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ON EXISTING OF FILTERED MULTIPLICATIVE BASES IN GROUP ALGEBRAS

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Abstract. We give an explicit list of all p-groups G of order at most p^4 or 2^5 such that the group algebra KG over the field K of characteristic p has a filtered multiplicative K-basis.

1. Introduction. In [8] Kupish introduced the following definition. Let A be a finite-dimensional algebra over a field K and B a K-basis of A. Suppose that B is a K-basis of A with properties:

1. if $u, v \in B$ then either $uv = 0$ or $uv \in B$;
2. $B \cap \operatorname{rad}(A)$ is a K-basis for $\operatorname{rad}(A)$, where $\operatorname{rad}(A)$ denotes the Jacobson radical of A.

Then B is called a *filtered multiplicative K-basis* of A.

R. Bautista, P. Gabriel, A. Roiter and L. Salmeron showed in [1] that if there are only finitely many isomorphism classes of indecomposable A-modules over an algebraically closed field K, then A has a filtered multiplicative K-basis.

In the present article we shall investigate the following question from [1]: *When have the group algebras KG got a filtered multiplicative K-basis*?

According to Higman's theorem the group algebra KG over a field of characteristic p has only finitely many isomorphism classes of indecomposable KG-modules if and only if all the Sylow p-subgroups of G are cyclic.

Let $G = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_s\rangle$ be a finite abelian p-group with factors $\langle a_i\rangle$ of order q_i. Then the set

$$B = \{(a_1 - 1)^{n_1}(a_2 - 1)^{n_2} \cdots (a_s - 1)^{n_s} \quad | \quad 0 \le n_i < q_i\}$$

forms a filtered multiplicative K-basis of the group algebra KG over the field K of characteristic p.

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Evidently, if B_1 and B_2 are filtered multiplicative K-bases of KG_1 and KG_2, respectively, then $B_1 \times B_2$ is a filtered multiplicative K-basis of the group algebra $K[G_1 \times G_2]$.

First L. Paris gave examples of nonabelian metacyclic p-groups G such that group algebras KG have a filtered multiplicative K-bases in [9].

In [10] P. Landrock and G.O. Michler proved that the group algebra of the smallest Janko group over a field of characteristic 2 does not have a filtered multiplicative K-basis.

In [2] the following theorem was proved:

Theorem. *Let G be a finite metacyclic p-group and K a field of characteristic p. Then the group algebra KG possesses a filtered multiplicative K-basis if and only if $p = 2$ and exactly one of the following conditions holds:*

1. *G is a dihedral group;*
2. *K contains a primitive cube root of the unity and G is a quaternion group of order 8.*

In [3] was given all p-groups G with a cyclic subgroup of index p^2 such that the group algebra KG over the field K of characteristic p has a filtered multiplicative K-basis.

For this question negative answer was given in [3], when G is either a powerful p-group or a two generated p-group ($p \neq 2$) with central cyclic commutator subgroup.

2. Main results.

Denote C_n the cyclic group of order n. For the sake of convenience we shall keep the indices of these groups as in GAP. We have obtained the following theorems:

Theorem 1. *Let KG be the group algebra of a finite nonabel p-group G of order p^n over the field K of characteristic p, where $n < 5$. Then KG possesses a filtered multiplicative K-basis if and only if $p = 2$ and one of the following conditions satisfy:*

1. *G is either dihedral group D_8 of order 8 or dihedral group D_{16} of order 16;*
2. *G is either Q_8 or $Q_8 \times C_2$ and K contains a primitive cube root of the unity;*
3. *G is either $D_8 \times C_2$, or the central product $D_8 Y C_4$ of D_8 with C_4;*
4. *G is $H_{16} = \langle \quad a, c \quad | \quad a^4 = b^2 = c^2 = 1,\ (a,b) = 1,\ (a,c) = b,\ (b,c) = 1 \quad \rangle$.*

Theorem 2. *Let K be a field of characteristic 2 and*

$$G = \langle \quad a, b \quad | \quad a^{2^n} = b^{2^m} = c^2 = 1,\ (a,b) = c,\ (a,c) = 1,\ (b,c) = 1 \quad \rangle,$$

with $n, m \geq 2$. Then KG possesses a filtered multiplicative K-basis.

Theorem 3. *Let G be the group*

$$G = \langle \quad a, b \quad | \quad a^{2^n} = b^2 = c^2 = d^2 = 1, (a, b) = c,\ (a, c) = d,$$
$$(a, d) = (b, c) = (b, d) = (c, d) = 1 \quad \rangle,$$

with $n > 1$, and K a field of characteristic 2. Then KG has no filtered multiplicative K-basis.

Theorem 4. *Let KG be the group algebra of a finite nonabel 2-group G of order 2^5 over a field K of characteristic 2. Then KG possesses a filtered multiplicative K-basis if and only if one of the following conditions satisfy:*

1. *G is $G_{18} = D_{32}$, $G_{25} = D_8 \times C_4$, $G_{39} = D_{16} \times C_2$ or $G_{46} = D_8 \times C_2 \times C_2$;*
2. *G is $G_{26} = Q_8 \times C_4$, or $G_{47} = Q_8 \times C_2 \times C_2$ and K contains a primitive cube root of the unity;*
3. *G is $G_{22} = H_{16} \times C_2$, $G_{48} = (D_8 Y C_4) \times C_2$*
4. *G is one of the following groups:*

$$G_2 = \langle \quad a, b \quad | \quad a^4 = b^4 = c^2 = 1,\ (a, b) = c,\ (a, c) = 1,\ (b, c) = 1 \quad \rangle;$$
$$G_5 = \langle \quad a, b \quad | \quad a^8 = b^2 = c^2 = 1,\ (a, b) = c,\ (a, c) = (b, c) = 1 \quad \rangle;$$
$$G_7 = \langle \quad a, b, c \mid \quad a^8 = b^2 = c^2 = 1,\ (a, c) = a^4,\ (a, b) = a^4 c,\ (b, c) = 1 \quad \rangle;$$
$$G_8 = \langle \quad a, b, c \mid \quad a^8 = c^2 = 1,\ b^2 = a^4,\ (a, c) = a^4,\ (a, b) = a^4 c,\ (b, c) = 1 \quad \rangle;$$
$$G_9 = \langle \quad a, b, c \mid \quad a^2 = b^8 = c^2 = 1,\ (b, c) = ab^6,\ (a, c) = (a, b) = 1 \quad \rangle;$$
$$G_{10} = \langle \quad a, b, c \mid \quad a^8 = b^4 = c^2 = 1,\ a^4 = b^2,\ (a, b) = a^6 c,\ (a, c) = (b, c) = 1 \quad \rangle;$$
$$G_{11} = \langle \quad a, b, c \mid \quad a^4 = b^4 = c^2 = 1,\ (b, c) = ab^2,\ (a, c) = (a, b) = 1 \quad \rangle;$$
$$G_{49} = \langle \quad a, b, c, d \mid a^4 = 1,\ b^2 = c^2 = d^2 = a^2,\ (a, b) = a^2,\ (c, d) = a^2,$$
$$(a, c) = (a, d) = (b, c) = (b, d) = 1 \quad \rangle.$$

3. Preliminary remarks and notation. Assume that B is a filtered multiplicative K-basis for a finite-dimensional K-algebra A. In the proof of the main results we use the following simple properties of B (see [2]):

(I) $B \cap rad(A)^n$ is a K-basis of $rad(A)^n$ for all $n \geq 1$.

(II) if $u, v \in B \setminus rad(A)^k$ and $u \equiv v \pmod{rad(A)^k}$ then $u = v$.

Recall that the *Frattini subalgebra* $\Phi(A)$ of A is defined as the intersection of all maximal subalgebras of A if those exist, and as A otherwise. If A is a nilpotent algebra over a field K, then $\Phi(A) = A^2$ by [5]. It implies that

(III) if B is a filtered multiplicative K-basis of A and if $B \setminus \{1\} \subseteq rad(A)$, then all elements of $B \setminus rad(A)^2$ are generators of A over K.

A p-group G is called *powerful*, if one of the following conditions holds:

1. G is a 2-group and G/G^4 is abelian;
2. G is a p-group ($p > 2$) and G/G^p is abelian.

Let G be a finite p-group. For $a, b \in G$ we define $a^b = b^{-1}ab$ and the commutator $(a, b) = a^{-1}b^{-1}ab$. Denote by Q_{2^n}, D_{2^n} and SD_{2^n} the *generalized quaternion group*, the *dihedral* and *semidihedral* 2-group of order 2^n, respectively, and

$$MD_{2^n} = \langle \quad a, b \quad | \quad a^{2^{n-1}} = b^2 = 1, \ (a, b) = a^{2^{n-2}} \quad \rangle.$$

We define the *Lazard-Jennings series* $M_i(G)$ of a finite p-group G by induction (see [6]). Put $M_1(G) = G$ and $M_i(G) = \langle \ (M_{i-1}(G), G), M^p_{[\frac{i}{p}]}(G)) \ \rangle$, where

— $[\frac{i}{p}]$ is the smallest integer not less than $\frac{i}{p}$;

— $\big(M_{i-1}(G), G\big) = \langle \ (u, v) \ \mid \ u \in M_{i-1}(G), \ v \in \ G \ \rangle$;

— $M^p_i(G)$ is the subgroup generated by p-powers of the elements of $M_i(G)$.

Evidently,

$$M_1(G) \supseteq M_2(G) \supseteq \cdots \supseteq M_t(G) = 1.$$

Let K be a field of characteristic p. The ideal

$$A(KG) = \left\{ \quad \sum_{g\in G} \alpha_g g \in KG \quad \mid \quad \sum_{g\in G} \alpha_g = 0 \quad \right\}$$

is called the *augmentation* ideal of KG. Since G is a finite p-group and K is a field of characteristic p, $A(KG)$ is nilpotent, and

$$A(KG) \supset A^2(KG) \supset \cdots \supset A^s(KG) \supset A^{s+1}(KG) = 0.$$

Moreover, $A(KG)$ is the radical of KG.

Then the subgroup $\mathfrak{D}_n(G) = \left\{ \ g \in G \quad \mid \quad g - 1 \in A^n(KG) \ \right\}$ is called the nth *dimensional subgroup* of KG.

