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GENERALISED TRIANGULAR GRIDS IN DIGITAL GEOMETRY

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Abstract. The hexagonal and the triangular grids are duals of each other. These two grids are the first and second ones in the family of triangular grids (we can call them as one- and two-planes triangular grids). This family comes from the mapping their points to $\mathbb{Z}^3$. Their symmetric properties are triangular. The next grid is the three-planes triangular grid, which looks like the mix of them. In this paper we analyze this grid and its dual from view of neighbouring conditions. After this we consider the n-planes grid. We prove that for $n > 3$ the n-planes triangular grids are non-planar. We also examine the 'circular' three-planes grid and the higher dimensional triangular grids.

1. Introduction

Digital geometry is a relatively new part of theoretical image processing, which develops in a fast way. In digital geometry the spaces we work in, consist of discrete points with integer coordinates. In digital geometry we use the concept of grid, while in geometry of numbers the concept of lattice is used with the same meaning [6].

The square grids $\mathbb{Z}^2$ and $\mathbb{Z}^3$ are well known [16,17,11,1]. In $\mathbb{Z}^n$ each coordinate value of a point is independent of each other. In n dimension we use number n coordinates. There are also many papers about hexagonal grid [8,9] and few papers use triangular grid [4]. In [9] the hexagonal grid was described with two independent coordinate values, but, because of triangular symmetry in [8] the description uses three coordinates which have zero sum. The triangular grid also has triangular symmetry therefore in [12,13] we introduced three coordinates to examine this system. In this paper, we analyze what is the relationship between $\mathbb{Z}^3$ in which we have three independent coordinate values and the hexagonal or the triangular grids (in which we use three coordinates, but they are not independent). As we will show, the hexagonal grid is a subplane of $\mathbb{Z}^3$. Opposite to the hexagonal case, the triangular grid is two parallel planes in $\mathbb{Z}^3$. We can call these grids as one- and two-planes triangular grids. We make some statements about them. After this we continue the sequence of triangular grids with the three-planes grid, in which the points are from three parallel planes from $\mathbb{Z}^3$. We will show the structure of this grid in two dimensions. After this we will define n-planes grids and we show that using more than three planes the results are non-planar. Also interesting the circular three-planes grid. We will extend our definition of triangular grids to higher dimensions. We will finish our examination with these grids.

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FIGURE 1. Neighbourhood relations on the square grid

2. The rectangular grid in two and three dimensions

The spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$ are widely used and they have huge literature. In this part we present a brief historical introduction and show some of the basic concepts.

In digital geometry, mainly the square grid (note, that it has an alternative name: 'rectangular array') was investigated, since this is the most usual space in image processing. One of the first results was the introduction of neighbourhood relations, defined by Rosenfeld and Pfaltz [16]. They gave two types of motions in the two dimensional square grid. The cityblock motion allows horizontal and vertical movements only, while the chessboard motion allows the diagonal directions also. So, in this grid we have two types of distances, based on these motions. Figure 1 shows a point together with those points, which have the distance 1 from it. Both cityblock and chessboard distances can be observed on the figure.

There are some surveys about the topic of digital rectangular-grids and about digital metrics [17,11]. In the case of square grid, each coordinate value of a point is independent from the others. In n dimensions we use n coordinates. The structure of the nodes of these grids is isomorphic with the structure of the n dimensional cubes. The number and the positions of the neighbours are the same. The grid of nodes, and the grid, obtained by representing each cube with its central point, are identical within a translation. So the dual (a well-known concept of graph theory [7, p. 235]) of the n dimensional square grid is an n dimensional square grid as well.

In n dimensions we have n-types of neighbourhood criteria according to the number of different coordinate values. If we use strict neighbouring relations then in two dimensions the points have four 1-neighbours and four 2-neighbours as a square has 4 sides and 4 corners. In 3 dimensions the points have six 1-neighbours, twelve 2-neighbours and eight 3-neigbours as a cube has 6 sides, 12 edges and 8 corners.

In the followings, we describe more or less the two dimensional spaces.

3. The hexagonal grid

One of the first papers about the hexagonal plane is [10], in which the name 'rhombic array' was introduced. In pattern recognition the idea of using non-Cartesian coordinate system can be found for example in [5]. Hence, there are some fairly old papers, the hexagonal grid also well described, we will use only those concepts that are useful for our point of view.

The dual of the triangular grid is not triangular, but hexagonal. [13] (Figure 2).

