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COMPLETE POLYNOMIAL VECTOR FIELDS IN BALL

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Abstract. We describe the complete polynomial vector fields in the unit ball of a Euclidean space.

1. Introduction

By definition, given any subset K in $\mathbb{R}^N$ the set real n-tuples and a mapping $v\colon R^N \to \mathbb{R}^N$, we say that v is a complete vector field in K if for every point $k_0 \in K$ there exists a curve $x\colon \mathbb{R} \to K$ such that $x(0) = k_0$ and $\frac{dx(t)}{dt} = v(x(t))$ for all $t \in \mathbb{R}$. The mapping $v\colon \mathbb{R}^N \to \mathbb{R}^N$ is said to be a polynomial vector field if $v(x) = (P_1(x), \dots, P_N(x))$, $x \in \mathbb{R}^N$ for some polynomials $P_1, \dots, P_N : \mathbb{R}^N \to \mathbb{R}$ of N variables (that is each p_i is a finite linear combination of functions of the form $x_1^{m_1} \dots x_N^{m_N}$ with non-negative integers m_j where $x_j\colon (\xi_1, \dots, \xi_j) \mapsto \xi_j$ denotes the j-th canonical coordinate function $\mathbb{R}^N$). By writing $\langle(\xi_1, \dots, \xi_N), (\eta_1, \dots, \eta_N)\rangle = \sum_{i=1}^N \xi_i\eta_i$ for the inner product in $\mathbb{R}^N$, it is easy to see [1] that a polynomial (or even smooth) vector field is complete in the ball $B := (\langle x, x\rangle < 1)$ iff it is complete in the sphere $S := (\langle x, x\rangle = 1)$. Furthermore, v is complete in S iff it is orthogonal to the radius vector on S, i.e. if $\langle v(x), x\rangle = 0$ for $x \in S$.

In 2001 L.L. Stachó [2] described the complete real polynomial vector fields on the unit disc $\mathbb{D}$ of the space $\mathbb{C}$ of complex numbers. He has shown that a real polynomial vector field $P\colon \mathbb{C} \to \mathbb{C}$ is complete in $\mathbb{D}$ iff P is a finite real linear combination of the functions iz, $\gamma\overline{z}^m - \overline{\gamma}z^{m+2} (\gamma \in \mathbb{C}, m = 0, 1, \dots)$ and $(1 - |z|^2)Q$ where Q is any real polynomial: $\mathbb{C} \to \mathbb{C}$. In this paper we describe the complete polynomial vector fields of the Euclidean unit ball B (or equivalently the unit Euclidean sphere S) of $\mathbb{R}^N$. We show that $P\colon \mathbb{R}^N \to \mathbb{R}^N$ is a complete polynomial vector field in B if and only if $P(x) = R(x) - \langle R(x), x\rangle x + (1 - \langle x, x\rangle)Q(x)$ for some polynomial vector fields Q, $R\colon \mathbb{R}^N \to \mathbb{R}^N$.

Our result not only generalizes the result of [2] on $\mathbb{D}$, but it even simplifies it by showing that the complete polynomial vector fields on the unit disc of $\mathbb{C}$ have the form $(ip(z)z + q(z)(1 - |z|^2))$ where $p, q\colon \mathbb{C} \to \mathbb{R}$ are any real polynomials. In our previous work [4] we represented complete vector fields on a simplex as polynomial combinations of finitely many basis complete vector fields. This idea motivated the formulation of our main result.

2. Main result

Lemma 2.1. *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be a polynomial such that $f(x) = 0$ for $x \in S$ where $S = (\langle x, x\rangle = 1)$. Then there exists a polynomial $Q\colon \mathbb{R}^N \to \mathbb{R}$, such that $f(x) = (1 - \langle x, x\rangle)Q(x)$.*

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Proof. Let $g\colon B \to \mathbb{R}$ be the function on the unit ball $B := (\langle x, x\rangle < 1)$, defined by $g(x) = \frac{f(x)}{(1-\langle x,x\rangle)}$. The function g is analytic, since it is the quotient of two polynomials. Thus $g(x) = \sum_{k=0}^{\infty} g_k(x)$ where g_k are k-homogeneous polynomials on $\mathbb{R}^N$. We have $f(\pm e) = 0$ if $\langle e, e\rangle = 1$, where e is the unit vector. So given $e \in \mathbb{R}^N$ with $\langle e, e\rangle = 1$, there exists a polynomial $P_e\colon \mathbb{R} \to \mathbb{R}$ of degree $\leq \deg f - 2$ such that $(1 - t^2)P_e(t) = f(te)$. It follows that, for every fixed unit vector $e \in \mathbb{R}^N$ $g(te) = \frac{f(te)}{(1-t^2)} = P_e(t) = \sum_{k=0}^{\deg\ f-2} \alpha_k(e)t^k$, with suitable constants $\alpha_0(e), \dots, \alpha_{\deg\ f-2}(e) \in \mathbb{R}$. Hence we deduce that $g_k(te) = 0$ for $k > \deg f - 2$ and for all $t \in R$ and unit vectors e. Then $g = \sum_{k=0}^{\deg\ f-2} g_k$ is a polynomial. $\square$

Remark 2.2. In classical algebraic geometry [3] an analogous results is known for irreducible sets in $\mathbb{K}^N$ where $\mathbb{K}$ is an algebraically closed field. However we do not know any reference for the simple case of the Lemma.

Theorem 2.3. *Let $P\colon \mathbb{R}^N \to \mathbb{R}^N$ be a polynomial. Then P is a complete polynomial vector field in the sphere $S := (\langle x, x\rangle = 1)$ if and only if*

$$[P(x) = R(x) - \langle \mathbb{R}(x), x\rangle x + (1 - \langle x, x\rangle)Q(x)]$$

for some polynomials; $R, Q\colon \mathbb{R}^N \to \mathbb{R}^N$.

Proof. Suppose $P(x) = R(x) - \langle R(x), x\rangle x + (1 - \langle x, x\rangle)Q(x)$ where $R, Q\colon \mathbb{R}^N \to \mathbb{R}^N$ are polynomials.

Then $\langle R(x) - \langle R(x), x\rangle, x, x\rangle = \langle (x), x\rangle - \langle R(x), x\rangle\langle x, x\rangle = 0$ on S. Since $\langle x, x\rangle = 1$ for $x \in S$ and also $(1 - \langle x, x\rangle)Q(x) = 0$ on S. Therefore $\langle P(x), x\rangle = 0$ for $x \in S$, that is P is tangent to S. Conversely, suppose the polynomial vector field $P\colon \mathbb{R}^N \to \mathbb{R}^N$ is complete in S. Let $\tilde{P}(x) = P(x) - \langle P(x), x\rangle x$. Since P is tangent to S, we have $P(x) \perp x$ (i.e. $\langle P(x), x\rangle = 0$). This implies that $P(x) = \tilde{P}(x)$ or $P(x) - \tilde{P}(x) = 0$ on S. By the Lemma $P(x) - \tilde{P}(x) = (1 - \langle x, x\rangle)Q(x)$ for some $Q \in Plo.(\mathbb{R}^N, \mathbb{R}^N)$. Therefore $P(x) = \tilde{P}(x) + (1 - \langle x, x\rangle)Q(x) = P(x) - \langle P(x), x\rangle x + (1 - \langle x, x\rangle)Q(x)$. $\square$

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