It is well known that for finite p-group G, $M_i(G) = \mathfrak{D}_i(G)$ for all i.

Let $\mathbb{I} = \{i \in \mathbb{N} \mid \mathfrak{D}_i(G) \neq \mathfrak{D}_{i+1}(G)\}$. For $i \in \mathbb{I}$, let p^{d_i} be the order of the elementary abelian p-group

$$\mathfrak{D}_i(G)/\mathfrak{D}_{i+1}(G) = \prod_{j=1}^{d_i} \langle u_{ij}\mathfrak{D}_{i+1}(G)\rangle.$$

Hence each $g \in G$ can be written uniquely in the form

$$g = u_{11}^{\alpha_{11}} u_{12}^{\alpha_{12}} \cdots u_{1d_1}^{\alpha_{1d_1}} u_{21}^{\alpha_{21}} \cdots u_{2d_2}^{\alpha_{2d_2}} \ldots u_{i1}^{\alpha_{i1}} \cdots u_{id_i}^{\alpha_{id_i}} \cdots u_{s1}^{\alpha_{s1}} \cdots u_{sd_s}^{\alpha_{sd_s}},$$

where the indices are in lexicographic order, $i \in \mathbb{I}$, $0 \le \alpha_{ij} < p$, and s is defined as above.

Let $w = \prod_{l\in\mathbb{I}}(\prod_{k=1}^{d_l}(u_{lk}-1)^{y_{lk}}) \in A(KG)$ be where $0 \le y_{lk} < p$, and the indices of the factors are in lexicographic order. Then w is called a regular element of weight $\mu(w) = \sum_{l\in\mathbb{I}}\left(\sum_{k=1}^{d_l} l y_{lk}\right)$. By Jennings Theorem (see [6]), regular elements which weight not less than t constitute a K-basis for the ideal $A^t(KG)$.

Clearly, $\{\ (u_{1j}-1)+A^2(KG)\ \mid\ j=1,\dots,d_1\ \}$ is a K-basis of $A(KG)/A^2(KG)$.

Note that $\mathfrak{D}_2(G)$ coincides with the Frattini subgroup of G, so the set $\{u_{11}, u_{12}, \dots, u_{1d_1}\}$ is a minimal generator system of G.

Suppose that $B_1 = \{1\} \cup \{b_1, b_2, \dots, b_{|G|-1}\}$ is a filtered multiplicative K-basis for KG. Then $B = B_1 \setminus \{1\}$ is a filtered multiplicative K-basis of $A(KG)$ and contains $|G|-1$ elements.

Let $B \setminus (B \cap A^2(KG)) = \{b_1, b_2, \dots, b_n\}$. Evidently, $n = d_1$ and

$$b_k \equiv \sum_{i=1}^{n} \alpha_{ki}(u_{1i}-1) \pmod{A^2(KG)},$$

where $\alpha_{ki} \in K$ and $\Delta = det(\alpha_{ki}) \neq 0$.

For units x, y of KG we have

$$\begin{aligned}(y-1)(x-1) = [(x-1)(y-1) + (x-1) + (y-1)](z-1)\\ + (x-1)(y-1) + (z-1),\end{aligned} \tag{1}$$

where $z = (y,x)$. Since $z_{ji} = (u_{1j}, u_{1i}) \in \mathfrak{D}_2(G)$ and $z_{ji} - 1 \in A^2(KG)$, using (1) we obtain that

$$(u_{1j}-1)(u_{1i}-1) \equiv (u_{1i}-1)(u_{1j}-1) + (z_{ji}-1) \pmod{A^3(KG)}. \tag{2}$$

Thus simple computations give that

$$\begin{aligned}b_k b_s \equiv \sum_{i=1}^{n} \alpha_{ki}\alpha_{si}(u_{1i}-1)^2 + \sum_{\substack{i,j=1\\ i<j}}^{n} (\alpha_{ki}\alpha_{sj} + \alpha_{kj}\alpha_{si})(u_{1i}-1)(u_{1j}-1)\\ + \sum_{\substack{i,j=1\\ i<j}}^{n} \alpha_{kj}\alpha_{si}(z_{ji}-1) \pmod{A^3(KG)},\end{aligned} \tag{3}$$

where $k, s = 1, \dots, n$.

Denote by $\mathfrak{A}$ the set of groups which belong to one of the following type of nonabelian p-groups:

1. either metacyclic or powerful;
2. p-group with cyclic subgroup of index p^2;
3. two generated p-group $(p \neq 2)$ with central cyclic commutator subgroup.

4. Proof of Theorem 1. Let K be a field of characteristic p (p is odd) and G a p-group of order p^4. The classification of these groups can be found in [7]. According to [2] and [3] if G belongs to $\mathfrak{A}$ then G has no filtered multiplicative basis. If G does not belong to $\mathfrak{A}$, then it is one of the following two groups:

$$H_1 = \langle \quad a, c \quad | \quad a^p = c^p = 1, \quad (a, c) = d, (d, c) = f,$$
$$(a, d) = (a, f) = (c, f) = (d, f) = 1 \quad \rangle \text{ with } p > 3;$$

and

$$H_2 = \langle \quad a, c \quad | \quad a^p = c^p = 1, (a, c) = d,$$
$$(c, d) = (a, d) = 1 \quad \rangle \times \langle \quad h \mid h^p = 1 \quad \rangle \text{ with } p \geq 3.$$

It is easy to check that in both group algebras KH_1 and KH_2:

$$(c-1)(a-1) \equiv (a-1)(c-1) - (d-1) \pmod{A^3(KG)}. \tag{4}$$

Let us consider the following cases:

Case 1. Let $G = H_1$. Since

$$M_1(G) = G, \quad M_2(G) = \langle G', G^p \rangle = \langle d, f \rangle, \quad M_3(G) = \langle \ (\langle d, f \rangle, G), \ G^p \ \rangle = \langle f \rangle$$

we have that $\mu(d) = 2$ and $\mu(f) = 3$, where μ is the weight of these elements. Using (4) and

$$(d-1)(c-1) \equiv (c-1)(d-1) + (f-1) \pmod{A^4(KG)},$$

let us compute $b_{i_1} b_{i_2} b_{i_3}$ modulo $A^4(KG)$ where ($i_k = 1, 2$). The results of our computations will be written in a table, consisting of the coefficients of the decomposition $b_{i_1} b_{i_2} b_{i_3}$ with respect to the basis

$$\{ \quad (a-1)^{j_1}(c-1)^{j_2}(d-1)^{j_3}(f-1)^{j_4} \quad | \quad j_1 + j_2 + 2j_3 + 3j_4 = 3;$$
$$j_1, j_2 = 0, 1, 2, 3; \ j_3, j_4 = 0, 1 \quad \}$$

of the ideal $A^3(KG)/A^4(KG)$. The coefficients of $b_{i_1}, b_{i_2}, b_{i_3}$ will be denoted $\alpha_i, \beta_i, \gamma_i$, respectively, and in the following we shall use these coefficients. We shall divide the table into two parts (the second part written below the first part). The coefficients corresponding to the first four basis elements will be in the first part of the table, while the next three will be in the second one. Thus

	$(a-1)^3$	$(a-1)^2(c-1)$	$(a-1)(d-1)$	$(a-1)(c-1)^2$
$b_1 b_2 b_1$	$\alpha_1^2\beta_1$	$2\alpha_1\alpha_2\beta_1 + \alpha_1^2\beta_2$	$-2\alpha_1\alpha_2\beta_1 - \alpha_1^2\beta_2$	$2\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1$
$b_1 b_2^2$	$\alpha_1\beta_1^2$	$2\alpha_1\beta_1\beta_2 + \alpha_2\beta_1^2$	$-2\alpha_2\beta_1^2 - \alpha_1\beta_1\beta_2$	$2\alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2$
$b_2 b_1^2$	$\alpha_1^2\beta_1$	$2\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1$	$-2\alpha_1^2\beta_2 - \alpha_1\alpha_2\beta_1$	$2\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1$
$b_2 b_1 b_2$	$\alpha_1\beta_1^2$	$2\alpha_1\beta_1\beta_2 + \alpha_2\beta_1^2$	$-2\alpha_1\beta_1\beta_2 - \alpha_2\beta_1^2$	$2\alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2$
b_1^3	α_1^3	$3\alpha_1^2\alpha_2$	$-3\alpha_1^2\alpha_2$	$3\alpha_1\alpha_2^2$
$b_1^2 b_2$	$\alpha_1^2\beta_1$	$2\alpha_1\alpha_2\beta_1 + \alpha_1^2\beta_2$	$-3\alpha_1\alpha_2\beta_1$	$2\alpha_1\alpha_2\beta_2 + \alpha_2^2\beta_1$
$b_2^2 b_1$	$\alpha_1\beta_1^2$	$2\alpha_1\beta_1\beta_2 + \alpha_2\beta_1^2$	$-3\alpha_1\beta_1\beta_2$	$2\alpha_2\beta_1\beta_2 + \alpha_1\beta_2^2$
b_2^3	β_1^3	$3\beta_1^2\beta_2$	$-3\beta_1^2\beta_2$	$3\beta_1\beta_2^2$

	$(c-1)(d-1)$	$(f-1)$	$(c-1)^3$
$b_1b_2b_1$	$-2\alpha_1\alpha_2\beta_2-\alpha_2^2\beta_1$	$-\alpha_2^2\beta_1-\alpha_1\alpha_2\beta_2$	$\alpha_2^2\beta_2$
$b_1b_2^2$	$-3\alpha_2\beta_1\beta_2$	$-2\alpha_2\beta_1\beta_2$	$\alpha_2\beta_2^2$
$b_2b_1^2$	$-3\alpha_1\alpha_2\beta_2$	$-2\alpha_1\alpha_2\beta_2$	$\alpha_2^2\beta_2$
$b_2b_1b_2$	$-2\alpha_2\beta_1\beta_2-\alpha_1\beta_2^2$	$-\alpha_1\beta_2^2-\alpha_2\beta_1\beta_2$	$\alpha_2\beta_2^2$
b_1^3	$-3\alpha_1\alpha_2^2$	$-2\alpha_1\alpha_2^2$	α_2^3
$b_1^2b_2$	$-2\alpha_2^2\beta_1-\alpha_1\alpha_2\beta_2$	$-\alpha_1\alpha_2\beta_2-\alpha_2^2\beta_1$	$\alpha_2^2\beta_2$
$b_2^2b_1$	$-2\alpha_1\beta_2^2-\alpha_2\beta_1\beta_2$	$-\alpha_2\beta_1\beta_2-\alpha_1\beta_2^2$	$\alpha_2\beta_2^2$
b_2^3	$-3\beta_2\beta_2^2$	$-2\beta_1\beta_2^2$	β_2^3

We have obtained 8 elements, but the K-dimension of $A^3(KG)/A^4(KG)$ equals 7. Since $\Delta \neq 0$, we can establish that one of this elements either equals to zero modulo ideal $A^4(KG)$ or coincides with another one.

