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ON THE CONJECTURE OF GÁT

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Abstract. In 2001 Gát conjectured that the integral of the maximal function of the Walsh – Kaczmarz – Fejér kernels to the pth power with $0 < p < 1$ is finite. We give positive answer to the question.

Let P denote the set of positive integers, $N = P\bigcup\{0\}$ the set of nonnegative integers and Z_2 the discrete cyclic group of order 2. That is, $Z_2 = \{0, 1\}$ the group operation is the mod 2 addition and every subset is open. Set

$$G := \underset{k=0}{\overset{\infty}{\times}} Z_2$$

the complete direct product. Thus, every $x \in G$ can be represented by a sequence $x = (x_i, i \in N)$, where $x_i \in \{0, 1\}, i \in N$.

The group operation on G is the coordinate-wise addition. The compact Abelian group G is called the Walsh group. A base for the neighborhoods of G can be given as follows

$$I_0(x) = G, I_n(x) = \{y = (y_i, i \in G : y_i = x_i \text{ for } i < n\}.$$

Let $n \in N$. Then $n = \sum_{i=0}^{\infty} n_i 2^i$, where $n_i \in Z_2$ Denote by

$$|n| = \max(j \in N : n_j \neq 0,$$

that is, $2^{|n|} \leq n < 2^{|n|+1}$. The Rademacher functions are defined as

$$r_n(x) = (-1)^{x_n} \qquad (x \in G, n \in N).$$

The Walsh – Paley system is defined as the set of Walsh – Paley function

$$\omega_n(x) = \prod_{k=0}^{\infty} (r_k(x))^{n_k}, (x \in G, n \in N).$$

The nth Walsh – Kaczmarz functions is

$$\kappa_n(x) = r_{|n|}(x) \sum_{i=0}^{|n|-1} (r_{|n|-1-i}(x))^{n_i}$$

for $n \in P, \kappa_0(x) = 1, x \in G$. The Walsh – Kaczmarz system $\kappa_n, n \in N$ can be obtained from the Walsh – Paley system by renumbering the functions with in the dyadic "block" with indices from the segment $[2^n, 2^{n+1})$. That is,

$$\{\kappa_n : 2^i \leq n < 2^{i+1}\} = \{\ \omega_n : 2^i \leq n < 2^{i+1}\}$$

for all $n \in N$. By means of the transformation $\tau_A : G \to G$

$$\tau_A(x) = (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots) \in G,$$

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which is clear measure preserving and such that $\tau_A(\tau_A(x)) = x$ we have

$$\kappa_n(x) = r_{|n|}(x)\omega_n(\tau_{|n|}(x))(n \in N).$$

Let us consider the Dirichlet and the Fejér kernel functions:

$$D_n^\alpha = \sum_{j=0}^{n-1} \alpha_j, \qquad K_n^\alpha = \frac{1}{n}\sum_{j=1}^{n} D_j$$

where α is either κ or κ and $n \in P$.

Fine [1] proved every Walsh – Paley-Fourier series is a.e. (C, β) summable for $\beta > 0$. Schipp [10] gave a simpler proof for the case $\beta = 1$. The theorem of Schipp are generalized by Taibleson [13], Pál and Simon [9], Gát [4] and Weisz [14].

Skvorcov [12] proved for continuous function f, that Fejér means converges uniformly to f. Gát proved [6] for integrable functions that the Fejér means(with respect to the Walsh – Kaczmarz system) converges a.e. to the function. The conception of quasi-locality is introduced by Schipp [11]. Behind most of the proof of the preceding result (except the Walsh – Kaczmarz case [7]) there is the quasi-locality of the maximal function of the Fejér means. The quasi-locality is the consequence of the following lemma

Lemma.
$$\int_{G\setminus I_k} \sup_{|n|\geq A} |K_n^\omega(x)|dx \leq \sqrt{2^{A-k}},$$
for all $A \geq k$.