It is well known that the grids of hexagonal regions and triangular nodes are duals of each other. We use the term hexagonal grid instead of triangular grid of nodes. So our grid is a kind of triangular grid (because of symmetry), but its name came from the history. In digital image processing the neighbourhood criterion,

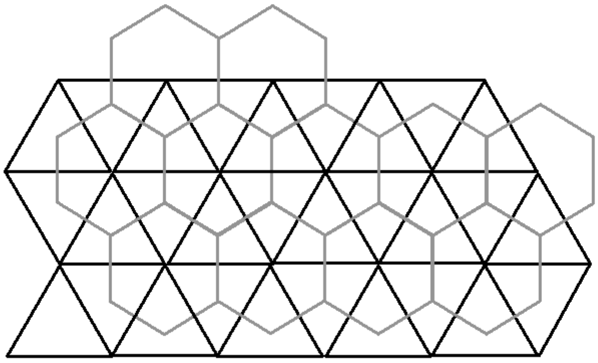


FIGURE 2. The triangular and the hexagonal grids are duals

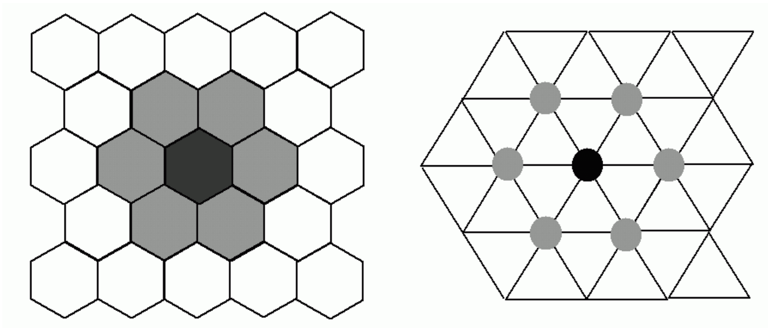


FIGURE 3. Neighbours in the hexagonal grid of regions and in the triangular grid of nodes

illustrated in Figure 3, is usually used, since this is the most natural for a human observer.

Two objects are neighbours

- in the grid of triangular nodes: if there is a direct connection between these nodes,
- in the grid of hexagonal regions: if these hexagons have a common side.

Considering Figure 3, we can easily check that every object has six neighbours.

These grids are equivalent, so from this moment we use the hexagonal grid. We consider the regions as points, and assign 3 coordinate values to them. The necessity of this procedure is to be able to handle mathematically this hexagonal structure, for example we can calculate numerically the distance of two objects. We introduce this coordinate system in the same way, as it was given in [8]. (In [8] the author used the triangular grid of nodes.) Figure 4 shows the result of this procedure. We use 3 coordinate values to reflect the geometrical symmetry of the grid.

We have zero sums for the coordinate values of every point, so there are only two independent values, any of them are removable. In [9] the authors use only two independent coordinate values. We use the coordinate value z only to preserve symmetry. According to the way the coordinates are introduced, moving in the direction z, the coordinate values are modified by (0,-1,+1), and to the opposite direction by (0,+1,-1). Two points are neighbours, if any of their coordinate value

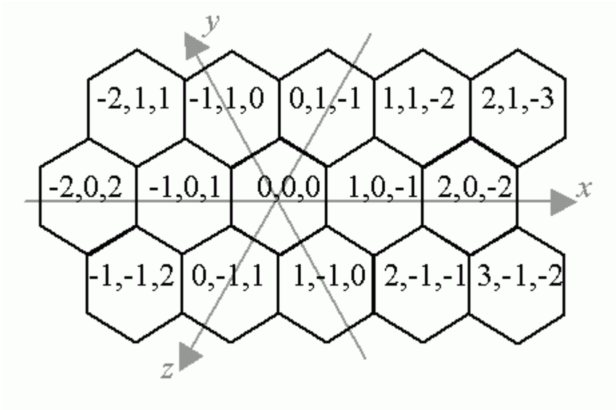


FIGURE 4. Coordinate values on the hexagonal grid

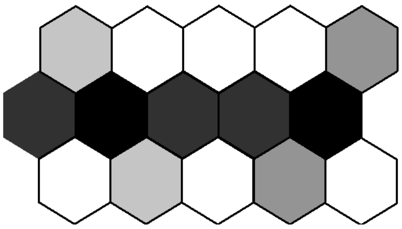


FIGURE 5. Lanes in the hexagonal grid

is the same, and the difference between the other two corresponding coordinate values are ± 1.

Definition 1. When we are taking a step, we are moving from an object to a neighbouring one. A path is a sequence of points, such that each point (except the first one) is a neighbour of the previous one. The first element is the starting point, and the last element is the end point. The length of the path is the number of steps in the path (it is less by 1 than the number of points in the path). The distance of two points is the length of a shortest path between them.

The sequence of objects, for which a coordinate value remains constant, forms a lane.

We can introduce only one hexagonal distance, based on the neighbourhood criterion at each step.

In Fig. 5 we can see some lanes.

Let us see, which points of $\mathbb{Z}^3$ have similar coordinate values as the points of the hexagonal grid. We use the triangular grid of nodes (which is equivalent to the hexagonal grid of regions.)

Figure 6 shows the points of $\mathbb{Z}^3$, and sign those ones which have zero-sum coordinate values.

As we can see the triangular plane of nodes is simply an oblique plane in $\mathbb{Z}^3$. Due to this fact, in [8] Her transferred some geometric properties to hexagonal plane from $\mathbb{Z}^3$.

