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STARLIKE AND CONVEX FUNCTIONS WITH RESPECT TO CONJUGATE POINTS

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Abstract. An analytic functions $f(z)$ defined on $\triangle = \{z : |z| < 1\}$ and normalized by $f(0) = 0,\ f'(0) = 1$ is starlike with respect to conjugate points if $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)+\overline{f}(\overline{z})}\right\} > 0,\ z \in \triangle$. We obtain some convolution conditions, growth and distortion estimates of functions in this and related classes.

1. Introduction

Let $\mathcal{A}$ denote the class of all analytic functions defined in the unit disk

$$\triangle = \{z : |z| < 1\}$$

and normalized by $f(0) = 0 = f'(0) - 1$. Let $S^*(\alpha), C(\alpha)$ and $K(\alpha)$ denote the classes of starlike, convex and close to convex functions of order α, $0 \leq \alpha < 1$, respectively. A function $f \in \mathcal{A}$ is starlike with respect to symmetric points in $\triangle$ if for every r close to 1, $r < 1$ and every z_0 on $|z| = r$ the angular velocity of $f(z)$ about $f(-z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. This class was introduced and studied by Sakaguchi[7]. He proved that the condition is equivalent to

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0,\ z \in \triangle.$$

A function $f \in \mathcal{A}$ is starlike with respect to conjugate points in $\triangle$ if f satisfies the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) + \overline{f}(\overline{z})}\right\} > 0,\ z \in \triangle.$$

A function $f \in \mathcal{A}$ is starlike with respect to symmetric conjugate points in $\triangle$ if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z) - \overline{f}(-\overline{z})}\right\} > 0,\ z \in \triangle.$$

Denote the classes consisting of these functions by S^*_c and S^*_{sc} respectively. These classes were introduced by El-Ashwah and Thomas[1]. The functions in these classes are close to convex and hence univalent. Sokol [11] introduced two more parameter in this class and obtained structural formula, the coefficient estimate, the radius of convexity and results about the neighborhoods of functions. See also Sokol [12].

If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, then the convolution of $f(z)$ and $g(z)$, denoted by $(f * g)(z)$, is the analytic function given by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

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The function $f(z)$ is subordinate to $F(z)$ in the disk Δ if there exits an analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = F(w(z))$ for $|z| < 1$. This is written as $f(z) \prec F(z)$. Notice that $f \in S^*(\alpha)$ if and only if

$$zf'(z)/f(z) \prec (1 + (1 - 2\alpha)z)/(1 - z)$$

and $f \in C(\alpha)$ if and only if $f * g \in S^*(\alpha)$ where $g(z) = z/(1 - z)^2$. This enables to obtain results about the convex class from the corresponding result of starlike class. Let $h(z)$ be analytic and $h(0) = 1$. A function $f \in \mathcal{A}$ is in the class $S^*(h)$ if

$$\frac{zf'(z)}{f(z)} \prec h(z), \ z \in \triangle.$$

The class $S^*(h)$ and a corresponding convex class $C(h)$ was defined by Ma and Minda[3]. But results about the convex class can be obtained easily from the corresponding result of functions in $S^*(h)$.

If $\phi(z) = (1 + z)/(1 - z)$, then the classes reduce to the usual classes of starlike and convex functions. If $\phi(z) = (1 + (1 - 2\alpha)z)/(1 - z)$, $0 \leq \alpha < 1$, then the classes reduce to the usual classes of starlike and convex functions of order α. If $\phi(z) = [(1 + z)/(1 - z)]^\alpha$, $0 < \alpha \leq 1$, then the classes reduce to the classes of strongly starlike and convex functions of order α. If $\phi(z) = (1 + Az)/(1 + Bz)$, $-1 \leq B < A \leq 1$, then the classes reduce to the classes $S^*[A, B]$ and $C[A, B]$.

Definition 1. A function $f \in \mathcal{A}$ is in the class $S_s^*(\phi)$ if

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \phi(z), \ z \in \triangle,$$

and is in the class $C_s(\phi)$ if

$$\frac{2(zf'(z))'}{f'(z) + f'(-z)} \prec \phi(z), \ z \in \triangle.$$

Let $S_c^*(\phi)$, $S_{sc}^*(\phi)$ denote the corresponding classes of starlike functions with respect to conjugate points and symmetric conjugate points respectively.

