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ITERATIVE CONVERGENCE OF RESOLVENTS OF MAXIMAL MONOTONE OPERATORS PERTURBED BY THE DUALITY MAP IN BANACH SPACES

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ABSTRACT. For a maximal monotone operator T in a Banach space an iterative solution of $0 \in Tx$ has been found through weak and strong convergence of resolvents of these operators. Identity mapping in the definition of resolvents has been replaced by the duality mapping. Solution after finite steps has also been established.

1. INTRODUCTION

Let E be a real Banach space and E^* its topological dual. Let $T : E \to E^*$ be a maximal monotone operator. Then J_r defined by $J_r = (I + rT)^{-1}$ for $r > 0$ is called resolvent of T. A well-known way to solve the inclusion $0 \in Tx$ through weak and strong convergence of resolvents of the maximal monotone operators T is to use the iteration scheme:

$$x_1 = x \in E, \;\; x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, 3, \ldots. \tag{1}$$

where $\{r_n\}$ is a sequence of positive real numbers. The convergence of the iteration scheme (1) in case of Hilbert spaces was studied by Rockafellar [13], Brézis and Lions [3], Lions [9] and Pazy [11]. In Banach spaces the problem was carried out by Bruck and Reich [6], Bruck and Passty [5] and Jung and Takahashi [7] among others.

The purpose of this paper is to find the solution of $0 \in Tx$ in the following manner. We replace the identity operator I by the duality mapping J in the definition of J_r above and define $P_r : E^* \to E$ as

$$P_r = (J + rT)^{-1}.$$

Since the duality mapping J is not linear, P_r is not nonexpansive as compared with J_r above. In case of a Hilbert space, both the definitions coincide. With the help of this P_r, we define

$$J_r = P_r \circ J \tag{2}$$

and

$$T_r = \frac{J - J \circ J_r}{r}, r > 0$$

where the symbol $\circ$ stands for the usual composition of functions. At first, we shall prove some of the properties of T_r. Afterwards, we shall give some weak and strong

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convergence theorems using this new J_r via the iteration scheme:

$$\begin{cases} x_0 = x \in E, x_{n+1} = J_{r_n} x_n, \ \ \|x_n - J_{r_n} x_n\| \le \epsilon_n, \ n = 0, 1, 2, \ldots, \\ \{r_n\} \subset (0, \infty), \ \ r_n \to \infty, \\ \{\epsilon_n\} \subset (0, \infty), \ \sum_{n=1}^{\infty} \epsilon_n < \infty. \end{cases}$$

At the end, following Rockafellar [13] we establish the solution of $0 \in Tx$ after a finite number of steps.

2. Preliminaries and Notation

Let E be a real Banach space and E^* its topological dual. The duality mapping $J : E \to E^*$ is defined as:

$$Jx = \left\{ y \in E^* : \langle x, y \rangle = \|x\|^2 = \|y\|^2 \right\}, \quad x \in E.$$

An operator $T : E \to E^*$ (generally multivalued) is called monotone if for any $x, y \in D(T), \ u \in Tx, \ v \in Ty$, we have $\langle u - v, x - y \rangle \ge 0$. T is termed as maximal monotone if it is monotone and for $(x, u) \in E \times E^*$, the inequalities $\langle u - v, x - y \rangle \ge 0$ for all $(y, v) \in G(T)$ imply $(x, u) \in G(T)$, where $G(T)$ denotes the graph of T.

In the sequel, the symbol $\rightharpoonup$ stands for the weak convergence and the symbol $\to$ for the strong convergence. In a uniformly convex Banach space E, for any sequence $\{x_n\} \in E$ satisfying $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, we have $x_n \to x$.

A Banach space E is said to satisfy Opial's condition [10] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ implies that

$$\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

We also know that for two nonnegative sequences $\{s_n\}$ and $\{t_n\}$ satisfying

$$s_{n+1} \le s_n + t_n \text{ for all } n \ge 1,$$

if $\sum_{n=1}^{\infty} t_n < \infty$ then $\lim_{n \to \infty} s_n$ exists.