It is easy to see that none of lines are equal to zero. Indeed, for example, if $b_1b_2b_1 \equiv 0 \pmod{A^4(KG)}$ then from second column of the first part and fourth column of the second part of the table we get that $\alpha_1^2\beta_1 = 0$ and $\alpha_2^2\beta_2 = 0$. Since $\Delta \neq 0$, this case is impossible by third column of the first part of this table. In a similar manner we can proof this statement for all lines.

The assumption that two of lines are equal also a contradiction. For instance, if $b_1b_2b_1 \equiv b_1b_2^2 \pmod{A^4(KG)}$, then from second column of the first part and fourth column of the second part of the table it follows that $\alpha_1\beta_1(\alpha_1-\beta_1) = 0$ and $\alpha_2\beta_2(\alpha_2-\beta_2) = 0$. Since $\Delta \neq 0$, the third column of the first part of the table leads to a contradiction.

Similar calculations for any two lines also lead to a contradiction, so we have got that KG has no filtered multiplicative basis.

Case 2. Let $G = H_2$. Using (4) let us compute $b_{i_1}b_{i_2}$ modulo $A^3(KG)$ where $(i_k = 1,2,3)$. The results of our computations will be written in a table, consisting of the coefficients of the decomposition $b_{i_1}b_{i_2}$ with respect to the basis

$$\left\{ \quad (a-1)^{j_1}(c-1)^{j_2}(h-1)^{j_3}(d-1)^{j_4} \quad \middle| \quad \begin{array}{l} j_1+j_2+j_3+2j_4 = 2; \\ j_1,j_2,j_3 = 0,1,2;\ j_4 = 0,1 \end{array} \quad \right\}$$

of the ideal $A^2(KG)/A^3(KG)$:

	$(a-1)^2$	$(a-1)(c-1)$	$(a-1)(h-1)$	$(c-1)^2$	$(c-1)(h-1)$	$(h-1)^2$	$(d-1)$
b_1b_2	$\alpha_1\beta_1$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_2$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_3\beta_3$	$-\alpha_2\beta_1$
b_2b_1	$\alpha_1\beta_1$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_2$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_3\beta_3$	$-\alpha_1\beta_2$
b_1b_3	$\gamma_1\alpha_1$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_2$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_3\gamma_3$	$-\alpha_2\gamma_1$
b_3b_1	$\gamma_1\alpha_1$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_2$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_3\gamma_3$	$-\alpha_1\gamma_2$
b_2b_3	$\beta_1\gamma_1$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_2$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_3\gamma_3$	$-\beta_2\gamma_1$
b_3b_2	$\beta_1\gamma_1$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_2$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_3\gamma_3$	$-\beta_1\gamma_2$
b_1^2	α_1^2	$2\alpha_1\alpha_2$	$2\alpha_1\alpha_3$	α_2^2	$2\alpha_2\alpha_3$	α_3^2	$-\alpha_1\alpha_2$
b_2^2	β_1^2	$2\beta_1\beta_2$	$2\beta_1\beta_3$	β_2^2	$2\beta_2\beta_3$	β_3^2	$-\beta_1\beta_2$
b_3^2	γ_1^2	$2\gamma_1\gamma_2$	$2\gamma_1\gamma_3$	γ_2^2	$2\gamma_2\gamma_3$	γ_3^2	$-\gamma_1\gamma_2$

We have obtained 9 elements, but the K-dimension of $A^2(KG)/A^3(KG)$ equals 7, so we conclude that some lines of the table either are equal to zero modulo the ideal $A^3(KG)$ or coincide with some other lines.

Since $\Delta \neq 0$, it is clear that $b_i^2 \not\equiv 0$ and $b_ib_j \not\equiv 0 \pmod{A^3(KG)}$. According to the last tree lines of the table if $b_i^2 \equiv b_j^2 \pmod{A^3(KG)}$, then either $b_i \equiv b_j$ or $b_i \equiv -b_j \pmod{A^3(KG)}$, so we have that $b_1b_2 \equiv b_2b_1$, $b_1b_3 \equiv b_3b_1$ and $b_2b_3 \not\equiv b_3b_2$ $(\mathrm{mod}\ A^3(KG))$, because the other cases are similar to this one.

Simple computations show that if either $\alpha_1 \neq 0$ or $\beta_1 \neq 0$, then KG is a commutative algebra which is a contradiction, so we can assume that $\alpha_1 = \beta_1 = 0$. From the 8th column we have $\alpha_2\gamma_1 = 0$. Since $\Delta \neq 0$ we conclude that $\alpha_2 = 0$ and we have a basis of $A(KG)/A^2(KG)$:

$$\begin{cases} b_1 & \equiv (h-1) \pmod{A^2(KG)}; \\ b_2 & \equiv (c-1) + \beta_3(h-1) \pmod{A^2(KG)}; \\ b_3 & \equiv (a-1) + \gamma_2(c-1) + \gamma_3(h-1) \pmod{A^2(KG)}. \end{cases}$$

Let us compute $b_{i_1}b_{i_2}b_{i_3}$ modulo $A^4(KG)$ where $i_k = 1,2,3$ with respect to the basis

$$\{ \quad (a-1)^{j_1}(c-1)^{j_2}(h-1)^{j_3}(d-1)^{j_4} \quad | \quad j_1+j_2+j_3+2j_4 = 3; \; j_1,j_2,j_3 = 0,1,2,3;\; j_4 = 0,1 \quad \}$$

of the ideal $A^3(KG)/A^4(KG)$.

Assume that $p = 3$. Since the dimension of $A^3(KG)/A^4(KG)$ is 10, so we conclude that

$$b_1^2b_2 \equiv b_1b_2^2, \quad b_2^2b_3 \equiv b_3b_2^2, \quad b_3^2b_2 \equiv b_3b_2b_3 \pmod{A^4(KG)},$$

and $b_1^3 \equiv 0, \quad b_2^3 \equiv 0 \pmod{A^4(KG)}$. From these congruences give that $\beta_3 = \gamma_2 = \gamma_3 = 0$.

Now suppose that $p > 3$. In this case the dimension of $A^3(KG)/A^4(KG)$ is 15, so we conclude that

$$b_1^2b_2 \equiv b_1b_2^2, \quad b_3^2b_2 \equiv b_3b_2b_3 \pmod{A^4(KG)},$$

and we also get that $\beta_3 = \gamma_2 = \gamma_3 = 0$.

Assume that KG has a filtered multiplicative basis. Since $KG = K[G_1 \times G_2]$, where

$$G_1 = \langle \quad a, c \quad | \quad a^p = c^p = 1, \quad (a,c) = d, (c,d) = (a,d) = 1 \quad \rangle$$

and $G_2 = \langle \quad h \quad | \quad h^p = 1 \quad \rangle$, and we have established that

$$\begin{cases} b_1 & \equiv (h-1) \pmod{A^2(KG)}; \\ b_2 & \equiv (c-1) \pmod{A^2(KG)}; \\ b_3 & \equiv (a-1) \pmod{A^2(KG)}, \end{cases}$$

with $b_1 \in KG_1$, $b_2, b_3 \in KG_2$, so we conclude that KG_2 also has filtered multiplicative basis, which is a contradiction by [3].

Let K be a field of characteristic 2. If $|G| < 2^5$, then KG has a filtered multiplicative basis (see [2,3]) if and only if G and K satisfy the conditions of Theorem 1, so the proof of the theorem is complete.

5. Proof of Theorem 2. Let

$$G = \langle \quad a, b \quad | \quad a^{2^n} = b^{2^m} = c^2 = 1,\ (a,b) = c,\ (a,c) = 1,\ (b,c) = 1 \quad \rangle,$$

and put

$$b_1^1 = u \equiv (1+a), \quad b_1^2 = v \equiv (1+b) \pmod{A^2(KG)}.$$

Using the identity:

$$(1+b)(1+a) \equiv (1+a)(1+b) + (1+c) \pmod{A^3(KG)},$$

we get that the set $\{\ b_2^1 = uv,\ b_2^2 = vu,\ b_2^3 = u^2,\ b_2^4 = v^2\ \}$ is a basis of $A^2(KG)/A^3(KG)$ and

$$\begin{aligned}
b_3^1 &= uvu \equiv (1+a)^2(1+b) + (1+a)(1+c) \pmod{A^3(KG)};\\
b_3^2 &= u^2v \equiv (1+a)^2(1+b) \pmod{A^3(KG)};\\
b_3^3 &= u^3 \equiv (1+a)^3 \pmod{A^3(KG)};\\
b_3^4 &= uv^2 \equiv (1+a)(1+b)^2 \pmod{A^3(KG)};\\
b_3^5 &= vuv \equiv (1+a)(1+b)^2 + (1+b)(1+c) \pmod{A^3(KG)};\\
b_3^6 &= v^3 \equiv (1+b)^3 \pmod{A^3(KG)};
\end{aligned}$$

is a basis for $A^3(KG)/A^4(KG)$ and its determinant $\Delta_3 = 1$. We shall construct a basis of $A^i(KG)/A^{i+1}(KG)$ by induction. Assume that $b_{i-1}^1, b_{i-1}^2, \cdots, b_{i-1}^{n-1}, b_{i-1}^n$ is a basis for $A^{i-1}(KG)/A^i(KG)$. Evidently, the determinant Δ_{i-1} of this basis is not zero. Simple computations show that the determinant Δ_i of the elements $b_i^j = ub_{i-1}^j$, for $j = \{1, 2, \cdots, n\}$ and $b_i^{n+1} = b_{i-1}^{n-1}v$, $b_i^{n+2} = b_{i-1}^n v$ is equal to $\Delta_{i-1} \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \neq 0$, so we got $n+2$ linearly independent elements. Since $dim\, A^i(KG)/A^{i+1}(KG)$ is also $n+2$ we have obtained that KG has a filtered multiplicative basis.