The functions $k_{\phi n}$ $(n = 2, 3, \ldots)$ defined by $k_{\phi n}(0) = k'_{\phi n}(0) - 1 = 0$ and

$$1 + \frac{zk''_{\phi n}(z)}{k'_{\phi n}(z)} = \phi(z^{n-1})$$

are examples of functions in $C(\phi)$. The functions $h_{\phi n}$ satisfying $zk'_{\phi n}(z) = h_{\phi n}$ are examples of functions in $S^*(\phi)$. The odd functions in $S^*(\phi)$ $(C(\phi))$ are in the class $S_s^*(\phi)$ $(C_s(\phi))$. The function with real coefficient belonging to $S^*(\phi)$ $(C(\phi))$ are in the class $S_c^*(\phi)$ $(C_c(\phi))$. Similarly, the odd function with real coefficient belonging to $S^*(\phi)$ $(C(\phi))$ are in the class $S_{sc}^*(\phi)$ $(C_{sc}(\phi))$.

In this paper, we obtain convolution conditions, growth and distortion inequalities for functions in our classes. Also we prove a convolution result.

2. Convolutions Conditions

Let $\mathcal{P} = \{p = 1 + cz + \cdots \,|\, \operatorname{Re} p(z) > 0\}$.

Theorem 1. *Let $f \in \mathcal{A}$, $\phi \in \mathcal{P}$ and $\phi(z) = 1/q(z)$. Then $f \in S^*(\phi)$ if and only if*

$$\frac{1}{z}\left[f(z) * \left(\frac{z + z^2/(q(e^{i\theta}) - 1)}{(1 - z)^2}\right)\right] \neq 0$$

for all $z \in \triangle$ and $0 \leq \theta < 2\pi$.

Proof. Since $\frac{zf'(z)}{f(z)} \prec \phi(z)$ if and only if

$$\frac{zf'(z)}{f(z)} \neq \phi(e^{i\theta})$$

it follows that

$$\frac{1}{z}(zf'(z) - f(z)\phi(e^{i\theta})) \neq 0$$

for $z \in \triangle$ and $0 \leq \theta < 2\pi$. Since $zf'(z) = f * \frac{z}{(1-z)^2}$ and $f(z) = f(z) * \frac{z}{1-z}$, the above inequality is equivalent to

$$\frac{1}{z}\left[f * \left(\frac{z}{(1-z)^2} - \frac{\phi(e^{i\theta})z}{1-z}\right)\right] \neq 0,$$

which proves the result. □

Corollary 1. *Let $f \in \mathcal{A}$, $\phi \in \mathcal{P}$ and $\phi(z) = 1/q(z)$. Then $f \in C(\phi)$ if and only if*

$$\frac{1}{z}\left[f(z) * \left(\frac{z + (1 + \frac{2}{q(e^{i\theta})-1})z^2}{(1-z)^3}\right)\right] \neq 0$$

for all $z \in \triangle$ and $0 \leq \theta < 2\pi$.

We state the following theorems without proof.

Theorem 2. *Let $f \in \mathcal{A}$ and $\phi \in \mathcal{P}$. Then $f \in S_s^*(\phi)$ if and only if*

$$\frac{1}{z}(f * h_\theta)(z) \neq 0$$

where

$$h_\theta(z) = \frac{z + \frac{1+\phi(e^{i\theta})}{1-\phi(e^{i\theta})}z^2}{(1-z)^2(1+z)}$$

for all $z \in \triangle$ and $0 \leq \theta < 2\pi$.

Corollary 2. *Let $f \in \mathcal{A}$ and $\phi \in \mathcal{P}$. Then $f \in C_s(\phi)$ if and only if*

$$\frac{1}{z}(f * k_\theta)(z) \neq 0$$

where $k_\theta = zh'_\theta(z)$, $h_\theta(z)$ is as in the previous Theorem, for all $z \in \triangle$ and

$$0 \leq \theta < 2\pi.$$

Theorem 3. *Let $f \in \mathcal{A}$ and $\phi \in \mathcal{P}$. Then $f \in S_c^*(\phi)$ if and only if*

$$\frac{1}{z}[(f * g_\theta)(z) + \overline{(f * e_\theta)(\overline{z})}] \neq 0$$

where

$$g_\theta(z) = \frac{2z - \phi(e^{i\theta})z(1-z)}{(1-z)^2},\ e_\theta = \frac{\phi(e^{-i\theta})z}{1-z}$$

for all $z \in \triangle$ and $0 \leq \theta < 2\pi$.