For the sake of simplicity we omit the symbol $\circ$. Thus the definitions of P_r, J_r and T_r can be rewritten as

$$\begin{cases} P_r = (J + rT)^{-1}, \\ J_r = P_r J, \\ T_r = \frac{J - J J_r}{r}, r > 0. \end{cases} \tag{3}$$

In the sequel, T will always stand for a maximal monotone operator, J for the duality map as defined above and P_r, J_r and T_r will be as defined in (3).

Before going to the weak and strong convergence theorems, we deal with some fundamental properties of T_r.

Proposition 1. *$T_r x \in T J_r x$, $r > 0$.*

Proof. Let $r > 0$ be arbitrary. Then for any $x \in E$,

$$\begin{aligned} & J_r x = (J + rT)^{-1} Jx \\ \text{or} \quad & Jx = (J + rT) J_r x \\ \text{or} \quad & \frac{Jx - J J_r x}{r} \in T J_r x \\ \text{or} \quad & T_r x \in T J_r x. \end{aligned}$$

□

Proposition 2. $0 \in Tx$ *if and only if* $T_r x = 0$. *In particular, we have* $T^{-1} 0 = F(J_r)$, *the set of fixed points of* $J_r, r > 0$.

Proof. Let $r > 0$ and $x \in E$. Then

$$\begin{aligned} 0 \in Tx \quad &\text{iff} \quad 0 \in rTx \\ &\text{iff} \quad Jx \in (J + rT)\,x \\ &\text{iff} \quad x = \big((J + rT)^{-1}J\big)\,x \\ &\text{iff} \quad x = J_r x \\ &\text{iff} \quad Jx = (JJ_r)x \\ &\text{iff} \quad 0 = (J - JJ_r)x \\ &\text{iff} \quad 0 = rT_r x \\ &\text{iff} \quad 0 = T_r x. \end{aligned}$$

□

3. Weak convergence of resolvents

Our purpose in this section is to prove a weak convergence theorem for resolvents of maximal monotone operators as follows.

Theorem 1. *Let E be a uniformly convex Banach space which satisfies Opial's condition. Let $x_0 = x \in E$ and $\{x_n\}$ be defined as $x_{n+1} = J_{r_n} x_n$ with $\|x_n - J_{r_n} x_n\| \leq \epsilon_n$ for all $n = 0, 1, 2, \ldots$, where $\{r_n\} \subset (0, \infty)$ such that $r_n \to \infty$ and $\{\epsilon_n\} \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. If $T^{-1}0 \neq \phi$ then $\{x_n\}$ converges weakly to a solution of $0 \in Tx$.*

Proof. Let $u \in T^{-1}0$. Then

$$\begin{aligned} \|x_{n+1} - u\| &\leq \|x_{n+1} - x_n\| + \|x_n - u\| \\ &\leq \epsilon_n + \|x_n - u\|. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \epsilon_n < \infty$ therefore $\lim_{n\to\infty} \|x_n - u\|$ exists and hence $\{\|x_n\|\}$ is bounded. Thus there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n = 0, 1, 2, \ldots$. We prove that $\{x_n\}$ has a unique weak subsequential limit in $T^{-1}0$. For, let p and q be the weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. We prove that $p = q \in T^{-1}0$. Since

$$\begin{aligned} \|T_{r_n} x_n\| &= \frac{1}{r_n}\|Jx_n - JJ_{r_n} x_n\| \\ &\leq \frac{1}{r_n}\left(\|Jx_n\| + \|JJ_{r_n} x_n\|\right) \\ &= \frac{1}{r_n}\left(\|x_n\| + \|J_{r_n} x_n\|\right) \\ &= \frac{1}{r_n}\left(\|x_n\| + \|x_{n+1}\|\right) \\ &\leq \frac{2M}{r_n} \\ &\to 0 \quad \text{as} \quad r_n \to \infty, \end{aligned}$$

and T is monotone, therefore

$$\langle x - J_{r_{n_i}} x_{n_i}, y - T_{r_{n_i}} x_{n_i}\rangle \geq 0 \tag{4}$$

for all $n_i = 0, 1, 2, \ldots$, $x \in E$ and $y \in Tx$.

