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ON THE HOMOGENEOUS GEOMETRICAL MODEL OF A RIEMANNIAN SPACE

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Abstract. In the paper we study the homogeneous geometrical model of a Riemannian space. The canonical connection is analyzed in details. On this model, the Einstein equations, the electromagnetic fields and generalized Einstein–Yang Mills equations are studied. We remark that the theory we proposed in this paper works only for the case when the space test is without charges. The Einstein equations of our model projected on the basis manifold M are perturbations of the classical Einstein equations on the basis manifold M.

1. Introduction

In the last thirty years, a lot of geometrical models for gravitational and electromagnetic theories have been proposed. We refer especially to the well known Riemannian, Finslerian, Lagrangian or, more general, Lagrangian of higher order theories [3, 8, 11, 12, 15, 14]. The differential geometry of the Lagrange spaces is now considerable developed and used in various fields to study the natural processes where the dependence on position, velocity or momentum are involved [4, 6, 5, 7, 9, 12, 14, 16]. The geometry of Lagrange spaces gives a model for both the gravitational and electromagnetic fields in a very natural blending of the geometrical structure of the space with the characteristic properties of the physical fields.

In [13] the notions of homogeneous lift of a Riemannian metric and the corresponding homogeneous complex structure were introduced (see also (2.20) within the paper). The pair consists from the above structures is a special Hermitian structure on $\widetilde{TM}$. In the present paper we will use these geometrical structures in order to give a picture of some physical aspects of a geometrical model generated by the Riemannian spaces.

The paper is organized as follows: In the next section the notions of almost $2-\pi$ structure and metrical almost $2-\pi$ structure will be presented. These structures were introduced in [17, 20] and were studied in details there. In Section 3 the homogeneous geometrical model of a Riemannian space will be studied. On this model, the Einstein equations, the electromagnetic fields and generalized Einstein-Yang Mills equations will be in attention. Finally, some conclusions will be given. The terminology and notations are those used in [3, 8, 11, 13, 14, 20].

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2. Preliminaries

Let $R^{(n)} = (M, \gamma)$ be a Riemannian space with a smooth, real, finite-dimensional manifold M $(dim(M) = n)$ and a Riemannian structure γ. The local coordinates on M are $x = \left(x^i\right)$, $i = 1, \ldots, n$ and $(x, y) = (x^i, y^i)$, $i = 1, \ldots, n$ are the local coordinates on the total space of the tangent manifold TM. The projection of TM on M will be denoted by τ. Wherever they will appear, the indices $i, j, k, \ldots$ will run from 1 to n. Moreover, the Einstein convention on summation is implied. The geometrical objects on TM, whose local components change in the same way like on M will be called d–objects. The kernel of the differential map $\tau^T : TTM \to TM$ is a vector subbundle of TTM, and it is called the vertical distribution on TM. The local vector fields $\left\{\frac{\partial}{\partial y^i}\right\}$ determine a local frame in VTM. A nonlinear connection in the tangent bundle $\tau : TM \to M$ is a distribution on TM, denoted by HTM, supplementary to the vertical distribution. That is

$$TTM = HTM \oplus VTM. \tag{2.1}$$

The position of the subspace H_uTM, $u \in TM$, can be given by n local vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^k(x, y)\frac{\partial}{\partial y^k}.$$

The real differentiable functions $\left(N_j^i(x, y)\right)$ completely determines a nonlinear connection which will be denoted in compact form N. For example we can take $N_j^i(x, y) := \gamma_{jk}^i(x)y^k$, where $\gamma_{jk}^i(x)$ are the Christoffel symbols of the Levi–Civita connection of the Riemannian space M. Therefore a nonlinear connection N determines a basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ associated to the decomposition (2.1). This basis will be called the adapted basis. The Sasaki lift (see for instance [14]) of the Riemannian structure γ on TM, denoted by G_S is defined as

$$G_S = \gamma_{ij}(x)dx^i \otimes dx^j + \gamma_{ij}(x)\delta y^i \otimes \delta y^j. \tag{2.2}$$