Theorem 4. *Let $f \in \mathcal{A}$ and $\phi \in \mathcal{P}$. Then $f \in S_{sc}^*(\phi)$ if and only if*

$$\frac{1}{z}[(f * g_\theta)(z) - \overline{(f * e_\theta)(\overline{-z})}] \neq 0$$

where

$$g_\theta(z) = \frac{2z - \phi(e^{i\theta})z(1-z)}{(1-z)^2},\ e_\theta = \frac{\phi(e^{-i\theta})z}{1-z}$$

for all $z \in \triangle$ and $0 \leq \theta < 2\pi$.

Similar results are true for the classes $C_c(\phi),\ C_{sc}(\phi)$.

In particular, if $\phi(z) = (1 + Az)/(1 + Bz), -1 \leq B < A \leq 1$, then the following results of Silverman and Silvia[10] are obtained as special cases of the previous Theorems.

Corollary 3 ([10])**.** *$f \in S^*[A, B]$ if and only if for all $z \in \triangle$ and all ζ, with $|\zeta| = 1$,*

$$\frac{1}{z}\left[f * \frac{z + \frac{\zeta - A}{A - B}z^2}{(1-z)^2}\right] \neq 0.$$

Corollary 4 ([10])**.** *$f \in C[A, B]$ if and only if for all $z \in \triangle$ and all ζ, with $|\zeta| = 1$,*

$$\frac{1}{z}\left[f * \frac{z + \frac{2\zeta - A - B}{A - B}z^2}{(1-z)^3}\right] \neq 0.$$

3. Growth, Distortion and Covering Theorems

For the purpose of this section, assume that the function $\phi(z)$ is an analytic function with positive real part in the unit disk $\triangle$, $\phi(\triangle)$ is convex and symmetric with respect to the real axis, $\phi(0) = 1$ and $\phi'(0) > 0$. The functions $k_{\phi n}$ $(n = 2, 3, \ldots)$ defined by $k_{\phi n}(0) = k'_{\phi n}(0) - 1 = 0$ and

$$1 + \frac{z k''_{\phi n}(z)}{k'_{\phi n}(z)} = \phi(z^{n-1})$$

are important examples of functions in $C(\phi)$. The functions $h_{\phi n}$ satisfying $zk'_{\phi n}(z) = h_{\phi n}$ are examples of functions in $S^*(\phi)$. Write $k_{\phi 2}$ simply as k_ϕ and $h_{\phi 2}$ simply as h_ϕ.

Theorem 5 ([3])**.** *Let $\min_{|z|=r} |\phi(z)| = \phi(-r),\ \max_{|z|=r} |\phi(z)| = \phi(r),\ |z| = r$. If $f \in C(\phi)$, then*

(i) $k'_\phi(-r) \leq |f'(z)| \leq k'_\phi(r)$
(ii) $-k_\phi(-r) \leq |f(z)| \leq k_\phi(r)$
(iii) $f(\triangle) \supset \{w : |w| \leq -k_\phi(-1)\}$.

The results are sharp.

If $f(z) = z + a_{k+1}z^{k+1} + \ldots \in C(\phi)$, then we can prove that

$$[k'_\phi(-r^k)]^{1/k} \leq |f'(z)| \leq [k'_\phi(r^k)]^{1/k}.$$

See [2].

We prove the following

Theorem 6. *Let $\min_{|z|=r} |\phi(z)| = \phi(-r),\ \max_{|z|=r} |\phi(z)| = \phi(r),\ |z| = r$. If $f \in C_c(\phi)$, then*

(i) $k'_\phi(-r) \leq |f'(z)| \leq k'_\phi(r)$
(ii) $-k_\phi(-r) \leq |f(z)| \leq k_\phi(r)$
(iii) $f(\triangle) \supset \{w : |w| \leq -k_\phi(-1)\}$.