Hence, the points has zero-sum coordinate values are in a plane in $\mathbb{Z}^3$ we can rename the hexagonal grid to one-plane triangular grid. (It is triangular because of its symmetry properties.)

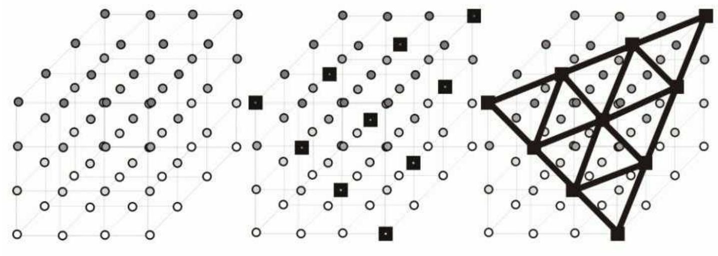


FIGURE 6. The hexagonal plane as part of $\mathbb{Z}^3$

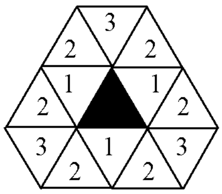


FIGURE 7. Types of neighbours on the triangular grid of regions

Easy to show that a lane is a line in $\mathbb{Z}^3$ which is the meeting of a plane in which a coordinate value constant and the hexagonal plane.

Remark 1. The neighbourhood criterion in hexagonal plane are according to the 2-neighbours in $\mathbb{Z}^3$.

4. The triangular grid

The triangular grid is also well known as the grids we mentioned earlier, but there are only few papers about the neighbouring relations of this grid and some of them are quite fresh. (More papers are about grid of triangular nodes which is the hexagonal grid.)

The grid of triangular regions is equivalent to the grid of hexagonal nodes. In this section, we work mostly with the triangular regions. It is the so-called triangular grid, which is completely different from the previous hexagonal grid. They have very different properties. Usually, we define three types of neighbours on triangular grid, as Figure 7 shows it. The triangular grid is relatively new field in digital geometry, however in [4] there are the neighbouring relations of this grid.

Each triangle has three 1-neighbours, nine 2-neighbours (the 1-neighbours, and six more 2-neighbours), and twelve 3-neighbours (nine 2-neighbours, and three more 3-neighbours).

For the similar reasons, as in the case of the hexagonal grid, we use three coordinate values to represent the triangles, see Figure 8. The objects assigned by coordinates are called points. For triangular grid the points are the triangles, and for hexagonal grid the points are the nodes. These coordinatization is very important part of this paper, because it is the main tool to make easy-writable system with numbers.

In [12, 13] the procedure of the coordinatization of the triangular grid is given, as shown in Figure 8. We define the parity of the points: if the sum of the coordinate values is zero then we call the point even, otherwise it is odd.

As we can see the definition of m-type neighbours (Fig. 7) harmonizes to the coordinate values, so we can use the following alternative definition.

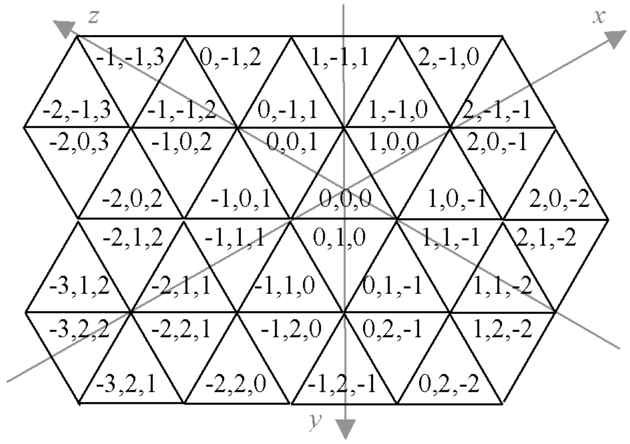


FIGURE 8. Coordinate values on the triangular grid

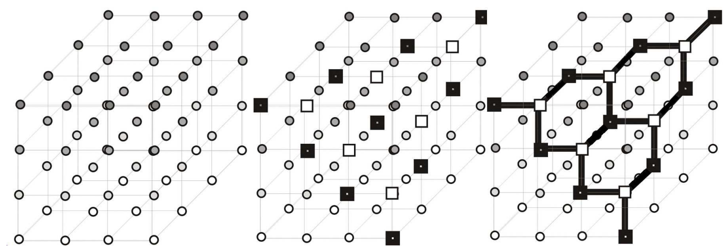


FIGURE 9. The hexagonal grid of nodes as two parallel planes in $\mathbb{Z}^3$

Let P and Q be two triangular regions. The points $P(x, y, z)$ and $Q(x', y', z')$ are m-neighbours ($m = 1, 2, 3$), if the following conditions hold:

$$|x - x'| \leqslant 1,\ |y - y'| \leqslant 1 \text{ and } |z - z'| \leqslant 1,$$
$$|x - x'| + |y - y'| + |z - z'| \leqslant m.$$

We will use the term strict m-neighbours if in the last condition we have equality.