Next we consider $G_{a,b,c,d}$ a $(h, v)-$ metrical structure on TM, given by

$$G_{a,b,c,d}(x, y) = g_{ij}(x, y)dx^i \otimes dx^j + h_{ij}(x, y)\delta y^i \otimes \delta y^j, \tag{2.3}$$

where

$$\left\{\begin{array}{l} g_{ij}(x, y) = \frac{a^2}{F^2}\gamma_{ij}(x) + \frac{b^2-a^2}{F^4}y_iy_j, \\ h_{ij}(x, y) = \frac{c^2}{F^2}\gamma_{ij}(x) + \frac{d^2-c^2}{F^4}y_iy_j, \end{array}\right. \tag{2.4}$$

where $F^2 = \gamma_{ij}(x)y^iy^j$, $y_i = \gamma_{ij}(x)y^j$ and a, b, c, d: $\mathrm{Im}(F^2) \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ are differentiable functions $(b \geq a > 0$, $d \geq c > 0)$. The analysis of $G_{a,b,c,d}$ can be found in [20]. Assume further that TM is endowed with a nonlinear connection determined by the local coefficients $\left(N_j^i(x, y) = \gamma_{jk}^i(x)y^k\right)$.

Definition 2.1. *Let D be a d–connection TM. D is called compatible with $G_{a,b,c,d}$ if it satisfies*

$$D_X G_{a,b,c,d} = 0, \forall X \in \chi(TM). \tag{2.5}$$

In the adapted basis, any d–connection on TM can be represented in the following way

$$\begin{array}{ll} D_{\frac{\delta}{\delta x^k}}\frac{\delta}{\delta x^j} = \overset{(H)}{F}{}^i_{jk}\frac{\delta}{\delta x^i} & , D_{\frac{\delta}{\delta x^k}}\frac{\partial}{\partial y^j} = \overset{(V)}{F}{}^i_{jk}\frac{\partial}{\partial y^i}, \\ D_{\frac{\partial}{\partial y^k}}\frac{\delta}{\delta x^j} = \overset{(H)}{C}{}^i_{jk}\frac{\delta}{\delta x^i}, & D_{\frac{\partial}{\partial y^k}}\frac{\partial}{\partial y^j} = \overset{(V)}{C}{}^i_{jk}\frac{\partial}{\partial y^i}, \end{array} \tag{2.6}$$

where the system of functions ($\overset{(H)}{F}{}^i_{jk}$, $\overset{(V)}{F}{}^i_{jk}$, $\overset{(H)}{C}{}^i_{jk}$, $\overset{(V)}{C}{}^i_{jk}$) represents the local coefficients of the above d–connection D.

Let $\mathbb{C}$ be the set of complex numbers and let I be the identity tensor field of type $(1,1)$ on TM. Recall that (see [20]) an almost $2-\pi$ structure is a tensor field Φ of type $(1,1)$ on TM such that $\Phi^2 = -\lambda I$. In the geometry of the tangent bundle there exists a special almost $2-\pi$ structure, which in the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, is given by

$$\Phi_S\left(\frac{\delta}{\delta x^i}\right) = -\lambda \cdot \frac{\partial}{\partial y^i}\ ,\ \Phi_S\left(\frac{\partial}{\partial y^i}\right) = \lambda \cdot \frac{\delta}{\delta x^i}\ ,\ \lambda \in \mathbb{C}, \tag{2.7}$$

Let G be a metrical structure on TM and let Φ be an almost $2-\pi$ structure on TM. By definition, a metrical almost $2-\pi$ structure is a pair (G,Φ) on TM for which:

$$\frac{1}{\lambda^2} G(\Phi X, \Phi Y) = G(X,Y),\ \forall X, Y \in \chi(TM), \tag{2.8}$$

and the 2-form $\Omega(X,Y) = G(\Phi(X),Y)$ is closed.