We shall now show that $x_{n_i} \rightharpoonup p$ implies $J_{r_{n_i}} x_{n_i} \rightharpoonup p$ as $n_i \to \infty$. Let $f \in E^*$. We know that $x_{n_i} \rightharpoonup p$ if and only if

$$\langle f, x_{n_i}\rangle \to \langle f, p\rangle.$$

Then

$$\begin{aligned}\langle f, x_{n_i}\rangle &= \langle f, J_{r_{n_i}} x_{n_i}\rangle + \langle f, x_{n_i} - J_{r_{n_i}} x_{n_i}\rangle \\ &\leq \langle f, J_{r_{n_i}} x_{n_i}\rangle + \|f\| \|x_{n_i} - J_{r_{n_i}} x_{n_i}\| \\ &\leq \langle f, J_{r_{n_i}} x_{n_i}\rangle + \|f\| \epsilon_{n_i}\end{aligned}$$

so that

$$\liminf_{n_i \to \infty} \langle f, x_{n_i}\rangle \leq \liminf_{n_i \to \infty} \langle f, J_{r_{n_i}} x_{n_i}\rangle + \|f\| \lim_{n_i \to \infty} \epsilon_{n_i}$$

or

$$\liminf_{n_i \to \infty} \langle f, x_{n_i}\rangle \leq \liminf_{n_i \to \infty} \langle f, J_{r_{n_i}} x_{n_i}\rangle$$

because f is bounded. Thus we obtain

$$\langle f, p\rangle \leq \liminf_{n_i \to \infty} \langle f, J_{r_{n_i}} x_{n_i}\rangle. \tag{5}$$

Similarly,

$$\limsup_{n_i \to \infty} \langle f, J_{r_{n_i}} x_{n_i}\rangle \leq \langle f, p\rangle. \tag{6}$$

By (5) and (6), we find that

$$\lim_{n_i \to \infty} \langle f, J_{r_{n_i}} x_{n_i}\rangle = \langle f, p\rangle$$

and in turn

$$J_{r_{n_i}} x_{n_i} \rightharpoonup p.$$

Hence (4) together with $J_{r_{n_i}} x_{n_i} \rightharpoonup p$ and $T_{r_{n_i}} x_{n_i} \to 0$ as $n_i \to \infty$ provides us with

$$\langle x - p, y\rangle \geq 0$$

for all $x \in E$ and $y \in Tx$. Since T is maximal therefore $0 \in Tp$. Again in the same fashion, we can prove that $0 \in Tq$. Next, we prove that $p = q$. To this end, if p and q are distinct then Opial's condition yields

$$\begin{aligned}\lim_{n \to \infty} \|x_n - p\| &= \lim_{n_i \to \infty} \|x_{n_i} - p\| \\ &< \lim_{n_i \to \infty} \|x_{n_i} - q\| \\ &= \lim_{n \to \infty} \|x_n - q\| \\ &= \lim_{n_j \to \infty} \|x_{n_j} - q\| \\ &< \lim_{n_j \to \infty} \|x_{n_j} - p\| \\ &= \lim_{n \to \infty} \|x_n - p\|,\end{aligned}$$

confuting our supposition $p \neq q$. This completes the proof. □

4. Strong convergence of resolvents

First, in this section, we prove a strong convergence theorem by using complete continuity of the duality mapping. Complete continuity is defined as follows. Let X and Y be two Banach spaces. A mapping $S\colon X \to Y$ is called completely continuous if it is continuous from the weak topology of X to the strong topology of Y, i.e.

$$x_n \rightharpoonup x \Rightarrow x_n \to x.$$

Now we prove our strong convergence theorem as follows.The method of proof is partially due to Kartsatos [8].