We can see that, if two points are 1-neighbours, then their parities are different. If two points are 2-neighbours and there are two different coordinate values (strict 2-neighbours) then their parities are the same, and if two points are strict 3-neighbours then their parities are different. In this grid we can define several types of distances [12, 13] according to the used neighbourhood criteria.

We can define the lanes as in hexagonal case. The points with a same-valued coordinate form a lane in the triangular grid [13].

The points and their coordinate values are assigned by a one-to-one mapping. We can see it in the following way. Let us fix two coordinate values. They define two non-parallel lanes, which intersection contains two points. The third coordinate values of these points should be given to have 0, and 1 as the sum of the coordinates, respectively.

If two points are 1-neighbours, then there exist two lanes containing both of them. If two points are strict 2-neighbours, then only one lane contains both of them. If two points are strict 3-neighbours, then no lane contains both of them.

Let us see how the triangular plane can be embedded into $\mathbb{Z}^3$, i.e. what the points of the triangular grid represent in $\mathbb{Z}^3$. (Fig. 9) There are 2 parallel planes (according to the parities of points.)

Place of the points	num. and	types of neighb.	coordinate difference		place of neighbours
A point in the top plane:	3	1-type	1 different coordinate:	+1	in middle plane
	6	2-type	2	+1, -1	in bottom
	3	2-type	2	+1, +1	in top
	3	3-type	3	+1, +1, -1	in middle
A point in the middle plane:	3	1-type	1 different coordinate:	+1	in top plane
	3	1-type	1	-1	in bottom plane
	6	2-type	2	+1, -1	in middle
	3	3-type	3	+1, +1, -1	in top plane
	3	3-type	3	+1, -1, -1	in bottom plane
A point in the bottom plane:	6	1-type	1 different coordinate:	-1	in middle
	3	2-type	2	+1, -1	in top plane
	3	2-type	2	-1, -1	in bottom plane
	3	3-type	3	+1, -1, -1	in middle

Table 1. Number and type of neighbours in the three-planes triangular grid

As we can see the considered points of $\mathbb{Z}^3$ are in the planes in which the sum of coordinate values are 0 or 1 hence the triangular grid is the two-planes triangular grid. In Fig. 9 we connect the nodes which are 1-neighbours. As we can see they form a hexagonal grid of nodes which is the so-called triangular grid.

The lanes of this grid are also parallel planes with a plane including two coordinate axes in $\mathbb{Z}^3$ joined with the two planes of this triangular grid.

5. The three-planes triangular grid

Now we are continuing our research. What is the next grid? We have described the one and the two-planes triangular grids as we have seen in Figure 6 and 9. We are going to extend their 'new definitions'. The next grid contains the points of three next parallel planes in $\mathbb{Z}^3$. Let these planes are (because of symmetry) the zero-sum and the ± 1 sum coordinate values planes. (Or for our convenient we can use planes with 0-, 1- and 2-sum coordinate values.) We will call top-plane the plane which has higher value in sum of coordinate values, similarly we use the bottom- and middle-planes. According to these three planes the points of this grid have three possible 'parities' opposite to the traditional triangular grid in which there are only two parities appeared.

We use our thinking way in backward direction as in previous sections.

The points have the following numbers of the different types of neighbours. (See Table 1.)

As we can see the points of the top plane are in the same situation as the points of the bottom plane because of their symmetry.

We can draw the grid based on these neighbourhood relations using symmetry. The middle plane with six 2-neighbours means the classical hexagonal grids. Considering the 1-neighbours we have 3-3 points both in top and bottom plane. It is easy to check that these conditions are satisfied by only the grid in Figure 10. We use 3 type of points according to the 3 planes (the signs 'o' mean the points of the middle plane, the squares are the points of the top-plane and the diamonds form

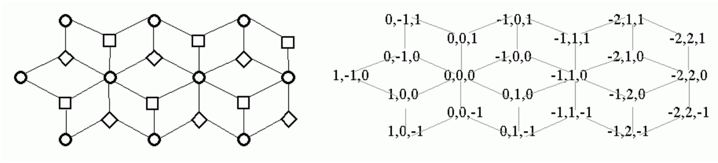


FIGURE 10. The points of the 3-planes triangular grid and their coordinate values