6. Proof of Theorem 3. Let G be the group

$$\begin{aligned}
G_6 = \langle \quad a, b \quad | \quad a^{2^n} = b^2 = c^2 = d^2 = 1, (a,b) = c,\ (a,c) = d,&\\
(a,d) = (b,c) = (b,d) = (c,d) = 1 \quad \rangle,&
\end{aligned}$$

with $n > 1$. Let us compute the Lazard-Jennings series of this group:

$$M_1(G) = G, \quad M_2(G) = \langle a^2, c, d\rangle, \quad M_3(G) = \langle d\rangle, \quad M_4(G) = \langle 1\rangle.$$

We conclude that $\mu(c) = 2$ and $\mu(d) = 3$. Using the identity

$$(1+b)(1+a) \equiv (1+a)(1+b) + (1+c) \pmod{A^3(KG)}, \tag{5}$$

it follows that

$$\begin{cases} b_1b_2 \equiv \alpha_1\beta_1(1+a)^2 + (\alpha_1\beta_2+\alpha_2\beta_1)(1+a)(1+b) + \alpha_2\beta_1(1+c) \pmod{A^3(KG)}; \\ b_2b_1 \equiv \alpha_1\beta_1(1+a)^2 + (\alpha_1\beta_2+\alpha_2\beta_1)(1+a)(1+b) + \alpha_1\beta_2(1+c) \pmod{A^3(KG)}; \\ b_1^2 \equiv \alpha_1^2(1+a)^2 + \alpha_1\alpha_2(1+c) \pmod{A^3(KG)}; \\ b_2^2 \equiv \beta_1^2(1+a)^2 + \beta_1\beta_2(1+c) \pmod{A^3(KG)}. \end{cases}$$

We have obtained 4 elements, but the K-dimension of $A^2(KG)/A^3(KG)$ equals 3. Since $\Delta \neq 0$, we get that $b_1b_2 \not\equiv b_2b_1, b_1^2, b_2^2$ and $b_1b_2,\ b_2b_1 \not\equiv 0$ and $b_1^2 \not\equiv b_2^2 \pmod{A^3(KG)}$. Thus either $b_1^2 \equiv 0$ or $b_2^2 \equiv 0 \pmod{A^3(KG)}$. It is easy to see that the second case is similar to the first one, so we consider the second one. Let $\beta_1 = 0$ and we can put $\alpha_1 = \beta_2 = 1$ and

$$u = b_1 \equiv (1+a) + \alpha_2(1+b) \pmod{A^2(KG)};$$
$$v = b_2 \equiv (1+b) \pmod{A^2(KG)}.$$

Using (5) and the identity

$$(1+c)(1+a) \equiv (1+a)(1+c) + (1+d) \pmod{A^4(KG)},$$

straightforward computations show that

$$\begin{cases} uvu^2 & \equiv (1+a)^3(1+b) + \alpha_2(1+a)(1+b)(1+c) + (1+a)(1+d) \pmod{A^5(KG)}; \\ vu^3 & \equiv (1+a)^3(1+b) + (1+a)^2(1+c) + \alpha_2(1+a)(1+b)(1+c) + \\ & \qquad (1+a)(1+d) \pmod{A^5(KG)}; \\ vuvu & \equiv (1+b)(1+d) + (1+a)(1+b)(1+c) \pmod{A^5(KG)}; \\ u^2vu & \equiv (1+a)^3(1+b) + (1+a)^2(1+c) + \alpha_2(1+a)(1+b)(1+c) + \\ & \qquad \alpha_2(1+b)(1+d) \pmod{A^5(KG)}; \\ uvuv & \equiv (1+a)(1+b)(1+c) \pmod{A^5(KG)}; \\ vu^2v & \equiv (1+b)(1+d) \pmod{A^5(KG)}; \\ u^3v & \equiv (1+a)^3(1+b) + \alpha_2(1+a)(1+b)(1+c) + \alpha_2(1+b)(1+d) \pmod{A^5(KG)}. \end{cases}$$

We have obtained 7 different element, but this is a contradiction because $dim(A^4(KG)/A^5(KG)) = 5$.

6. Proof of Theorem 4.

Let G be a nonabelian 2-group of order 2^5. According to [3] if G is one of the following groups $\{G_5, G_7, G_8, G_9, G_{10}, G_{11}\}$, then G has a cyclic subgroup of index p^2 and KG has filtered multiplicative basis, but if G is one of the following groups:

$$G_{40} = SD_{16} \times C_2;$$
$$G_{41} = Q_{16} \times C_2;$$
$$G_{42} = \langle\ a,b,c \mid a^8 = b^4 = c^4 = 1, a^4 = b^2 = c^2, (a,b) = a^6, (a,c) = (b,c) = 1\rangle;$$
$$G_{43} = \langle\ a,b,c \mid a^8 = b^2 = c^2 = 1, (a,b) = a^6, (a,c) = a^4, (b,c) = 1\rangle;$$
$$G_{44} = \langle\ a,b,c \mid a^8 = c^2 = 1, b^2 = a^4, (a,c) = a^4, (a,b) = a^6, (b,c) = 1\rangle,$$

then KG has no filtered multiplicative basis.

If G is one of the following groups:

$$\begin{aligned} G_4 =& \langle \quad a,b \quad | \quad a^8 = b^4 = 1,\ (a,b) = a^4 \quad \rangle;\\ G_{37} =& MD_{16} \times C_2;\\ G_{38} =& \langle \quad a,b,c \quad | \quad a^8 = b^2 = c^2 = 1,\ (b,c) = a^4,\ (a,b) = (a,c) = 1 \quad \rangle. \end{aligned}$$

then they are powerful groups and by [3] KG has no a filtered multiplicative basis. If G is one of the following groups:

$$\begin{aligned} G_{12} =& \langle \quad a,b \quad | a^4 = b^8 = 1, (a,b) = a^2 \rangle;\\ G_{13} =& \langle \quad a,b \quad | a^8 = b^4 = 1, (a,b) = a^2 \rangle;\\ G_{14} =& \langle \quad a,b \quad | a^8 = b^4 = 1, (a,b) = a^6 \rangle;\\ G_{15} =& \langle \quad a,b \quad | a^8 = 1, b^4 = a^4, (a,b) = a^6 \rangle;\\ G_{17} =& MD_{32},\ G_{18} = D_{32},\ G_{19} = SD_{32},\ G_{20} = Q_{32}, \end{aligned}$$

then G is a metacyclic group and KG has a filtered multiplicative basis if and only if $G = G_{18}$ by [2].

According to [2] and [3] we get that for the following direct products KG has a filtered multiplicative basis: $G_{22} = H_{16} \times C_2, G_{25} = D_8 \times C_4, G_{26} = Q_8 \times C_4, G_{39} = D_{16} \times C_2, G_{46} = D_8 \times C_2 \times C_2, G_{47} = Q_8 \times C_2 \times C_2, G_{48} = (D_8 Y C_4) \times C_2$.

If $G = G_2$, then for $n = m = 2$ Theorem 2 asserts that KG has a filtered multiplicative K-basis.

Let G be the group

$$\begin{aligned} G_6 = \langle \quad a,b \quad | \quad a^4 = b^2 = 1, (a,b) = c,\ (a,c) = d,\\ (a,d) = (b,c) = (b,d) = (c,d) = 1 \quad \rangle. \end{aligned}$$

For $n = 2$ the group in Theorem 3 is isomorphic to G_6, so the group algebra KG_6 has no filtered multiplicative K-basis.

Now, we shall consider the following 7 cases.

Case 1. Let G be the group

$$G_{23} = \langle \quad a,b,c \quad | \quad a^4 = b^4 = c^2 = 1,\ (a,c) = (b,c) = 1,\ (a,b) = a^2 \quad \rangle.$$

Using the identity

$$(1+b)(1+a) \equiv (1+a)(1+b) + (1+a)^2 \pmod{A^3(KG)},$$

let us compute $b_{i_1}b_{i_2}$ modulo $A^3(KG)$ where $(i_k = 1, 2, 3)$. The results of our computation will be written in a table, consisting of the coefficients of the decomposition $b_{i_1}b_{i_2}$ with respect to the basis

$$\{ \quad (1+a)^{j_1}(1+b)^{j_2}(1+c)^{j_3} \quad \mid \quad j_1+j_2+j_3=2; \\ j_1, j_2 = 0,1,2;\ j_3 = 0,1 \quad \}$$

of the ideal $A^2(KG)$.

	$(1+a)^2$	$(1+a)(1+b)$	$(1+a)(1+c)$	$(1+b)(1+c)$	$(1+b)^2$
b_1b_2	$\alpha_1\beta_1+\alpha_2\beta_1$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_2\beta_2$
b_2b_1	$\alpha_1\beta_1+\alpha_1\beta_2$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_2\beta_2$
b_1b_3	$\alpha_1\gamma_1+\alpha_2\gamma_1$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_2\gamma_2$
b_3b_1	$\alpha_1\gamma_1+\alpha_1\gamma_2$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_2\gamma_2$
b_2b_3	$\beta_1\gamma_1+\beta_2\gamma_1$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_2\gamma_2$
b_3b_2	$\beta_1\gamma_1+\beta_1\gamma_2$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_2\gamma_2$
b_1^2	$\alpha_1^2+\alpha_2\alpha_3$	0	0	0	α_2^2
b_2^2	$\beta_1^2+\beta_2\beta_3$	0	0	0	β_2^2
b_3^2	$\gamma_1^2+\gamma_2\gamma_3$	0	0	0	γ_2^2

Since $\Delta \neq 0$, it is easy to see that the first six lines not equal neither zero nor the last three lines. Note that the dimension of $A^2(KG)/A^3(KG)$ equal to 5 and KG is not a commutative algebra. From the fact $b_i^2 \equiv b_j^2 \equiv 0 \pmod{A^3(KG)}$, $i \neq j$ it implies that b_i linearly depends on b_j, so we shall consider two interesting cases.