The pair $(G_{a,b,c,d},\ \Phi_S)$, where the metrical structure $G_{a,b,c,d}$ is defined by (2.3) and (2.4), is not a metrical almost 2-π structure. In [20], it was proved that there exists a class of almost $2-\pi$ structures Φ such that the pair $(G_{a,b,c,d},\ \Phi)$ should be a metrical almost $2-\pi$ structure. In fact it was shown that the pairs $(G_{a,b,c,d},\ \Phi_{a,b,c,d})$ and $(G_{a,a,c,c},\ \Phi_{a,c})$ are metrical almost $2-\pi$ structures on the tangent bundle, where the almost $2-\pi$ structures $\Phi_{a,b,c,d}$ and $\Phi_{a,c}$ are as follows

$$\Phi_{a,b,c,d} = \lambda A^k_i \frac{\partial}{\partial y^k} \otimes dx^i + \lambda B^k_i \frac{\delta}{\delta x^k} \otimes \delta y^i, \tag{2.9}$$

and

$$\Phi_{a,c} = \lambda \overset{\smile}{A}{}^k_i \frac{\partial}{\partial y^k} \otimes dx^i + \lambda \overset{\smile}{B}{}^k_i \frac{\delta}{\delta x^k} \otimes \delta y^i. \tag{2.10}$$

The coefficients $A^k_i, \overset{\smile}{A}{}^k_i, B^k_i, \overset{\smile}{B}{}^k_i$ are

$$A^k_i = -\frac{a}{c}\delta^k_i + \frac{ad+bc}{dcF^2} y_i y^k, \quad \overset{\smile}{A}{}^k_i = -\frac{a}{c}\delta^k_i, \tag{2.11}$$

and,

$$B^k_i = \frac{c}{a}\delta^k_i - \frac{ad+bc}{abF^2} y_i y^k, \quad , \overset{\smile}{B}{}^k_i = \frac{c}{a}\delta^k_i. \tag{2.12}$$

Definition 2.2. a. *Let D be a linear connection on TM. D is said to be compatible with the almost $2-\pi$ structure Φ if it satisfies*

$$D_X \Phi = 0,\ \forall\, X \in \chi(TM). \tag{2.13}$$

b. *A linear connection on TM is said to be compatible with the metrical almost $2-\pi$ structure (G,Φ), if it satisfies the conditions*

$$D_X G = 0,\ D_X \Phi = 0,\ \ \forall\, X \in \chi(TM). \tag{2.14}$$

Before we state the next result let us to make some notations: $A = \frac{2a'F^2 - a}{aF^2}$ and $B = \frac{2c'F^2 - c}{cF^2}$. With respect to the above notions, one obtains

Theorem 2.3. *The set of all d–connections compatible with the metrical almost $2-\pi$ structure $(G_{a,a,c,c},\ \Phi_{a,c})$ is determined by the following local coefficients*

$$\overset{(H)}{F}{}^i_{jk} = \gamma^i_{jk} + \Omega^{ei}_{jm} \cdot X^m_{ek}, \tag{2.15}$$

$$\overset{(V)}{F}{}^i_{jk} = \gamma^i_{jk} + \Omega^{ei}_{jm} \cdot X^m_{ek}, \tag{2.16}$$

$$\overset{(H)}{C}{}^i_{jk} = A \cdot \delta^i_j \cdot y_k + \Omega^{ei}_{jm} \cdot U^m_{ek}, \tag{2.17}$$

$$\overset{(V)}{C}{}^i_{jk} = B \cdot \delta^i_j \cdot y_k + \Omega^{ei}_{jm} \cdot U^m_{ek}, \tag{2.18}$$

where X^m_{ek}, U^m_{ek} are arbitrary d–tensor fields and Ω^{ei}_{jm} is the Obata operator of the Riemannian structure γ.

Particular cases.

$\mathbf{1^0}$. In the case $X^m_{ek} = U^m_{ek} = 0$ one obtains a d–connection compatible with the metrical almost 2–π structure $(G_{a,a,c,c}, \Phi_{a,c})$, which depends only on the Riemannian structure γ and the functions a, c. The local coefficients of this d–connection are as follows

$$\overset{(H)}{F}{}^i_{jk} = \overset{(V)}{F}{}^i_{jk} = \gamma^i_{jk}, \quad \overset{(H)}{C}{}^i_{jk} = A \cdot \delta^i_j \cdot y_k, \quad \overset{(V)}{C}{}^i_{jk} = B \cdot \delta^i_j \cdot y_k. \tag{2.19}$$

The simplicity of this d–connection and the fact that it is determined only by the Riemannian structure γ and by the functions a, c allows us to call it the canonical d–connection of the space $\left(\widetilde{TM}, G_{a,a,c,c}, \Phi_{a,c}\right)$.