Theorem 2. *Let E be a uniformly convex Banach space satisfying Opial's condition. Suppose that J is completely continuous. Let $x_0 = x \in E$ and $\{x_n\}$ be defined as $x_{n+1} = J_{r_n} x_n$ with*

$$\|x_n - J_{r_n} x_n\| \le \epsilon_n$$

for all $n = 0, 1, 2, \ldots$, where $\{r_n\} \subset (0, \infty)$ such that $r_n \to \infty$ and $\{\epsilon_n\} \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. If $T^{-1}0 \neq \phi$ then $\{x_n\}$ converges strongly to a solution of $0 \in Tx$.

Proof. $x_n \rightharpoonup x_0 \in T^{-1}0$ follows from Theorem 1. Thus, in view of uniform convexity of E, it is sufficient to prove that $\|x_n\| \to \|x_0\|$ to reach our goal. To this end, notice that

$$r_n T_{r_n} x_n = Jx_n - JJ_{r_n} x_n \in r_n T J_{r_n} x_n = r_n T x_{n+1}.$$

Thus for some $y_n^* \in Tx_{n+1}$, $r_n y_n^* = Jx_n - JJ_{r_n} x_n$. Since $y_n^* \in Tx_{n+1}$, $0 \in Tx_0$ and T is monotone therefore we have

$$\begin{aligned}
0 &\le r_n \langle y_n^* - 0, x_{n+1} - x_0 \rangle \\
&= \langle Jx_n - JJ_{r_n} x_n, x_{n+1} - x_0 \rangle \\
&= \langle Jx_n - Jx_{n+1}, x_{n+1} - x_0 \rangle \\
&= \langle -Jx_{n+1}, x_{n+1} - x_0 \rangle + \langle Jx_n, x_{n+1} - x_0 \rangle \\
&= -\langle Jx_{n+1} - Jx_0, x_{n+1} - x_0 \rangle + \langle Jx_n - Jx_0, x_{n+1} - x_0 \rangle \\
&= -(\langle Jx_{n+1}, x_{n+1} \rangle + \langle Jx_0, x_0 \rangle - \langle Jx_{n+1}, x_0 \rangle \\
&\quad - \langle Jx_0, x_{n+1} \rangle) + \langle Jx_n - Jx_0, x_{n+1} - x_0 \rangle \\
&\le -(\|x_{n+1}\| - \|x_0\|)^2 + \langle Jx_n - Jx_0, x_{n+1} - x_0 \rangle \\
&= -(\|x_{n+1}\| - \|x_0\|)^2 + \langle Jx_n, x_{n+1} - x_0 \rangle - \langle Jx_0, x_{n+1} - x_0 \rangle.
\end{aligned}$$

That is,

$$(\|x_{n+1}\| - \|x_0\|)^2 \le \langle Jx_n, x_{n+1} - x_0 \rangle - \langle Jx_0, x_{n+1} - x_0 \rangle.$$

Here we make use of complete continuity of J to assure that the right hand side of the above inequality vanishes so that

$$\limsup_{n \to \infty} (\|x_{n+1}\| - \|x_0\|)^2 \le 0$$

which means that $\|x_n\| \to \|x_0\|$ thereby showing that $x_n \to x_0$ as desired. □

Next we prove our strong convergence theorem using Lipschitz continuity of T^{-1}. Lipschitz continuity is defined as follows.

An operator $S^{-1} \colon E^* \to E$ is said to be Lipschitz continuous at origin, with modulus $a > 0$, if there is a unique solution x' to $0 \in Sx$ (i.e. $S^{-1}0 = \{x'\}$), and for some $\tau > 0$, we have

$$\|x - x'\| \le a\|y\| \text{ whenever } x \in S^{-1}y \text{ and } \|y\| \le \tau.$$

Note that this condition guarantees the uniqueness of the solution. This condition turns out to be very natural in applications to convex programming. For details, see [12, 13].

Theorem 3. *Let E be a uniformly convex Banach space and let T^{-1} be Lipschitz continuous at origin with modulus $a > 0$. Suppose that $x_0 = x \in E$ and $\{x_n\}$ defined by $x_{n+1} = J_{r_n} x_n$ satisfies*

$$\|x_n - J_{r_n} x_n\| \le \epsilon_n \tag{7}$$

for all $n = 0, 1, 2, \ldots$, where $\{r_n\} \subset (0, \infty)$ such that $r_n \to \infty$ and $\{\epsilon_n\} \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \epsilon_n < \infty$. If $T^{-1}0 \neq \phi$ then $\{x_n\}$ converges strongly to a unique solution of $0 \in Tx$.