Consequently, $\int_{G\setminus I_k} \sup_{|n|\in N} |K_n^\omega(x)|dx \leq \infty$ for all $k \in N$. The proof of this lemma can be found for the Walsh – Paley system in [5], for the Vilenkin system in [2] and for the character system of the group of 2-adic integers in [4]. In [7] Gát proved that this lemma does not hold for the Walsh – Kaczmarz system and he conjectured that the integral of the maximal function of the Walsh – Kaczmarz – Fejér kernel to the pth power is finite with $0 < p < 1$. In this paper we give positive answer to the question.

Theorem. *Let $p \in (0, 1)$. Then*
$$\int_{G\setminus I_k} \sup_{|n|\in N} |K_n^\kappa(x)|^p dx < \infty.$$

Proof. It is shown in [11] that

$$K_{2^n}^\omega(x) \leq c\sum_{j=0}^{n} 2^{j-n} D_{2^n}(x \oplus 2^{-j-1}).$$

Applying the inequality

$$\left(\sum_{k=1}^{\infty} a_k\right)^p \leq \sum_{k=1}^{\infty} a_k^p \qquad (a_k \geq 0, 0 < p \leq 1) \tag{1}$$

and from

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n(x), \\ 0, & \text{if } x \notin I_n(x), \end{cases} \tag{2}$$

we have

$$\int_G (2^n |K_{2^n}^\omega(x)|)^p dx \leq c\sum_{j=0}^{n} 2^{jp} \int_G D_{2^n}^p(x \oplus 2^{-j-1}) \leq c2^{n(2p-1)}. \tag{3}$$

First we prove that

$$\int_G \sup_{n\in N} |K_{2^n}^{\kappa}(x)|^p dx < \infty, \qquad 0<p<1. \tag{4}$$

Skvorcov in [12] proved that for any $n \in P$ and $x \in G$

$$2^n K_{2^n}^{\kappa}(x) = 1 + \sum_{i=0}^{n-1} 2^i D_{2^i}(x) + \sum_{i=0}^{n-1} 2^i r_i(x) K_{2^i}^{\omega}(\tau_i(x)).$$

Then from (1) and (2) we have

$$\begin{aligned}
\int_G \sup_{n\in N} |K_{2^n}^{\kappa}(x)|^p dx &\le \sum_{\nu=1}^{\infty} \int_G |K_{2^\nu}^{\kappa}(x)|^p dx \\
&\le \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} \int_G 1dx + \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} \sum_{i=1}^{\nu-1} 2^{ip} \int_G D_{2^i}^p(x) dx \\
&\quad + \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} \sum_{i=0}^{\nu-1} 2^{ip} \int_G |K_{2^i}^{\omega}(\tau_i(x))|^p dx \\
&\le C_p + C_p \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} \sum_{i=0}^{\nu-1} 2^{(2p-1)i} < C_p < \infty.
\end{aligned} \tag{5}$$

which proves (4).

Since [12]

$$\begin{aligned}
nK_n^{\kappa}(x) &= 2^{|n|} K_{2^{|n|}}^{k}(x) + (n - 2^{|n|}) D_{2^{|n|}}(x) \\
&\quad + (n-2^{|n|}) r_n(x) K_{n-2^{|n|}}^{\omega}(\tau_{|n|}(x))
\end{aligned}$$

and [8]

$$(n-2^{|n|})|K_{n-2^{|n|}}^{\omega}(u)| \le 3 \sum_{j=0}^{|n|} 2^j K_{2^j}(u),$$

from (4) we obtain

$$\begin{aligned}
\int_G \sup_{n\ge 1} |K_n^k(x)|^p dx &\le \int_G \sup_{n\in N} |K_{2^n}^k(x)|^p dx \\
&\quad + \sum_{\nu=1}^{\infty} \int_G D_{2^\nu}^p(x) dx + \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} \sum_{j=0}^{\nu} \int_G (2^j K_{2^j}(x))^p dx \\
&\le C_p + \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu(1-p)}} + \sum_{\nu=1}^{\infty} \frac{1}{2^{\nu p}} \sum_{j=0}^{\nu} 2^{j(2p-1)} \le C_p < \infty.
\end{aligned}$$

Theorem is proved. □

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