The results are sharp.

Proof. Since $f \in C_c(\phi)$ and ϕ is convex and symmetric with respect to real axis, it follows that $g(z) = [f(z) + \overline{f}(\overline{z})]/2$ is in $C(\phi)$. Since $g \in C(\phi)$, it follows that $g'(z) \prec k'_\phi(z)$. Now,

$$\begin{aligned} rk'_\phi(-r) = k'_\phi(-r) - rk''_\phi(-r) &\leq k'_\phi(-r)\phi(-r) \\ &\leq |(zf'(z))'| \end{aligned}$$

and

$$\begin{aligned} |(zf'(z))'| &= |\tfrac{(zf'(z))'}{g'(z)} g'(z)| \\ &\leq \phi(r)k_\phi'(r) = k_\phi'(r) + rk_\phi''(r) \\ &\leq (rk_\phi(r))'. \end{aligned}$$

By integrating from 0 to r, it follows that

$$k_\phi'(-r) \leq |f'(z)| \leq k_\phi'(r).$$

Part (ii) follows from (i). Also part (iii) follows from part (ii), since $-k_\phi(-r)$ is increasing in $(0,1)$ and bounded by 1. Here $-k_\phi(-1) = \lim_{r\to 1} -k_\phi(-r)$.

The results are sharp for the function $f(z) = k_\phi(z) \in C_c(\phi)$ since it has real coefficients and is in $C(\phi)$. □

Theorem 7. *Let* $\min_{|z|=r} |\phi(z)| = \phi(-r)$, $\max_{|z|=r} |\phi(z)| = \phi(r)$, $|z| = r$. *If* $f \in S_c^*(\phi)$, *then*

(i) $h_\phi'(-r) \leq |f'(z)| \leq h_\phi'(r)$
(ii) $-h_\phi(-r) \leq |f(z)| \leq h_\phi(r)$
(iii) $f(\triangle) \supset \{w : |w| \leq -h_\phi(-1)\}$.

The results are sharp.

Proof. Part (i) follows from above Theorem and the fact $zf' \in S_c^*(\phi)$ if and only if $f \in C_c(\phi)$. Let

$$p(z) = \frac{2zf'(z)}{f(z) + \overline{f}(\overline{z})} = \frac{zf'(z)}{g(z)},$$

where $g(z) = [f(z) + \overline{f}(\overline{z})]/2$. Since $g \in S^*(\phi)$, and hence,

$$-h_\phi(-r) \leq |g(z)| \leq h_\phi(r).$$

Therefore, for $|z| = r < 1$,

$$h_\phi'(-r) = \frac{\phi(-r)h_\phi(-r)}{-r} \leq \left| p(z)\frac{g(z)}{z} \right| = |f'(z)| \leq \frac{\phi(r)h_\phi(r)}{r} = h_\phi(r).$$

This proves (ii). The other part follows easily. □

Similar theorems are true for the classes of functions with respect to symmetric conjugate points.

Theorem 8. *Let* $\min_{|z|=r} |\phi(z)| = \phi(-r)$, $\max_{|z|=r} |\phi(z)| = \phi(r)$, $|z| = r$. *If* $f \in C_s(\phi)$, *then*

$$\frac{1}{r}\int_0^r \phi(-r)[k_\phi'(-r^2)]^{1/2} dr \leq |f'(z)| \leq \frac{1}{r}\int_0^r \phi(r)[k_\phi'(r^2)]^{1/2} dr$$

The other results for this class may be obtained easily and hence omitted.

Proof. The function $g(z) = [f(z) - f(-z)]/2 = z + a_3 z^3 + \ldots$ is in $C(\phi)$. Then the result follows easily. □

The following theorem gives a growth and distortion estimate for functions subordinate to starlike functions with respect to conjugate points.

Theorem 9. *If* $f(z)$ *is starlike with respect to conjugate points in* $\triangle$ *and* $g(z) \prec f(z)$, *then*

$$|g(z)| \leq \frac{r}{(1-r)^2} \text{ and } |g'(z)| \leq \frac{1+r}{(1-r)^3}$$

for $|z| = r < 1$.