In the first case $b_1^2 \equiv 0$, $b_2^2 \equiv b_3^2 \not\equiv 0 \pmod{A^3(KG)}$ and we get that $b_1 \equiv \alpha_3(1+c) \pmod{A^2(KG)}$ and by property (II) of the filtered multiplicative K-basis, $b_2^2 = b_3^2$. From the condition $b_2^2 \equiv b_3^2 \pmod{A^3(KG)}$ we have that $\beta_2 = \gamma_2 \neq 0$ and $(\beta_1+\gamma_1)(\beta_1+\gamma_1+\gamma_2) = 0$. Since $\Delta \neq 0$ so $\beta_1 = \gamma_1+\gamma_2$ and we conclude that $b_2 = (\lambda+1)(1+a)+(1+b)+\mu(1+c)$ and $b_3 = \lambda(1+a)+(1+b)+\eta(1+c)$, where $\lambda = \frac{\gamma_1}{\gamma_2}$, $\mu = \frac{\beta_3}{\gamma_2}$ and $\eta = \frac{\gamma_3}{\gamma_2}$. The fact $b_2^2 = b_3^2$ gives that $1+a^2+ab+a^3b = 0$, which is impossible.

In the second case $b_1^2 \equiv b_2^2 \equiv b_3^2 \not\equiv 0 \pmod{A^3(KG)}$ and we can assume that $b_1b_2 \equiv b_2b_1$, $b_1b_3 \equiv b_3b_1 \pmod{A^3(KG)}$ and $b_3b_2 \not\equiv b_2b_3 \pmod{A^3(KG)}$. Since $b_1b_2 \equiv b_2b_1$ and $b_1b_3 \equiv b_3b_1 \pmod{A^3(KG)}$ we have that $\alpha_2\beta_1 = \alpha_1\beta_2$ and $\alpha_2\gamma_1 = \alpha_1\gamma_2$. From the fact that $b_1^2 \equiv b_2^2 \equiv b_3^2 \not\equiv 0 \pmod{A^3(KG)}$ the sixth column asserts that $\alpha_2 = \beta_2 = \gamma_2$ and the second column give that $\alpha_1 = \beta_1 = \gamma_1$, so we conclude that $b_3b_2 \equiv b_2b_3 \pmod{A^3(KG)}$, which is a contradiction. These facts give that KG has no filtered multiplicative basis.

Case 2. Let G be the group

$$G_{24} = \langle \quad a, b, c \quad \mid \quad a^4 = b^4 = c^2 = 1,\ (a,b) = (a,c) = 1,\ (b,c) = a^2 \quad \rangle.$$

Using the identity

$$(1+c)(1+b) \equiv (1+b)(1+c) + (1+a)^2 \qquad \pmod{A^3(KG)},$$

let us compute $b_{i_1}b_{i_2}$ modulo $A^3(KG)$ where $(i_k = 1, 2, 3)$. The results of our computation will be written in a table, consisting of the coefficients of the decomposition $b_{i_1}b_{i_2}$ with respect to the basis

$$\{ \quad (1+a)^{j_1}(1+b)^{j_2}(1+c)^{j_3} \quad | \quad j_1+j_2+j_3=2; \; j_1, j_2 = 0,1,2; \; j_3 = 0,1 \quad \}$$

of the ideal $A^2(KG)$.

	$(1+a)^2$	$(1+a)(1+b)$	$(1+a)(1+c)$	$(1+b)(1+c)$	$(1+b)^2$
b_1b_2	$\alpha_1\beta_1+\alpha_3\beta_2$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_2\beta_2$
b_2b_1	$\alpha_1\beta_1+\alpha_2\beta_3$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_2\beta_2$
b_1b_3	$\alpha_1\gamma_1+\alpha_3\gamma_2$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_2\gamma_2$
b_3b_1	$\alpha_1\gamma_1+\alpha_2\gamma_3$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_2\gamma_2$
b_2b_3	$\beta_1\gamma_1+\beta_3\gamma_2$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_2\gamma_2$
b_3b_2	$\beta_1\gamma_1+\beta_2\gamma_3$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_2\gamma_2$
b_1^2	$\alpha_1^2+\alpha_2\alpha_3$	0	0	0	α_2^2
b_2^2	$\beta_1^2+\beta_2\beta_3$	0	0	0	β_2^2
b_3^2	$\gamma_1^2+\gamma_2\gamma_3$	0	0	0	γ_2^2

It is obvious that the first six lines not equal neither zero nor the last three lines. Since the dimension of $A^2(KG)/A^3(KG)$ equals 5 and KG is not commutative we have either $b_1b_2 \equiv b_2b_1$, $b_1b_3 \not\equiv b_3b_1$, $b_2b_3 \not\equiv b_3b_2 \pmod{A^3(KG)}$ or $b_1b_2 \equiv b_2b_1$, $b_1b_3 \equiv b_3b_1$, $b_2b_3 \not\equiv b_3b_2 \pmod{A^3(KG)}$, because the other cases are analogous to these.

In the first case we get that $b_1^2 \equiv b_2^2 \equiv b_3^2 \equiv 0 \pmod{A^3(KG)}$, so $\Delta = 0$ which is impossible. In the second case consider the following subcases:

a) $b_1^2 \equiv b_2^2 \equiv b_3^2 \not\equiv 0 \pmod{A^3(KG)}$;

b) $b_i^2 \equiv b_j^2 \equiv 0$ and $b_k^2 \not\equiv 0 \pmod{A^3(KG)}$;

c) $b_i^2 \equiv b_j^2 \not\equiv 0$ and $b_k^2 \equiv 0 \pmod{A^3(KG)}$.

Since $\Delta \neq 0$ the subcase $a)$ is impossible. Consider the subcase $b)$, and for example put $b_1^2 \equiv b_2^2 \equiv 0$ and $b_3^2 \not\equiv 0 \pmod{A^3(KG)}$. We get that $\alpha_2 = \beta_2 = 0$ and $\alpha_1 = \beta_1 = 0$ by second and sixth columns, so $\Delta = 0$ which is a contradiction. The other cases also lead to contradictions.

Assume that $b_i^2 \equiv b_j^2 \not\equiv 0$ and $b_k^2 \equiv 0 \pmod{A^3(KG)}$, for instance $b_1^2 \equiv b_2^2 \not\equiv 0$ and $b_3^2 \equiv 0 \pmod{A^3(KG)}$. According to second and sixth columns $\alpha_2 = \beta_2 \neq 0$ and $(\alpha_1+\beta_1)^2 = \alpha_2(\alpha_3+\beta_3)$. Since $b_1b_2 \equiv b_2b_1$ the second column gives that $\alpha_3\beta_2 = \alpha_2\beta_3$, so $\Delta = 0$ which is a contradiction. Thus KG has no a filtered multiplicative basis.

Case 3. Let

$$G = G_{27} = \langle \quad a, b, c \quad | \quad a^2 = b^2 = c^2 = 1, \; (a,c) = d, \; (b,c) = e,$$
$$(a,b) = (a,d) = (a,e) = (b,d) = (b,e) = (c,d) = (c,e) = (d,e) = 1 \quad \rangle.$$

Since

$$M_1(G) = G, \quad M_2(G) = \langle d, e \rangle, \quad M_3(G) = \langle 1 \rangle$$

we obtained that $\mu(d) = \mu(e) = 2$. Let us compute $b_{i_1} b_{i_2}$ modulo $A^3(KG)$ where $(i_k = 1, 2, 3)$. The results of our computation will be also written in a table, consisting of the coefficients of the decomposition $b_{i_1} b_{i_2}$ with respect to the basis

$$\{ \quad (1+a)^{j_1}(1+b)^{j_2}(1+c)^{j_3}(1+d)^{j_4}(1+e)^{j_5} \quad | \quad j_1 + j_2 + j_3 + 2j_4 + 2j_5 = 2; \quad j_1, j_2, j_3 = 0, 1;\ j_4, j_5 = 0, 1 \quad \}$$

of the ideal $A^2(KG)$. Using the identities:

$$(1+c)(1+a) \equiv (1+a)(1+c) + (1+d) \pmod{A^3(KG)};$$
$$(1+c)(1+b) \equiv (1+b)(1+c) + (1+e) \pmod{A^3(KG)},$$

we get

	$(1+a)(1+b)$	$(1+a)(1+c)$	$(1+b)(1+c)$	$(1+d)$	$(1+e)$
b_1b_2	$\alpha_1\beta_2 + \alpha_2\beta_1$	$\alpha_1\beta_3 + \alpha_3\beta_1$	$\alpha_2\beta_3 + \alpha_3\beta_2$	$\alpha_3\beta_1$	$\alpha_3\beta_2$
b_2b_1	$\alpha_1\beta_2 + \alpha_2\beta_1$	$\alpha_1\beta_3 + \alpha_3\beta_1$	$\alpha_2\beta_3 + \alpha_3\beta_2$	$\alpha_1\beta_3$	$\alpha_2\beta_3$
b_1b_3	$\alpha_1\gamma_2 + \alpha_2\gamma_1$	$\alpha_1\gamma_3 + \alpha_3\gamma_1$	$\alpha_2\gamma_3 + \alpha_3\gamma_2$	$\alpha_3\gamma_1$	$\alpha_3\gamma_2$
b_3b_1	$\alpha_1\gamma_2 + \alpha_2\gamma_1$	$\alpha_1\gamma_3 + \alpha_3\gamma_1$	$\alpha_2\gamma_3 + \alpha_3\gamma_2$	$\alpha_1\gamma_3$	$\alpha_2\gamma_3$
b_2b_3	$\beta_1\gamma_2 + \beta_2\gamma_1$	$\beta_1\gamma_3 + \beta_3\gamma_1$	$\beta_2\gamma_3 + \beta_3\gamma_2$	$\beta_3\gamma_1$	$\beta_3\gamma_2$
b_3b_2	$\beta_1\gamma_2 + \beta_2\gamma_1$	$\beta_1\gamma_3 + \beta_3\gamma_1$	$\beta_2\gamma_3 + \beta_3\gamma_2$	$\beta_1\gamma_3$	$\beta_2\gamma_3$
b_1^2	0	0	0	$\alpha_1\alpha_3$	$\alpha_2\alpha_3$
b_2^2	0	0	0	$\beta_1\beta_3$	$\beta_2\beta_3$
b_3^2	0	0	0	$\gamma_1\gamma_3$	$\gamma_2\gamma_3$

It is easy to see that the first six lines not equal neither zero nor the last three lines. Since the dimension of $A^2(KG)/A^3(KG)$ equals 5 and KG is not commutative we have either $b_1b_2 \equiv b_2b_1$, $b_1b_3 \not\equiv b_3b_1$, $b_2b_3 \not\equiv b_3b_2 \pmod{A^3(KG)}$ or $b_1b_2 \equiv b_2b_1$, $b_1b_3 \equiv b_3b_1$, $b_2b_3 \not\equiv b_3b_2 \pmod{A^3(KG)}$, because the other cases are similar to these.