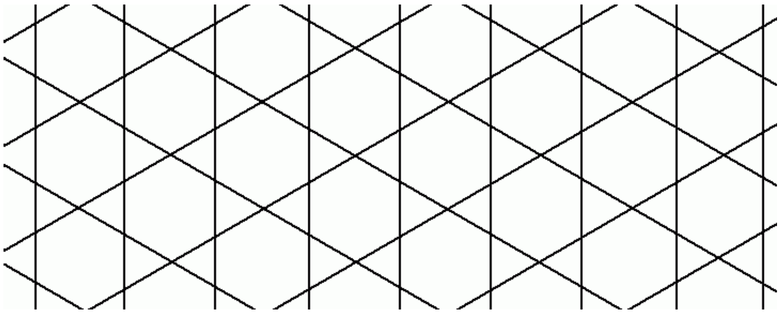


FIGURE 11. The 3-planes triangular grid in plane

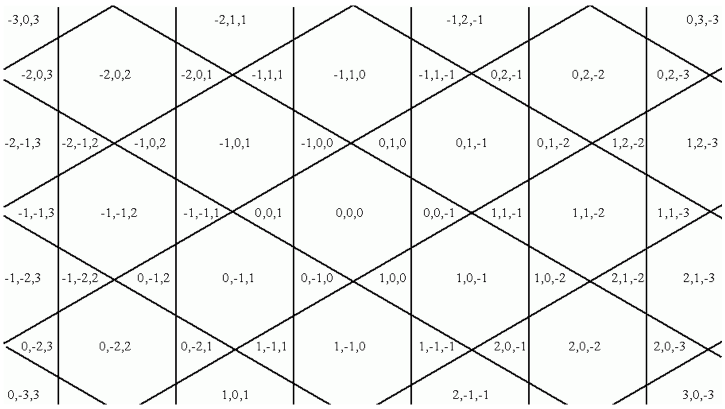


FIGURE 12. The coordinatization of 3-planes triangular grid of regions as planar graph

the bottom-plane), and we marked them by coordinate values at the right hand side of Figure 10.

Now we make the dual of this graph. (We connect the centres of the neighbouring regions.) In Figure 11 we can see the three-planes triangular grid of regions in plane.

This grid looks like a mix of the hexagonal and the triangular grids (actually it is the union of a one- and a two-planes grid). This grid is also known, it is drawn in [8] and in [15], where Radványi examines it as regular planar graph T(6,3,6,3). This grid is not examined for our viewpoint yet, we introduce coordinate values (they can be seen in Figure 12) and neighbouring criteria on this grid.

Unfortunately, in this grid the relation of neighbourhood is not looking so naturally, but in the dual (the grid of nodes) are nice (Figure 10). As we can see in Figure 10 there are 3 types of nodes and accordingly in Fig. 11 and 12 there are 3 types of regions, they represent the points of the 3 different planes, respectively.

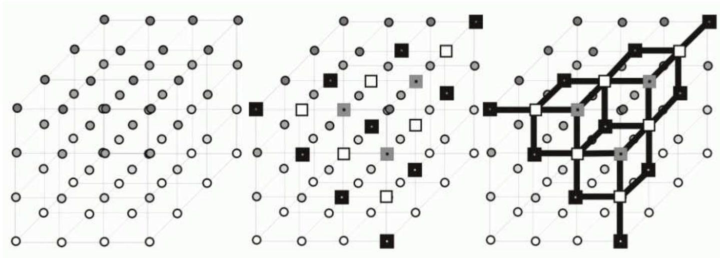


FIGURE 13. The 3-planes triangular grid of nodes as three parallel planes in $\mathbb{Z}^3$

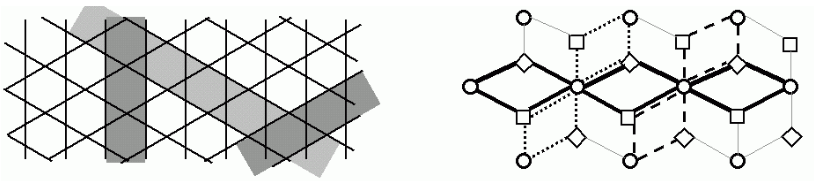


FIGURE 14. Examples for lanes in the 3-planes triangular grid of regions and in the grid of nodes

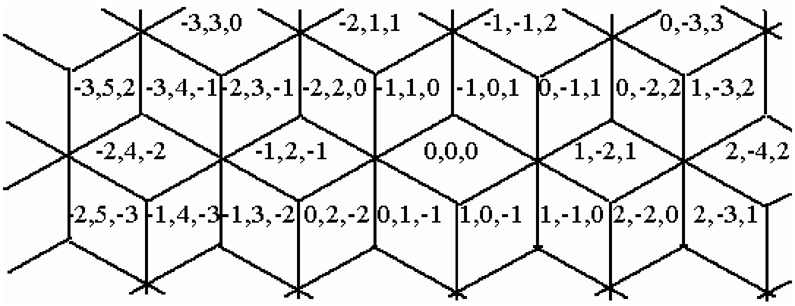


FIGURE 15. Coordinate values of the dual-grid of 3-planes triangular grid

Remark 2. Our coordinatization is in harmony with the 3 dimensional neighbourhood criteria.

Figure 13 shows how we can embed this grid to the 3 dimensional cubic grid.

We can define the lanes as in previous grids (Fig. 14.). The regions where a coordinate value fixed form a lane. (It corresponds to Figure 12.) Similarly the nodes in Figure 10 form lanes as we show in the second figure of Fig. 14. The lanes in the grid of regions seem to be straight lanes, and in grid of nodes they seem to be section planes in 3 dimensional grid.