Proof. Since $g(z) \prec f(z)$ implies $g(z) = f(w(z))$ for some analytic function $w(z)$ with $|w(z)| \le |z|$,

$$|g(z)| = |f(w(z))| \le \frac{|w(z)|}{(1-|w(z)|)^2} \le \frac{r}{(1-r)^2},$$

for $|z| = r < 1$.

To prove the other inequality, note that

$$g'(z) = f'(w(z))w'(z)$$

and

$$|w'(z)| \le \frac{1-|w(z)|^2}{1-|z|^2}.$$

Now, for $|z| = r < 1$,

$$\begin{aligned} |g'(z)| &= |f'(w(z))||w'(z)| \\ &\le \frac{1+|w(z)|}{(1-|w(z)|)^3}\frac{1-|w(z)|^2}{1-|z|^2} \\ &= \left[\frac{1+|w(z)|}{1-|w(z)|}\right]^2 \frac{1}{1-|z|^2} \\ &\le \frac{1+r}{(1-r)^3}. \end{aligned}$$

□

Theorem 10. *If $f(z)$ is starlike with respect to symmetric conjugate points in $\triangle$ and $g(z) \prec f(z)$, then*

$$|g(z)| \le \frac{r}{(1-r)^2} \text{ and } |g'(z)| \le \frac{1+r}{(1-r)^3}$$

for $|z| = r < 1$.

4. Convolution Theorems

Let $\alpha \le 1$. The class R_α of prestarlike functions of order α consists of functions $f(z) \in \mathcal{A}$ satisfying the following condition: For $\alpha < 1$,

$$f * \frac{z}{(1-z)^{2-2\alpha}} \in S^*(\alpha)$$

and for $\alpha = 1$

$$\operatorname{Re} \frac{f(z)}{z} \ge \frac{1}{2}, z \in \triangle.$$

To prove our results we need the following

Theorem 11. *For $\alpha \le 1$, let $f \in R_\alpha$, $g \in S^*(\alpha)$, $F \in \mathcal{A}$. Then*

$$\left(\frac{f * gF}{f * g}\right)(\triangle) \subset \overline{\mathrm{Co}}(F(\triangle)),$$

where $\overline{\mathrm{Co}}(F(\triangle))$ denotes the closed convex hull of $F(\triangle)$.

Unless or otherwise stated, in this section we assume that $\phi(z) = 1 + cz + \dots$ is convex, $\operatorname{Re}\phi(z) > \alpha$, $0 \le \alpha < 1$. We now prove that the class of starlike functions with respect to conjugate points is closed under convolution with convex functions.

Theorem 12. *Let $\phi(z)$ is convex, $\phi(0) = 1$, $\operatorname{Re}\phi(z) > \alpha$, $0 \le \alpha < 1$. If $f \in S^*(\phi)$, $g \in R_\alpha$, then $f * g \in S^*(\phi)$.*

Proof. Since $g \in S^*(\phi)$, the function $F(z) = \frac{zg'(z)}{g(z)}$ is analytic in $\triangle$ and $F(z) \prec \phi(z)$. Also $\operatorname{Re}\phi(z) > \alpha$ implies $\operatorname{Re}(zf'(z)/f(z)) > \alpha$. This means that $g \in S^*(\alpha)$. Let $f \in R_\alpha$. Then by an application of Theorem 11, we have

$$\left(\frac{f * gF}{f * g}\right)(\triangle) \subset \overline{\mathrm{Co}}(F(\triangle)).$$

Since $\phi(z)$ is convex in $\triangle$ and $F(z) \prec \phi(z)$, $\overline{\mathrm{Co}}(F(\triangle)) \subset \phi(\triangle)$. Also $(f * gF)(z) = (f * zg')(z) = z(f * g)'(z)$. Therefore,

$$\frac{z(f * g)'(z)}{(f * g)(z)} \prec \phi(z)$$

and hence $f * g \in S^*(\phi)$. $\square$

It should be noted that the class $C(\phi)$ is also closed under convolution with prestarlike functions of order α. This follows directly from the above result. Also the other four classes $S_c^*(\phi)$, $C_c(\phi)$, $S_{sc}^*(\phi)$ $C_{sc}(\phi)$ are all closed under convolution with prestarlike functions of order α having real coefficients. We omit the details.

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