In the first case we get that $b_1^2 \equiv b_2^2 \equiv b_3^2 \equiv 0 \pmod{A^3(KG)}$ and $\alpha_3 = \beta_3 = \gamma_1 = \gamma_2 = 0$. Let us compute $b_{i_1} b_{i_2} b_{i_3}$ modulo $A^4(KG)$ where $(i_k = 1, 2, 3)$. Since the dimension of $A^3(KG)/A^4(KG)$ equal to 7 but we have got 8 different elements, this case is impossible. In the second case $b_1b_2 \equiv b_2b_1$ and $b_1b_3 \equiv b_3b_1 \pmod{A^3(KG)}$. Assume that $\alpha_3 = 0$. Fifth and sixth columns give that $\beta_3 = \gamma_3 = 0$ which is impossible, so $\alpha_3, \beta_3, \gamma_3 \neq 0$. These columns gives that $b_2 \equiv \beta_3\alpha_3^{-1} b_1 \pmod{A^2(KG)}$ which is a contradiction, therefore KG has no a filtered multiplicative basis.

Case 4. Let G be one of the following groups:

$$G_{28} = \langle \quad a, b, c \quad | \quad a^4 = b^2 = c^2 = 1,\ (a, c) = a^2,\ (b, c) = d, \quad (a, b) = (a, d) = (b, d) = (c, d) = 1 \quad \rangle;$$

$$G_{29} = \langle \quad a, b, c \quad | \quad a^4 = b^2 = 1,\ a^2 = c^2,\ (a, c) = a^2,\ (b, c) = d, \quad (a, b) = (a, d) = (b, d) = (c, d) = 1 \quad \rangle;$$

$$G_{30} = \langle \quad a, b, c \quad | \quad a^4 = b^2 = c^2 = 1,\ (a, c) = d,\ (b, c) = a^2, \quad (a, b) = (a, d) = (b, d) = (c, d) = 1 \quad \rangle.$$

If G is either G_{28} or G_{29} then we have

$$(1+c)(1+a) \equiv (1+a)(1+c) + (1+a)^2 \pmod{A^3(KG)};$$
$$(1+c)(1+b) \equiv (1+b)(1+c) + (1+d) \pmod{A^3(KG)}.$$

If $G = G_{30}$ then we have

$$(1+c)(1+a) \equiv (1+a)(1+c) + (1+d) \pmod{A^3(KG)};$$
$$(1+c)(1+b) \equiv (1+b)(1+c) + (1+a)^2 \pmod{A^3(KG)}.$$

Using the last four identities let us compute $b_{i_1}b_{i_2}$ modulo $A^3(KG)$ where $(i_k = 1,2,3)$. The results of our computation will be written in a table as above, consisting of the coefficients of the decomposition $b_{i_1}b_{i_2}$ with respect to the basis

$$\{ \quad (1+a)^{j_1}(1+b)^{j_2}(1+c)^{j_3}(1+d)^{j_4} \quad | \quad j_1+j_2+j_3+2j_4 = 2; \quad j_1,j_2,j_3 = 0,1,2;\ j_4 = 0,1 \quad \}$$

of the ideal $A^3(KG)$:

	$(1+a)^2$	$(1+a)(1+b)$	$(1+a)(1+c)$	$(1+b)(1+c)$	$(1+d)$
b_1b_2	$\alpha_1\beta_1 + \Delta(\alpha,\beta)$	$\alpha_1\beta_2 + \alpha_2\beta_1$	$\alpha_1\beta_3 + \alpha_3\beta_1$	$\alpha_2\beta_3 + \alpha_3\beta_2$	$\Omega(\alpha,\beta)$
b_2b_1	$\alpha_1\beta_1 + \Delta(\beta,\alpha)$	$\alpha_1\beta_2 + \alpha_2\beta_1$	$\alpha_1\beta_3 + \alpha_3\beta_1$	$\alpha_2\beta_3 + \alpha_3\beta_2$	$\Omega(\beta,\alpha)$
b_1b_3	$\alpha_1\gamma_1 + \Delta(\alpha,\gamma)$	$\alpha_1\gamma_2 + \alpha_2\gamma_1$	$\alpha_1\gamma_3 + \alpha_3\gamma_1$	$\alpha_2\gamma_3 + \alpha_3\gamma_2$	$\Omega(\alpha,\gamma)$
b_3b_1	$\alpha_1\gamma_1 + \Delta(\gamma,\alpha)$	$\alpha_1\gamma_2 + \alpha_2\gamma_1$	$\alpha_1\gamma_3 + \alpha_3\gamma_1$	$\alpha_2\gamma_3 + \alpha_3\gamma_2$	$\Omega(\gamma,\alpha)$
b_2b_3	$\beta_1\gamma_1 + \Delta(\beta,\gamma)$	$\beta_1\gamma_2 + \beta_2\gamma_1$	$\beta_1\gamma_3 + \beta_3\gamma_1$	$\beta_2\gamma_3 + \beta_3\gamma_2$	$\Omega(\beta,\gamma)$
b_3b_2	$\beta_1\gamma_1 + \Delta(\gamma,\beta)$	$\beta_1\gamma_2 + \beta_2\gamma_1$	$\beta_1\gamma_3 + \beta_3\gamma_1$	$\beta_2\gamma_3 + \beta_3\gamma_2$	$\Omega(\gamma,\beta)$
b_1^2	$\alpha_1^2 + \Delta(\alpha,\alpha)$	0	0	0	$\Omega(\alpha,\alpha)$
b_2^2	$\beta_1^2 + \Delta(\beta,\beta)$	0	0	0	$\Omega(\beta,\beta)$
b_3^2	$\gamma_1^2 + \Delta(\gamma,\gamma)$	0	0	0	$\Omega(\gamma,\gamma)$

where if $G = G_{28}$ then $\Delta(\delta,\epsilon) = \delta_3\epsilon_1$, $\Omega(\delta,\epsilon) = \delta_3\epsilon_2$, if $G = G_{29}$ then $\Delta(\delta,\epsilon) = \delta_3\epsilon_1 + \delta_3\epsilon_3$, $\Omega(\delta,\epsilon) = \delta_3\epsilon_2$ and if $G = G_{30}$ then $\Delta(\delta,\epsilon) = \delta_3\epsilon_2$, $\Omega(\delta,\epsilon) = \delta_3\epsilon_1$.

It is clearly that the first six lines not equal neither zero nor the last three lines. Since the dimension of $A^2(KG)/A^3(KG)$ equal to 5 and KG is not commutative we have either $b_1b_2 \equiv b_2b_1$, $b_1b_3 \not\equiv b_3b_1$, $b_2b_3 \not\equiv b_3b_2 \pmod{A^3(KG)}$ or $b_1b_2 \equiv b_2b_1$, $b_1b_3 \equiv b_3b_1$, $b_2b_3 \not\equiv b_3b_2 \pmod{A^3(KG)}$, because the other cases are analogous to these.

In the first case we get that $b_1^2 \equiv b_2^2 \equiv b_3^2 \equiv 0 \pmod{A^3(KG)}$, so $\Delta = 0$ which is impossible. In the second case $b_1b_2 \equiv b_2b_1$ and $b_1b_3 \equiv b_3b_1 \pmod{A^3(KG)}$. Assume that $\alpha_3 = 0$. Second and sixth columns give that $\beta_3 = \gamma_3 = 0$ which is impossible, so $\alpha_3, \beta_3, \gamma_3 \neq 0$. Consequences of columns 2 and 6 are that $b_2 \equiv \beta_3\alpha_3^{-1}b_1 \pmod{A^2(KG)}$ which is a contradiction. Thus these group algebras have no filtered multiplicative bases.

Case 5. Let G one of the following groups:

$$G_{31}=\langle\ a,b,c\ \mid\ a^4=b^4=c^2=1,\ (b,c)=a^2b^2,\ (a,c)=a^2,\ (a,b)=1\ \rangle;$$
$$G_{32}=\langle\ a,b,c\ \mid\ a^4=b^4=1,\ c^2=a^2b^2,\ (b,c)=a^2b^2,\ (a,c)=a^2,\ (a,b)=1\ \rangle;$$
$$G_{33}=\langle\ a,b,c\ \mid\ a^4=b^4=c^2=1,\ (b,c)=a^2,\ (a,c)=a^2b^2,\ (a,b)=1\ \rangle;$$
$$G_{34}=\langle\ a,b,c\ \mid\ a^4=b^4=c^2=1,\ (b,c)=b^2,\ (a,c)=a^2,\ (a,b)=1\ \rangle;$$
$$G_{35}=\langle\ a,b,c\ \mid\ a^4=b^4=1,\ c^2=a^2,\ (b,c)=b^2,\ (a,c)=a^2,\ (a,b)=1\ \rangle.$$