$\mathbf{2^0}$. If $a = F$, $c = k$, $k \in R^*$, one obtains the so called homogeneous metrical almost $2 - \pi$ structure $\left(\overset{(0)}{G}, \overset{(0)}{\Phi}\right)$, where the metrical structure

$$\overset{(0)}{G} = \gamma_{ij}(x)dx^i \otimes dx^j + \frac{k^2}{F^2}\gamma_{ij}(x)\delta y^i \otimes \delta y^j, \; k \in R, \tag{2.20}$$

is the Miron metrical structure from [13], and the almost $2 - \pi$ structure $\overset{(0)}{\Phi}$ is defined by

$$\overset{(0)}{\Phi} = -\lambda \cdot \tfrac{F}{k} \cdot \tfrac{\partial}{\partial y^i} \otimes dx^i + \lambda \cdot \tfrac{k}{F} \cdot \tfrac{\delta}{\delta x^i} \otimes \delta y^i. \tag{2.21}$$

The canonical d-connection of the space $\left(\widetilde{TM}, \overset{(0)}{G}, \overset{(0)}{\Phi}\right)$ is determined by the following local coefficients

$$\overset{(H)}{F}{}^i_{jk} = \overset{(V)}{F}{}^i_{jk} = \gamma^i_{jk}, \quad \overset{(H)}{C}{}^i_{jk} = 0, \quad \overset{(V)}{C}{}^i_{jk} = -\tfrac{1}{F^2} \cdot \delta^i_j \cdot y_k. \tag{2.22}$$

Remark. The space $\overset{(2-\pi)}{M}{}^{2n} = \left(\widetilde{TM}, G_{a,a,c,c}, \Phi_{a,c}\right)$ is called the (a, c)–geometrical model of the Riemannian space (M, γ) with respect to the metrical almost $2 - \pi$ structure $(G_{a,a,c,c}, \Phi_{a,c})$ (see [20]). On the other side, $\overset{(0)}{M}{}^{2n} = \left(\widetilde{TM}, \overset{(0)}{G}, \overset{(0)}{\Phi}\right)$ is called the homogeneous geometrical model of the Riemannian space (M, γ) with respect to the metrical almost $2 - \pi$ structure $\left(\overset{(0)}{G}, \overset{(0)}{\Phi}\right)$ (see [20]), and it will be the subject of the next section.

3. The homogeneous geometrical model

In this section we shall study in details the homogeneous geometrical model of the Riemannian space (M, γ). Let r^i_{jhk} be the local components of the curvature tensor field of the Riemannian space (M, γ). The notation $t^{a_1,a_2,\ldots,a_r}_{b_1,\ldots,b_{k-1},0,b_{k+1},\ldots b_s}$ means the contraction of the tensor $t^{a_1,a_2,\ldots,a_r}_{b_1,\ldots,b_{k-1},i,b_{k+1},\ldots,b_s}$ with y^i. The next results deal with the canonical connection of the homogeneous geometrical model $\overset{(0)}{M}{}^{(2n)}$.

Proposition 3.1. *Let* $\overset{(0)}{D}$ *be the canonical connection of the homogeneous geometrical model of a Riemannian space* (M, γ).

a. *In the adapted basis* $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, *the torsion tensor field of* $\overset{(0)}{D}$ *has the following components*

$$T^i_{jk} = 0,\ R^i_{jk} = r^i_{0jk}, \tag{3.1}$$

$$\overset{(H)}{P}{}^i_{\ jk} = 0,\ \overset{(V)}{P}{}^i_{\ jk} = 0,\ S^i_{jk} = 0, \tag{3.2}$$

and the curvature tensor field of $\overset{(0)}{D}$ *has the following components*

$$\overset{(H)}{R}{}^i_{\ hjk} = r^i_{hjk} + r^m_{0jk} \cdot \gamma^i_{hm}, \tag{3.3}$$

$$\overset{(V)}{R}{}^i_{\ hjk} = r^i_{hjk} - \frac{1}{F^2} \cdot \delta^i_h \cdot r^0_{0jk}, \tag{3.4}$$