Proof. Since T^{-1} is Lipschitz continuous at origin, so by definition, the inclusion $0 \in Tx$ has a unique solution, say x'. As in Theorem 1, $T_{r_n}x_n \to 0$. Choose a positive integer n_0 such that

$$\|T_{r_n}x_n\| \le \tau \text{ for all } n \ge n_0$$

where τ is same as in the definition of Lipschitz continuity. We also have from Proposition 1 that

$$J_{r_n}x_n \in T^{-1}(T_{r_n}x_n), \quad n = 0, 1, 2, \ldots.$$

Thus by Lipschitz continuity, we have

$$\|J_{r_n}x_n - x'\| \le a\|T_{r_n}x_n\|, \quad n = 0, 1, 2, \ldots \tag{8}$$

which enables us to write

$$\|J_{r_n}x_n - x'\| \to 0.$$

Finally, using the triangle inequality

$$\|x_n - x'\| \le \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - x'\|,$$

we obtain

$$\|x_n - x'\| \to 0.$$

Eventually, $\{x_n\}$ converges strongly to a unique solution of $0 \in Tx$. □

Following [13], we establish the solution of $0 \in Tx$ after a finite number of steps. By $Int(D)$ we mean the interior of a set D. In this connection we prove the following theorem.

Theorem 4. *Let E be a uniformly convex Banach space. Suppose that there exists $x' \in E$ such that $0 \in Int(Tx')$. Let $x_0 = x \in E$ and $\{x_n\}$ defined by $x_{n+1} = J_{r_n}x_n$ for all $n = 0, 1, 2, \ldots$ be bounded where $\{r_n\} \subset (0, \infty)$ such that $r_n \to \infty$. Then there exists a positive integer n_0 such that $x_n = x'$ for all $n \ge n_0$.*

Proof. We first show that $T^{-1}\colon E^* \to E$ is single-valued and constant on a neighbourhood of 0. That is, we prove that

$$T^{-1}y = x' \quad \text{if} \quad \|y\| < \epsilon. \tag{9}$$

Let $\epsilon > 0$ be chosen so that $\|y\| < \epsilon$ implies $y \in Int(Tx')$. Taking any $x, y \in Tx$, and y' with $\|y'\| < \epsilon$, we have by monotonicity of T that

$$\langle x - x', y - y' \rangle \ge 0.$$

This yields

$$\langle x - x', y' \rangle \le \langle x - x', y \rangle.$$

So that

$$\sup_{\|y'\|<\epsilon} \langle x - x', y' \rangle \le \langle x - x', y \rangle \quad \text{whenever} \quad y \in Tx$$

implies

$$\epsilon\|x - x'\| \le \|x - x'\|\|y\| \quad \text{whenever} \quad y \in Tx$$

and hence if $x \ne x'$,

$$\epsilon \le \|y\| \quad \text{whenever} \quad y \in Tx.$$

This means that if $\|y\| < \epsilon$ and $x \in T^{-1}y$ then $x = x'$. Virtually, $T^{-1}\colon E^* \to E$ is single-valued and constant on a neighbourhood of 0. Next we know from Proposition 1 that

$$J_{r_n}x_n \in T^{-1}(T_{r_n}x_n), \quad n = 0, 1, 2, \ldots.$$

Thus, as in Theorem 1, $T_{r_n}x_n \to 0$ so that for all $\epsilon > 0$ there exists a positive integer n_0 such that $\|T_{r_n}x_n\| < \epsilon$ for all $n \geq n_0$. Using (9) with $y = T_{r_n}x_n$, we obtain

$$x' = T^{-1}(T_{r_n}x_n)$$

or

$$x' = J_{r_n}x_n$$

Hence $x_n = x'$ for all $n \geq n_0$ as desired. □

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