The dual-grid is also an interesting one. We can coordinatize it using the coordinates of the (2-planes) triangular grid such a way that we represent each object by the values of zero-sum part of it. See Fig. 15. The neighbourhood criteria are not very nice in formal way, but we cannot use nicer coordinatization. We can interpret three kinds of neighbours in this grid. Fig. 16 shows them, they come from the 2-planes triangular grid, two rhombuses are such-type neighbours as the minimal neighbouring criterion of their triangles. The mathematical definition using coordinate values is more complex, because this coordinatization cannot use the symmetry of this grid.

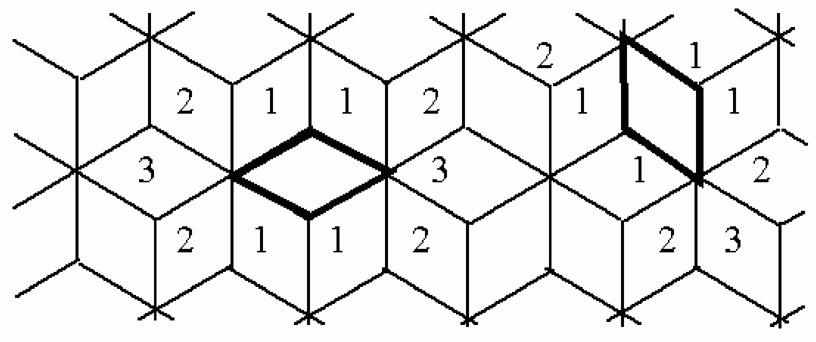


FIGURE 16. Neighbourhood relation in dual-grid of 3-planes triangular grid

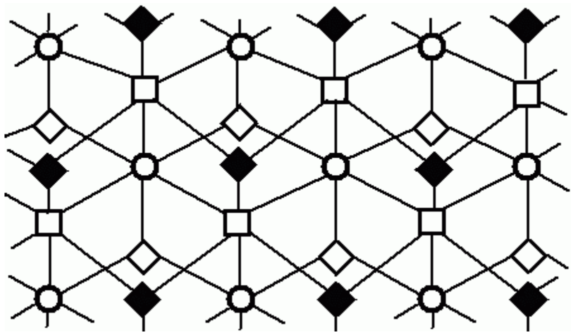


FIGURE 17. The 4-type of points of 4-planes triangular grid (the diamonds the points of edge-planes)

In these grids according to the different neighbourhood criteria we can define several kind of distances based on steps to the neighbouring points.

6. Four- and more-planes triangular grids

Try to imagine the next triangular grids. It is easy in $\mathbb{Z}^3$, but not so easy in the two dimensional plane.

Definition 2. The points of $\mathbb{Z}^3$ for which the sum of coordinate values are in $\{0, 1, \dots n-1\}$ form the points of the n-planes triangular grid of nodes.

First, let us investigate the four-planes triangular grid. As we can see in Figure 17 the points in the bottom- and top-plane (the filled and empty diamonds hide each other.

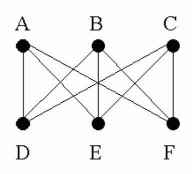
We cannot draw it to the plane, so we cannot use its dual.

Theorem 1. *The four-planes triangular grid is non-planar.*

Proof. We show that the four-planes grid has topological subgraph $K_{3,3}$, i.e., the Kuratowski graph is homeomorphic part of the grid. (It is sufficient condition to our statement [theorem 10.5.1. in 7, pp. 243].) In Figure 18 we can see an example satisfying this property. As we can see the nodes written in the table are 1-neighbours of each-others, respectively. $\square$

We also can define triangular grids using by more planes than four. So we take a general definition for n-planes triangular grids in $\mathbb{Z}^3$.

Corollary 1. *n-planes triangular grids for $n > 4$ are non-planar.*



A	B	C
(0,0,0)	(1,1,1)	(1,1,-1),(1,2,-1),(0,2,-1) (2,1,-1),(2,1,0),(2,0,0)

(1,0,0),(1,1,0)	(0,2,0),(0,1,0),(0,1,1)	(0,0,1),(1,0,1),(2,0,1)
D	E	F

FIGURE 18. The Kuratowski graph ($K_{3,3}$) and the coordinate values of the nodes of the grid respectively

type of neighbours	number of neighbours	place of neighbours	different
1	3	+1 plane	(+)
	3	-1 plane	(-)
2	6	same plane	(+,-)
	3	+2 plane	(+,+)
	3	-2 plane	(-,-)
3	1	+3 plane	(+,+,+)
	1	-3 plane	(-,-,-)
	3	+1 plane	(+,+,-)
	3	-1 plane	(+,-,-)

TABLE 2. Possible neighbours in the n-planes grids. If the plane in the last but one column does not part of our system in any row then those neighbours are missing.