Let us compute $b_{i_1}b_{i_2}$ modulo $A^3(KG)$, $(i_k=1,2,3)$. The results of our computations will be written in a table, as before, with respect to the basis

$$\{\ (1+a)^{j_1}(1+b)^{j_2}(1+c)^{j_3}\ \mid\ j_1+j_2+j_3=2;\ j_1,j_2,j_3=0,1,2\ \}$$

of the ideal $A^3(KG)$. If $G=G_{31}$ then

$$(1+c)(1+a)\equiv(1+a)(1+c)+(1+a)^2 \pmod{A^3(KG)};$$
$$(1+c)(1+b)\equiv(1+b)(1+c)+(1+a)^2+(1+b)^2 \pmod{A^3(KG)},$$

if $G=G_{32}$ then

$$(1+c)(1+a)\equiv(1+a)(1+c)+(1+a)^2 \pmod{A^3(KG)};$$
$$(1+c)(1+b)\equiv(1+b)(1+c)+(1+a)^2+(1+b)^2 \pmod{A^3(KG)};$$
$$(1+c)^2\equiv(1+a)^2+(1+b)^2 \pmod{A^3(KG)},$$

if $G=G_{33}$ then

$$(1+c)(1+a)\equiv(1+a)(1+c)+(1+a)^2+(1+b)^2 \pmod{A^3(KG)};$$
$$(1+c)(1+b)\equiv(1+b)(1+c)+(1+a)^2 \pmod{A^3(KG)},$$

if $G=G_{34}$ then

$$(1+c)(1+a)\equiv(1+a)(1+c)+(1+a)^2 \pmod{A^3(KG)};$$
$$(1+c)(1+b)\equiv(1+b)(1+c)+(1+b)^2 \pmod{A^3(KG)},$$

if $G=G_{35}$ then

$$(1+c)(1+a)\equiv(1+a)(1+c)+(1+a)^2 \pmod{A^3(KG)};$$
$$(1+c)(1+b)\equiv(1+b)(1+c)+(1+b)^2 \pmod{A^3(KG)};$$
$$(1+c)^2\equiv(1+a)^2.$$

Using the last 12 identities we get

	$(1+a)^2$	$(1+a)(1+b)$	$(1+a)(1+c)$	$(1+b)(1+c)$	$(1+b)^2$
b_1b_2	$\alpha_1\beta_1+\Delta(\alpha,\beta)$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_2\beta_2+\Omega(\alpha,\beta)$
b_2b_1	$\alpha_1\beta_1+\Delta(\beta,\alpha)$	$\alpha_1\beta_2+\alpha_2\beta_1$	$\alpha_1\beta_3+\alpha_3\beta_1$	$\alpha_2\beta_3+\alpha_3\beta_2$	$\alpha_2\beta_2+\Omega(\beta,\alpha)$
b_1b_3	$\alpha_1\gamma_1+\Delta(\gamma,\alpha)$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_2\gamma_2+\Omega(\gamma,\alpha)$
b_3b_1	$\alpha_1\gamma_1+\Delta(\alpha,\gamma)$	$\alpha_1\gamma_2+\alpha_2\gamma_1$	$\alpha_1\gamma_3+\alpha_3\gamma_1$	$\alpha_2\gamma_3+\alpha_3\gamma_2$	$\alpha_2\gamma_2+\Omega(\alpha,\gamma)$
b_2b_3	$\beta_1\gamma_1+\Delta(\gamma,\beta)$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_2\gamma_2+\Omega(\gamma,\beta)$
b_3b_2	$\beta_1\gamma_1+\Delta(\beta,\gamma)$	$\beta_1\gamma_2+\beta_2\gamma_1$	$\beta_1\gamma_3+\beta_3\gamma_1$	$\beta_2\gamma_3+\beta_3\gamma_2$	$\beta_2\gamma_2+\Omega(\beta,\gamma)$
b_1^2	$\alpha_1^2+\Delta(\alpha,\alpha)$	0	0	0	$\alpha_2^2+\Omega(\alpha,\alpha)$
b_2^2	$\beta_1^2+\Delta(\beta,\beta)$	0	0	0	$\beta_2^2+\Omega(\beta,\beta)$
b_3^2	$\gamma_1^2+\Delta(\gamma,\gamma)$	0	0	0	$\gamma_2^2+\Omega(\gamma,\gamma)$

where $\Delta(\delta,\epsilon)$ and $\Omega(\delta,\epsilon)$ is the following:

	$\Delta(\delta,\epsilon)$	$\Omega(\delta,\epsilon)$
G_{31}	$\delta_3\epsilon_1+\delta_3\epsilon_2$	$\delta_3\epsilon_2$
G_{32}	$\delta_3\epsilon_1+\delta_3\epsilon_2+\delta_3\epsilon_3$	$\delta_3\epsilon_2+\delta_3\epsilon_3$
G_{33}	$\delta_3\epsilon_1+\delta_3\epsilon_2$	$\delta_3\epsilon_1$
G_{34}	$\delta_3\epsilon_1$	$\delta_3\epsilon_2$
G_{35}	$\delta_3\epsilon_1+\delta_3\epsilon_3$	$\delta_3\epsilon_2$

Evidently the first six lines not equal neither zero nor the last three lines. Since the dimension of $A^2(KG)/A^3(KG)$ equal to 5 and KG is not commutative, we have either $b_1b_2\equiv b_2b_1$, $b_1b_3\not\equiv b_3b_1$, $b_2b_3\not\equiv b_3b_2 \pmod{A^3(KG)}$ or $b_1b_2\equiv b_2b_1$, $b_1b_3\equiv b_3b_1$, $b_2b_3\not\equiv b_3b_2 \pmod{A^3(KG)}$, because the other cases are similar to these.

In both of cases we can see that $b_1b_2\equiv b_2b_1 \pmod{A^3(KG)}$. Assume that $\alpha_3=0$. Second and sixth columns give that $\beta_3=\gamma_3=0$ which is impossible, so $\alpha_3,\beta_3,\gamma_3$ are not zero. Columns 2 and 6 imply b_2 depends on b_1 modulo $A^2(KG)$ which is a contradiction so these group algebras have no filtered multiplicative bases.

Case 6. Let $G=G_{49}$ be and put $u\equiv(1+a)+(1+c)$, $v\equiv(1+b)+(1+d)$, $w\equiv(1+b)+(1+c)+(1+d)$ and $z\equiv(1+a)+(1+b)+(1+c) \pmod{A^2(KG)}$.

Using the identities:

$$(1+b)(1+a)\equiv(1+a)(1+b)+(1+a)^2 \pmod{A^3(KG)};$$
$$(1+d)(1+c)\equiv(1+c)(1+d)+(1+a)^2 \pmod{A^3(KG)};$$
$$(1+a)^2\equiv(1+b)^2\equiv(1+c)^2\equiv(1+d)^2 \pmod{A^3(KG)};$$

we get that

- $\{uv, uw, uz, zu, vw, vz, wz\}$ is a basis of $A^2(KG)/A^3(KG)$;
- $\{uzu, uvw, vzu, wzu, vuz, uzw, vwz, zuz\}$ is a basis for $A^3(KG)/A^4(KG)$;
- $\{vuzu, wuzu, zuzu, vzuz, wzuz, uvwz, vwzu\}$ is a basis of $A^4(KG)/A^5(KG)$;
- $\{vzuz, wzuz, vwuzu, vwzuz\}$ is a basis of $A^5(KG)/A^6(KG)$,

and the element $vwzuzu$ is a basis for $A^6(KG)$.

Case 7. Let

$$G = G_{50} = \langle \quad a, b, c, d \quad | \quad a^4 = b^2 = c^2 = d^4 = 1,\ a^2 = d^2,$$
$$(a, d) = (b, c) = (c, d) = a^2,\ (a, b) = (a, c) = (b, d) = 1 \quad \rangle.$$

Using the identities:

$$(1 + d)(1 + a) \equiv (1 + a)(1 + d) + (1 + a)^2 \pmod{A^3(KG)};$$
$$(1 + c)(1 + b) \equiv (1 + b)(1 + c) + (1 + a)^2 \pmod{A^3(KG)};$$
$$(1 + d)(1 + c) \equiv (1 + c)(1 + d) + (1 + a)^2 \pmod{A^3(KG)};$$
$$(1 + a)^2 \equiv (1 + d)^2 \pmod{A^3(KG)},$$

let us compute $b_{i_1} b_{i_2}$ modulo $A^3(KG)$ where $i_k = 1, 2, 3, 4\}$. The results of our computations we shall write in a table, similar to previous cases with respect to the basis

$$\{ \quad (1 + a)^{j_1}(1 + b)^{j_2}(1 + c)^{j_3}(1 + d)^{j_4} \quad | \quad j_1 + j_2 + j_3 + j_4 = 2;$$
$$j_1 = 0, 1, 2;\ j_2, j_3, j_4 = 0, 1 \quad \}$$

of the ideal $A^2(KG)$:

	$(1+a)(1+b)$	$(1+a)(1+c)$	$(1+a)(1+d)$	$(1+b)(1+c)$	$(1+b)(1+d)$	$(1+c)(1+d)$	$(1+a)^2$
b_1b_2	$\Delta^{1,2}(\alpha,\beta)$	$\Delta^{1,3}(\alpha,\beta)$	$\Delta^{1,4}(\alpha,\beta)$	$\Delta^{2,3}(\alpha,\beta)$	$\Delta^{2,4}(\alpha,\beta)$	$\Delta^{3,4}(\alpha,\beta)$	$\Omega_{\alpha,\beta}+\alpha_3\beta_2+\alpha_4\beta_1+\alpha_4\beta_3$
b_2b_1	$\Delta^{1,2}(\alpha,\beta)$	$\Delta^{1,3}(\alpha,\beta)$	$\Delta^{1,4}(\alpha,\beta)$	$\Delta^{2,3}(\alpha,\beta)$	$\Delta^{2,4}(\alpha,\beta)$	$\Delta^{3,4}(\alpha,\beta)$	$\Omega_{\alpha,\beta}+\alpha_2\beta_3+\alpha_1\beta_4+\alpha_3\beta_4$
b_1b_3	$\Delta^{1,2}(\alpha,\gamma)$	$\Delta^{1,3}(\alpha,\gamma)$	$\Delta^{1,4}(\alpha,\gamma)$	$\Delta^{2,3}(\alpha,\gamma)$	$\Delta^{2,4}(\alpha,\gamma)$	$\Delta^{3,4}(\alpha,\gamma)$	$\Omega_{\alpha,\gamma}+\alpha_3\gamma_2+\alpha_4\gamma_1+\alpha_4\gamma_3$
b_3b_1	$\Delta^{1,2}(\alpha,\gamma)$	$\Delta^{1,3}(\alpha,\gamma)$	$\Delta^{1,4}(\alpha,\gamma)$	$\Delta^{2,3}(\alpha,\gamma)$	$\Delta^{2,4}(\alpha,\gamma)$	$\Delta^{3,4}(\alpha,\gamma)$	$\Omega_{\alpha,\gamma}+\alpha_2\gamma_3+\alpha_1\gamma_4+\alpha_3\gamma_4$
b_1b_4	$\Delta^{1,2}(\alpha,\delta)$	$\Delta^{1,3}(\alpha,\delta)$	$\Delta^{1,4}(\alpha,\delta)$	$\Delta^{2,3}(\alpha,\delta)$	$\Delta^{2,4}(\alpha,\delta)$	$\Delta^{3,4}(\alpha,\delta)$	$\Omega_{\alpha,\delta}+\alpha_3\delta_2+\alpha_4\delta_1+\alpha_4\delta_3$
b_4b_1	$\Delta^{1,2}(\alpha,\delta)$	$\Delta^{1,3}(\alpha,\delta)$	$\Delta^{1,4}(\alpha,\delta)$	$\Delta^{2,3}(\alpha,\delta)$	$\Delta^{2,4}(\alpha,\delta)$	$\Delta^{3,4}(\alpha,\delta)$	$\Omega_{\alpha,\delta}+\alpha_2\delta_3+\alpha_1\delta_4+\alpha_3\delta_4$
b_2b_3	$\Delta^{1,2}(\beta,\gamma)$	$\Delta^{1,3}(\beta,\gamma)$	$\Delta^{1,4}(\beta,\gamma)$	$\Delta^{2,3}(\beta,\gamma)$	$\Delta^{2,4}(\beta,\gamma)$	$\Delta^{3,4}(\beta,\gamma)$	$\Omega_{\beta,\gamma}+\beta_3\gamma_2+\beta_4\gamma_1+\beta_4\gamma_3$
b_3b_2	$\Delta^{1,2}(\beta,\gamma)$	$\Delta^{1,3}(\beta,\gamma)$	$\Delta^{1,4}(\beta,\gamma)$	$\Delta^{2,3}(\beta,\gamma)$	$\Delta^{2,4}(\beta,\gamma)$	$\Delta^{3,4}(\beta,\gamma)$	$\Omega_{\beta,\gamma}+\beta_2\gamma_3+\beta_1\gamma_4+\beta_3\gamma_4$
b_2b_4	$\Delta^{1,2}(\beta,\delta)$	$\Delta^{1,3}(\beta,\delta)$	$\Delta^{1,4}(\beta,\delta)$	$\Delta^{2,3}(\beta,\delta)$	$\Delta^{2,4}(\beta,\delta)$	$\Delta^{3,4}(\beta,\delta)$	$\Omega_{\beta,\delta}+\beta_3\delta_2+\beta_4\delta_1+\beta_4\delta_3$
b_4b_2	$\Delta^{1,2}(\beta,\delta)$	$\Delta^{1,3}(\beta,\delta)$	$\Delta^{1,4}(\beta,\delta)$	$\Delta^{2,3}(\beta,\delta)$	$\Delta^{2,4}(\beta,\delta)$	$\Delta^{3,4}(\beta,\delta)$	$\Omega_{\beta,\delta}+\beta_2\delta_3+\beta_1\delta_4+\beta_3\delta_4$
b_3b_4	$\Delta^{1,2}(\gamma,\delta)$	$\Delta^{1,3}(\gamma,\delta)$	$\Delta^{1,4}(\gamma,\delta)$	$\Delta^{2,3}(\gamma,\delta)$	$\Delta^{2,4}(\gamma,\delta)$	$\Delta^{3,4}(\gamma,\delta)$	$\Omega_{\gamma,\delta}+\gamma_3\delta_2+\gamma_4\delta_1+\gamma_4\delta_3$
b_4b_3	$\Delta^{1,2}(\gamma,\delta)$	$\Delta^{1,3}(\gamma,\delta)$	$\Delta^{1,4}(\gamma,\delta)$	$\Delta^{2,3}(\gamma,\delta)$	$\Delta^{2,4}(\gamma,\delta)$	$\Delta^{3,4}(\gamma,\delta)$	$\Omega_{\gamma,\delta}+\gamma_2\delta_3+\gamma_1\delta_4+\gamma_3\delta_4$
b_1^2	0	0	0	0	0	0	$\Omega_{\alpha,\alpha}+\alpha_2\alpha_3+\alpha_1\alpha_4+\alpha_3\alpha_4$
b_2^2	0	0	0	0	0	0	$\Omega_{\beta,\beta}+\beta_2\beta_3+\beta_1\beta_4+\beta_3\beta_4$
b_3^2	0	0	0	0	0	0	$\Omega_{\gamma,\gamma}+\gamma_2\gamma_3+\gamma_1\gamma_4+\gamma_3\gamma_4$
b_4^2	0	0	0	0	0	0	$\Omega_{\delta,\delta}+\delta_2\delta_3+\delta_1\delta_4+\delta_3\delta_4$

where $\Omega_{\alpha,\beta} = \alpha_1\beta_1 + \alpha_4\beta_4$ and $\Delta^{i,j}(\alpha, \beta) = \alpha_i\beta_j + \alpha_j\beta_i$.

It is easy to see that the first twelve lines not equal neither zero nor the last four lines.

Since $\Delta^{i,j}(\varepsilon, \eta)$ is a subdeterminant of Δ and $\Delta \neq 0$, by expansion theorem of determinant b_ib_j cannot be equivalent other else $b_kb_l \pmod{A^3(KG)}$ apart from the case when $k = j$ and $l = i$.

Assume that $\{u \equiv b_1, v \equiv b_2, w \equiv b_3, z \equiv b_4 \pmod{A^2(KG)}\}$ and the coefficients of u, v, w, z will be denoted by $\alpha_i, \beta_i, \gamma_i, \delta_i$, respectively.

Since the dimension of $A^2(KG)/A^3(KG)$ is equal to 7 and this group algebra is not commutative, we have that

$$\begin{aligned} uv \equiv vu, \quad uw \equiv wu, \quad uz \equiv zu, \quad vw \equiv wv, \quad vz \equiv zv, \\ wz \not\equiv zw, \quad u^2 \equiv v^2 \equiv w^2 \equiv z^2 \equiv 0 \pmod{A^3(KG)}, \end{aligned} \tag{6}$$

and the other cases are analogous to this one.

Assume that KG has a filtered multiplicative basis and $\{u, v, w, z\}$ form a basis of $A(KG)/A^2(KG)$, satisfies (6) and

$$\begin{aligned} &\{uv, uw, uz, vw, vz, wz, zw\} \quad \text{is a basis for} \quad A^2(KG)/A^3(KG);\\ &\{uvw, uvz, uwz, uzw, vwz, vzw, wzw, zwz\} \quad \text{is a basis of} \quad A^3(KG)/A^4(KG);\\ &\{uvwz, uvzw, uwzw, uzwz, vwzw, vzwz, wzwz\} \text{ is a basis of } A^4(KG)/A^5(KG);\\ &\{uvwzw, uvzwz, uwzwz, vwzwz\} \quad \text{is a basis for} \quad A^5(KG)/A^6(KG);\\ &\{uvwzwz\} \quad \text{is a basis for} \quad A^6(KG). \end{aligned} \tag{7}$$

Suppose that $\alpha_4 = 0$ and there exists $b \in \{v, w, z\}$ such that b is congruent with $\varepsilon_1(1+a) + \varepsilon_2(1+b) + \varepsilon_3(1+c) + \varepsilon_4(1+d) \pmod{A^2(KG)}$ and $\varepsilon_4 = 0$. The facts $u^2 \equiv b^2 \equiv 0$ and $ub \equiv bu \pmod{A^3(KG)}$ give that $\alpha_3\varepsilon_2 + \alpha_2\varepsilon_3 = 0$ and $\alpha_1^2 + \alpha_2\alpha_3 = \varepsilon_1^2 + \varepsilon_2\varepsilon_3 = 0$. It is very simple to prove that either $b_1 \equiv 0$ or $b_1 \equiv b_i \pmod{A^2(KG)}$, which is impossible.

Now, we shall consider two subcases.

Subcase 1. Suppose that $\alpha_4 = 0$ and $\beta_4 = \gamma_4 = 1$. For $\alpha_2 = 0$ it follows that $\Delta = 0$, so we can also assume that $\alpha_2 = 1$. According to eighth column of the previous table

$$\begin{aligned} \alpha_1^2 + \alpha_3 &= 0;\\ \beta_1 + \beta_3 + 1 &= \beta_1^2 + \beta_2\beta_3;\\ \gamma_1 + \gamma_3 + 1 &= \gamma_1^2 + \gamma_2\gamma_3; \end{aligned} \tag{8}$$

Since $vw \equiv wv \pmod{A^3(KG)}$ we get $\beta_3\gamma_2 + \gamma_1 + \gamma_3 + 1 = \beta_2\beta_3 + \beta_1 + \beta_3 + 1$ and using (8) it follows that

$$(\beta_1 + \gamma_1)^2 = (\beta_2 + \gamma_2)(\beta_3 + \gamma_3). \tag{9}$$

Also eighth column of the previous table and $uv \equiv vu$, $uw \equiv wu \pmod{A^3(KG)}$ give that $\alpha_1^2\beta_2 + \beta_3 = \alpha_1^2\gamma_2 + \gamma_3$, so

$$\alpha_1^2(\beta_2 + \gamma_2) = \beta_3 + \gamma_3. \tag{10}$$

Thus (9) and (10) give the equation $\beta_1 + \gamma_1 = \alpha_1(\beta_2 + \gamma_2)$.

Since $\alpha_1^2 = \alpha_3$ we have established $v + w \equiv (\beta_2 + \gamma_2)u \pmod{A^3(KG)}$ which is a contradiction.

Subcase 2. Suppose that $\alpha_4, \beta_4 \neq 0$ and without loss of generality we can assume that $\alpha_4 = \beta_4 = 1$. Simple computations show that $\{u \equiv b_1 + b_2, v \equiv b_2, w \equiv b_3, z \equiv b_4 \pmod{A^2(KG)}\}$ form a basis of $A(KG)/A^2(KG)$, satisfies conditions (6) and (7), but it is a contradiction to subcase 1, so this group algebra has no filtered multiplicative basis. This completes the proof of the theorem.

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