$$\overset{(H)}{P}{}^i_{\ hjk} = \overset{(V)}{P}{}^i_{\ hjk} = \overset{(H)}{S}{}^i_{\ hjk} = \overset{(V)}{S}{}^i_{\ hjk} = 0. \tag{3.5}$$

b. *In the adapted basis* $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ *the Ricci tensor field of* $\overset{(0)}{D}$ *has the following components*

$$R_{ij} = r_{ij} + \overset{(1)}{r}_{ij}, \tag{3.6}$$

$$\overset{(1)}{P}_{ij} = \overset{(2)}{P}_{ij} = S_{ij} = 0, \tag{3.7}$$

where r_{ij} *are the local components of the Ricci tensor field of* (M, γ) *Riemannian space and* $\overset{(1)}{r}_{ij} = r^m_{0jn} \cdot \gamma^n_{im}$.

c. *The scalar curvature of* $\overset{(0)}{D}$ *is given by*

$$R = r + r_1, \tag{3.8}$$

where r *is the scalar curvature of* (M, γ) *Riemannian space and* $r_1 = \gamma^{ij} \cdot r^m_{0jn} \cdot \gamma^n_{im}$.

Concerning the homogeneous geometrical model of a Riemannian space we will answer to the following questions

A. *Which are the Einstein equations of the model?*
B. *Which are the physical implications of the electromagnetic fields?*
C. *Which are the EYM equations of the model?*

A. In the adapted basis $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$, the Einstein equations (for these equations we refer to [14]) of the test space, are as follows

$$R_{\alpha\beta} - \frac{1}{2} R \cdot G_{\alpha\beta} K \cdot W_{\alpha\beta}, \tag{3.9}$$

where $W_{\alpha\beta}$ are the local components of the stress–energy tensor field. The above results together with (3.9) lead to

Theorem 3.2. *In the adapted basis* $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)$ *the Einstein equations of the homogeneous geometrical model are*

$$r_{ij} + \overset{(1)}{r}_{ij} - \frac{1}{2} \cdot (r + r_1) \cdot \gamma_{ij} = K \cdot \overset{(H)}{W}_{ij}, \tag{3.10}$$

$$\frac{1}{2F^2}\cdot(r+r_1)\cdot\gamma_{ij}=-K\cdot\overset{(V)}{W}_{ij} \tag{3.11}$$

$$\overset{(H,V)}{W}_{ij}=0,\ \overset{(V,H)}{W}_{ij}=0, \tag{3.12}$$

where $\overset{(H)}{W}_{ij}$, $\overset{(H,V)}{W}_{ij}$, $\overset{(V,H)}{W}_{ij}$ *and* $\overset{(V)}{W}_{ij}$ *are the local components of stress–energy tensor field in the adapted basis* $\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\right)$.

We remark that the mixed stress–energy tensor fields vanishing. Moreover, the projection of Einstein equation (3.9) on the basis manifold M (namely (3.10)) is a perturbation of the classical Einstein equation on the Riemannian manifold (M,γ). In fact the term of the perturbation is given by $p_{ij}=\overset{(1)}{r}_{ij}-\frac{1}{2}\cdot r_1\cdot\gamma_{ij}$.

B. We will determine the local components of the electromagnetic field of the our model. For this reason, we will calculate the local components of so-called deflection tensor field. It is known that in Lagrange geometry approach the above mentioned tensor field has the local components given by

$$D^i_m=y^i_{|m},\ d^i_m=y^i\,|_m\,. \tag{3.13}$$

In the our case we have the following particular expresion for the deflection tensor field

$$D^i_m=0,\ d^i_m=\delta^i_m-\frac{1}{F^2}\cdot y_m\cdot y^i. \tag{3.14}$$

The covariant expresion of the deflection tensor field is useful in order to determine the form of the electromagnetic tensor field. Its local components are as follows

$$D_{ij}=g_{ik}\cdot D^k_j,\ d_{ij}=h_{ik}\cdot d^k_j. \tag{3.15}$$

We recall that, in the general setting of the Lagrange spaces, the h– and v– local components of the electromagnetic tensor fields are given by

$$F_{ij}=\frac{1}{2}\left(D_{ij}-D_{ji}\right),\ f_{ij}=\frac{1}{2}\left(d_{ij}-d_{ji}\right). \tag{3.16}$$

Using the above formulas we remark that in the our geometrical model, the electromagnetic tensor fields vanishing

$$F_{ij}=0, f_{ij}=0. \tag{3.17}$$

C. In the last part of this section we deal with the generalized EYM equations of the model $\overset{(0)}{M}{}^{(2n)}$ endowed with the canonical linear connection $\overset{(0)}{D}$. The basic ideas for the development of the gauge theory on the total space of the tangent bundle are given in [3, 6, 7, 8, 10, 14, 16, 18]. For a better understanding of the next results, we recall some basic definitions from the Lagrange gauge theory.