Proof. Since the four-planes triangular grid is proper part of n-planes triangular grid for $n > 4$ the statement follows. □

Definition 3. We can say that two planes α and β cover each other if their points are in the following relation: each point $P(x, y, z)$ of α has image $P'(x + m, y + m, z + m)$ in β where m is an integer.

Lemma 1. *Every plane has covering planes, in which the sum of coordinate values is greater or less by* $3m$ *where* m *is an arbitrary integer.*

Proof. It is easy to show by using definition 3. □

Remark 3. The covering is equivalence relation among the planes, and it can be represented by the translation that is perpendicular the planes. According to the previous lemma we have 3 classes of planes according to the sum of coordinate values of their points (mod 3).

We will use the term '+1 plane' in which each point has greater coordinate-sum by 1 then the previous plane. Similarly we will use '+2'-, '+3'-, '-1'-, '-2'- and '-3 planes'.

Theorem 2. *Table 2 shows all type of possible neighbours (we use strict neighbouring criteria) in* n*-planes triangular grids.*

Proof. There are 3 possible differences according to the coordinate values. Each of them is $+1$, -1 or 0 (same value). With the same point (which can be represented as the 0-neighbour of itself) we have $3^3 = 27$ possibilities, and they are given in Table 2. □

type of neighbours	number of neighbours	place of neighbours	different
1	3	+1 plane	(+)
2	6	same (bottom)	(+,-)
	3	+2 plane	(+,+)
3	3	+1 plane	(+,+,-)
	1	+3 plane	(+,+,+)

Table 3. A point of the bottom plane of 5-planes grid has the neighbours above.

So, for example Table 3 shows the case of the points on the bottom-plane of 5-planes grid (there are only $+n$ planes).

All the neighbours of a point are in the same and in the next six parallel $(\pm 1, \pm 2, \pm 3)$ planes, because we use 3 coordinate values, the maximum of different of the sum of the coordinate values is 3.

We can define distances in the same way as in the previous grids.

7. "Circular grids"

When one try to imagine the structure of an n-planes triangular grid in the plane it is obvious, that the points of the bottom and the top-planes are not in the same situation as the points of the middle planes. Now, we construct such a grids, in which all the planes have equal role.

In this section we will use the neighbourhood criterion such a way that the top-plane and the bottom-plane (i.e. which has the maximal and minimal sum of coordinate values) are neighbours of each other. The first two grids, the hexagonal and the triangular has only one and two planes, respectively. There are already neighbour points on these planes, and in the 2-planes triangular case the two planes are neighbours, i.e. they are the closest two parallel planes. How we can extend this property to the n-planes grids?

In case of 3 planes we have two planes which are not neighbours of each other (they are not ± 1-planes of each other). The bottom and the top-plane are in this situation, because the middle plane is between them. We extract our definition such a way that we use these planes as neighbours, i.e. the next plane in upward direction after the top-plane is the bottom allowing 1-neighbour point-pairs on these planes. Similarly, in this case the previous plane of the bottom is the top and we get the circular 3-planes grid.

Let us analyze the three-planes circular triangular grid in particular. Assume that our planes consist points which has 0, 1 and 2 sum of coordinate values mod 3. In this case we join those points which cover each other.

There is an equivalence relation on $\mathbb{Z}^3$, the classes contain exactly those points which cover each other, i.e. the points $P(x, y, z)$ and $Q(x', y', z')$ are in the same class if and only if $x - x' = y - y' = z - z'$. These classes form the circular 3-planes triangular grid. Two classes are k-neighbours (for $k = 1, 2, 3$) if they contain k-neighbour point-pair, respectively.

The circular 3-planes grid is planar. Our grid is the same as we shown in Figure 17, but we unite the diamond-type nodes respectively as we show in Figure 19. In Figure 20 the coordinate values (each class is represented by the points which have coordinate values with sum $-1, 0, 1$ or 2) ordered to the nodes are shown.

Figure 21 shows that the dual-grid is the hexagonal. In this figure one can see also that this grid is planar, and how the coordinate values can represent the

