) = \frac{1}{2}\sum_{i\in\{i_1,\dots,i_k\}} \mathrm{Ric}_{L_{i_1\dots i_k}}(e_i), \tag{3.6}$$

$$\tau(p) = \frac{1}{C_{n-2}^{k-2}}\sum_{1\leq i_1<\dots<i_k\leq n} \tau(L_{i_1\dots i_k}). \tag{3.7}$$

Combining (1.9), (3.6) and (3.7), we find

$$\tau(p) \geq \frac{n(n-1)}{2}\Theta_k(p). \tag{3.8}$$

□

From (3.6), (3.7) and (3.1), we get the following.

Theorem 3.2. *Let M be an $(n = 2d_1 + 2d_2 + 1)-$dimensional bi-slant submanifold satisfying $g(X, \phi Y) = 0$, for any $X \in \mathcal{D}_1$ and any $X \in \mathcal{D}_2$, in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3}{4} - \frac{[3(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - n + 1](c-1)}{2n(n-1)}. \tag{3.9}$$

Proof. The proof is similar with their corresponding statements of Theorem 3.1. □

Theorem 3.3. *Let M be an $(n = 2d_1 + 2d_2 + 1)-$dimensional semi-slant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ. Then we have*

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - \frac{c+3}{4} - \frac{[3(d_1 + d_2\cos^2\theta) - n + 1](c-1)}{2n(n-1)}. \tag{3.10}$$

Proof. The proof is similar with their corresponding statements of Theorem 3.1. □

Theorem 3.4. *Let M be an $(n = 2k+1)-$dimensional $\theta-$ slant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{[3(n-1)\cos^2\theta - 2n + 2](c-1)}{4n(n-1)}. \tag{3.11}$$

Theorem 3.5. *Let M be an $(n = 2d_1 + 2d_2 + 1)-$dimensional bi-slant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{[3(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) - n + 1](c-1)}{2n(n-1)}. \tag{3.12}$$

Theorem 3.6. *Let M be an $(n = 2d_1 + 2d_2 + 1)-$dimensional semi-slant submanifold in a $(2m+1)$-dimensional Sasakian space form $\tilde{M}(c)$ tangent to ξ. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{[3(d_1 + d_2\cos^2\theta) - n + 1](c-1)}{2n(n-1)}. \tag{3.13}$$

Corollary 3.7. *Let M be an n-dimensional invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\Theta_k(p) \leq \frac{c+3}{4} + \frac{c-1}{4n}. \tag{3.14}$$

Corollary 3.8. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} + \frac{c-1}{2n}. \tag{3.15}$$

Corollary 3.9. *Let M be an n-dimensional contact CR submanifold ($\theta_1 = 0$, $\theta_2 = \frac{\pi}{2}$) of a Sasakian space form $\tilde{M}(c)$. Then, for any integer $k, 2 \leq k \leq n$, and any point $p \in M$, we have*

$$\|H\|^2(p) \geq \Theta_k(p) - \frac{c+3}{4} - \frac{(3d_1 - n + 1)(c-1)}{2n(n-1)}. \tag{3.16}$$

where $2d_1 = \dim \mathcal{D}_1$.

References

[1] D. E. Blair. *Contact manifolds in Riemannian geometry.* Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 509.

[2] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernández. Semi-slant submanifolds of a Sasakian manifold. *Geom. Dedicata*, 78(2):183–199, 1999.

[3] J. L. Cabrerizo, A. Carriazo, L. M. Fernández, and M. Fernández. Slant submanifolds in Sasakian manifolds. *Glasg. Math. J.*, 42(1):125–138, 2000.

[4] B.-Y. Chen. Some pinching and classification theorems for minimal submanifolds. *Arch. Math. (Basel)*, 60(6):568–578, 1993.

[5] B.-Y. Chen. Mean curvature and shape operator of isometric immersions in real-space-forms. *Glasgow Math. J.*, 38(1):87–97, 1996.

[6] B.-Y. Chen. Relations between Ricci curvature and shape operator for submanifolds with arbitrary codimensions. *Glasg. Math. J.*, 41(1):33–41, 1999.

[7] D. Cioroboiu and A. Oiagă. B. Y. Chen inequalities for slant submanifolds in Sasakian space forms. *Rend. Circ. Mat Palermo.*, 51, 2002. to appear.

[8] F. Defever, I. Mihai, and L. Verstraelen. B.-Y. Chen's inequality for C-totally real submanifolds of Sasakian space forms. *Boll. Un. Mat. Ital. B (7)*, 11(2):365–374, 1997.

[9] A. Lotta. Slant submanifolds in contact geometry. *Bull. Math. Soc. Roumanie*, 39:183–198, 1996.

[10] K. Matsumoto, I. Mihai, and A. Oiagă. Ricci curvature of submanifolds in complex space forms. *Rev. Roumaine Math. Pures Appl.*, 46(6):775–782 (2002), 2001.

[11] I. Mihai. Ricci curvature of submanifolds in Sasakian space forms. *J. Aust. Math. Soc.*, 72(2):247–256, 2002.

[12] I. Mihai, R. Rosca, and L. Verstraelen. *Some aspects of the differential geometry of vector fields*, volume 2 of *Centre for Pure and Applied Differential Geometry (PADGE).* Katholieke Universiteit Brussel Group of Exact Sciences, Brussels, 1996. On skew symmetric Killing and conformal vector fields, and their relations to various geometrical structures.

[13] K. Yano and M. Kon. *Structures on manifolds*, volume 3 of *Series in Pure Mathematics.* World Scientific Publishing Co., Singapore, 1984.

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