Let M be a real, $n-$dimensional, differentiable manifold, and let

$$\begin{cases} x^i=x^i\left(\overline{x}\right), & \det\left(\frac{\partial x^i}{\partial\overline{x}^j}\right)\neq 0,\\ y^a=\frac{\partial x^a}{\partial\overline{x}^b}\overline{y}^b, & i,a=\overline{1,n}, \end{cases} \tag{3.18}$$

be the local coordinate transformations on TM. If $\gamma_{ij}\left(x\right)$ is a Riemannian metric tensor field on M, then M can be endowed with the nonlinear connection $N=\left\{N^i_j\left(x,y\right)\right\}$ (see [11, 14]), where

$$\begin{array}{l} N^i_j\left(x,y\right)=\gamma^i_{jk}\left(x\right)\cdot y^k\\ \gamma^i_{jk}=\frac{1}{2}\cdot\gamma^{is}\cdot\left(\partial_j\gamma_{sk}+\partial_k\gamma_{js}-\partial_s\gamma_{jk}\right),\ \partial_k=\frac{\partial}{\partial x^k}. \end{array} \tag{3.19}$$

This provides the corresponding adapted basis for $\chi(TM)$, namely

$$\begin{aligned} &\delta_i = \tfrac{\delta}{\delta x^i} = \tfrac{\partial}{\partial x^i} - N_i^j(x,y)\tfrac{\partial}{\partial y^j} \\ &\overset{\bullet}{\partial}_a = \tfrac{\partial}{\partial y^a},\ i,a=\overline{1,n}. \end{aligned} \tag{3.20}$$

A gauge transformation on TM (see [3, 4, 6, 14]) is an element of a fixed subgroup of automorphisms of the tangent bundle, locally given by

$$\begin{cases} x^i = X^i(\widetilde{x}), & \det\left(\frac{\partial X^i}{\partial \widetilde{x}^j}\right) \neq 0 \\ y^a = Y_b^a(\widetilde{x}) \cdot \widetilde{y}^b, & \det\left(Y_b^a(\widetilde{x})\right) \neq 0 \end{cases} \tag{3.21}$$

The natural compatibility with respect to the coordinate transformations (3.18) is satisfied.

A linear d–connection on TM (see [8, 14]), whose associated h– and v– covariant derivatives preserve the gauge tensorial character of the gauge tensor fields, is called a generalized gauge linear d–connection. The nonlinear connection N, which provides the local adapted basis $\left(\delta_i, \overset{\bullet}{\partial}_a\right)$, is called a generalized gauge nonlinear connection (see also [6, 5, 7, 16, 18]). A gauge d–connection D on TM is said to be compatible with the metrical structure G if it satisfies the condition $D_X G = 0$, for all $X \in \mathcal{X}(TM)$. In the adapted basis, any gauge d–connection on TM can be represented as in (2.6) (the system of functions $(\overset{(H)}{F}{}^i_{jk}, \overset{(V)}{F}{}^i_{jk}, \overset{(H)}{C}{}^i_{jk}, \overset{(V)}{C}{}^i_{jk})$ represents the local coefficients of the above d–gauge connection D). It is not difficult to prove that if the canonical connection is a gauge one then the corresponding torsion and curvature tensor fields are gauge d–tensor fields.

From now on let us suppose that the canonical connection of the model is a gauge one with respect to the gauge transformations (3.21). As concerns the gauge theory on the our model one can prove the following results

Proposition 3.3. *The following Lagrangians*

$$L_2 = R^i_{jk} \cdot R^{jk}_i,\ L_{11} = R^{ij}_{ij},\ L_{12} = \overset{(V)}{R}{}^i_{hjk} \cdot \overset{(V)}{R}{}^{hjk}_i,\ L_{21} = \overset{(H)}{R}{}^i_{hjk} \cdot \overset{(H)}{R}{}^{hjk}_i$$

are gauge invariants with respect to the gauge transformations (3.21) *and to the local coordinates transformations* (3.18)*, respectively.*

Next, we will work with a full gauge Lagrangian, defined by

$$L = n_2 L_2 + n_{11} L_{11} + n_{12} L_{12} + n_{21} L_{21}, \tag{3.22}$$

where n_2, n_{11}, n_{12}, n_{21} are arbitrary real numbers. We consider the following Lagrangian density:

$$l = L \cdot G,\ G = \left(\det(g_{ij})\right)^{\frac{1}{2}} \cdot \left(\det(h_{ab})\right)^{\frac{1}{2}} \tag{3.23}$$

which depends only on the generalized gauge fields $\Phi \in \{g_{ij}, h_{ij}, N^i_j, \gamma^i_{jk}, \overset{(V)}{C}{}^i_{jk}\}$.

Proposition 3.4. *The solution of the variational problem $\delta \int d^n x d^n y = 0$, is the same with the solution of the following system of equations*

$$\frac{\delta L}{\delta \Phi} \equiv \frac{\partial}{\partial x^j}\left(\frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial x^j}\right)}\right) + \frac{\partial}{\partial y^j}\left(\frac{\partial L}{\partial\left(\frac{\partial \Phi}{\partial y^j}\right)}\right) - \frac{\partial L}{\partial \Phi} = 0 \tag{3.24}$$

Moreover, the gauge derivatives $\frac{\delta L}{\delta \Phi}$ are invariant to the gauge transformations (3.21) and to the local coordinates transformations (3.18), respectively. Before we state the last result of this paper we make the following notations

$$\overline{D}_m = D_m + \frac{D_m G}{G} \tag{3.25}$$

$$\omega_i^{jkl} = \gamma^{jk} \cdot \delta_i^l \tag{3.26}$$

Theorem 3.5. *Let α, β be the arbitrary real constants. The Einstein Yang Mills equations generated by the full Lagrangian L, the d–canonical connection $\overset{(0)}{D}$ and the generalized gauge fields $\Phi \in \{g_{ij}\ , h_{ij},\ N_j^i,\ \gamma_{jk}^i,\ \overset{(V)}{C}{}^i_{jk}\}$ are the following*

$$n_2 \overline{D}_m R_i^{mj} + n_{12} \cdot \frac{1}{F^6} \cdot y_i \cdot \left(\overline{\overset{(V)}{R}}{}^{hj0}_{h} + \gamma_{0r}^0 \cdot \overline{\overset{(V)}{R}}{}^{hrj}_{h} \right) = 0 \tag{3.27}$$

$$\overline{D}_m \left[4n_{21} \cdot \alpha \cdot \overset{(H)}{R}{}^{jkm}_{i} + 4n_{12} \cdot \beta \cdot \overset{(V)}{R}{}^{jkm}_{i} + n_{11} \cdot \alpha \cdot \left(\omega_i^{jkm} - \omega_i^{jmk} \right) \right] = 0 \tag{3.28}$$

$$n_{12} \cdot R_{mn}^k \cdot \overset{(V)}{R}{}^{jmn}_{i} = 0 \tag{3.29}$$

$$\frac{\partial L}{\partial \gamma_{ij}} + \frac{1}{2} \cdot \gamma^{ij} \cdot L = 0 \tag{3.30}$$

$$\frac{\partial L}{\partial h_{ij}} + \frac{1}{2} \cdot h^{ij} \cdot L = 0 \tag{3.31}$$

4. Conclusions

We remark that the theory we proposed in this paper works only for the case when the space test is without charges (see (3.17)). The Einstein equations of the our model projected on the basis manifold M (see (3.10)) are perturbations of the classical Einstein equations on the basis manifold M. For this reason the our Einstein equations can explain some unknown aspects in classical Einstein's theory (for another approach see [16]). The gauge equations (3.27)–(3.31) look like those from [6]. Consequently, they may be solved in the same manner. Finally we remark that, if on the basis manifold a Finsler function is given, the our theory could be developed too.

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