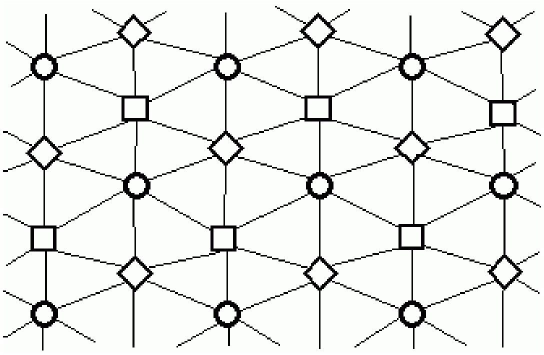


FIGURE 19. The circular 3-planes triangular grid

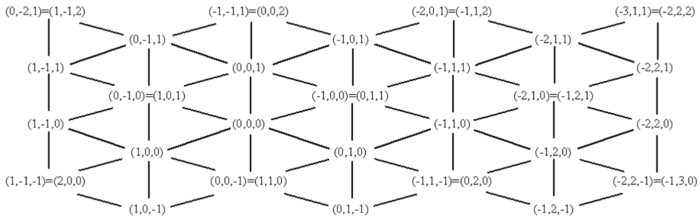


FIGURE 20. Possible coordinate values of the nodes representing classes

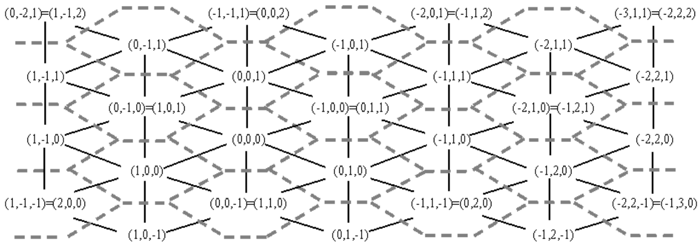


FIGURE 21. Another coordinatization of the hexagonal grid using the circular 3-planes triangular grid

regions. Moreover it is obvious that this possible coordinatization is different from the previous one (Fig. 4).

We can use the number of possible neighbours as in the previous section, but using the circularity. Fig. 22 shows for example the neighbours of the Origin.

Theorem 3. *Table 4 shows all types of possible neighbours (we use strict neighbouring criterion) of the circular triangular grids.*

Proof. We know the facts that the $+3$ and -3 planes is the same as the original, -1 is the same as $+2$, and -2 is the same as the $+1$ plane. Using Theorem 2, we have all the six 1-neighbours, and six 2-neighbours in the original plane, as well. The other three-three 2-neighbours of Table 2 are exactly the previous 1-neighbour points (i.e. same classes). The 3-neighbour point in which each coordinate value differs by $+1$ means the class of the original point by the definition of classes.

type of neighbours	number of neighbours	place of neighbours	different
1	3	+1 plane	(+)
	3	+2 plane	(-)
2	6	same plane	(+,-)
3	3	+1 plane	(+,+,-)
	3	+2 plane	(+,-,-)

TABLE 4. Number and place of neigbours in 3-planes circular grid.

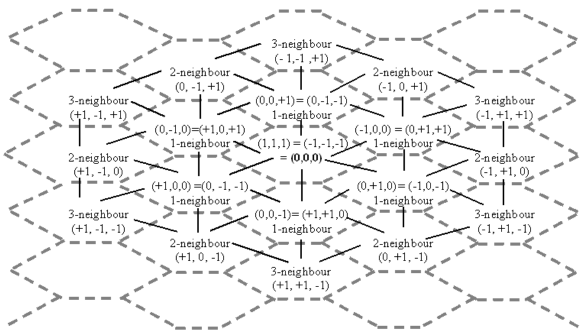


FIGURE 22. Neighbouring relations in the circular 3-planes triangular grid

Similarly the 3-neighbour, which represented as $(-,-,-)$ is also missing. The other six 3-neighbours of Table 2 present in this case too. □

As we can see in Fig. 21 and Fig. 2 the circular 3-planes grid isomorphic to the hexagonal plane. The difference is only in the coordinatization and (therefore) the neighbourhood criteria.

Now, we assume that we use the planes including points with sum of their coordinate values in $\{0, 1, 2, \ldots n-1\}$.

Definition 4. In n-planes circular grid the points P and Q are neighbours, if one of the following conditions is true:

– they are neighbours in the n-planes triangular grid or
– they are neighbours from circularity.

In the latter case there are three possibilities:

1: if the number of planes divisible by 3 (i.e. $n = 3k$), then we have nice properties (the next plane, the $(n+1)$st, would be covering the first, hence our extension is natural). Let us use the $(n+1)$st plane but descend each coordinate value by k. So the points of the $(n+1)$st plane go to the first(= the bottom) plane, i.e. they represent the same classes. Therefore those neighbours, which are not in any of the original planes in the n-planes grid, but they would be in the plane where the points have m-sum coordinate values, are in the plane with sum values ($m \mod n$) in the circular n-planes grid.

2: if the number of planes is $n = 3k+1$, then our top- and bottom plane are covering each other. Let the covering points of these planes are 1-neighbours of each other by definition.

3: if the number of planes is $n = 3k+2$ then our top- and bottom plane are not covering each other, but the bottom plane and the $(3k+1)$st plane cover of each other, and the 2nd and the top ones cover each other. We can define the 1-neighbourhood between the bottom and top-plane in the following way. If a point P on the $(3k+1)$st plane are 1-neighbour of a point Q in $(3k+2)$nd plane then the point R in the first plane which is covered by P is 1-neighbour of Q, by definition.

As we can see according to the number of planes we have three possibilities to get circular triangular grids. Easy to check that the neighbouring relation above is symmetric.

The first case, when the grid contains $n = 3k$ planes is a natural extension of the 3-planes circular triangular grid. One can imagine that the classes of points of $\mathbb{Z}^3$ representing a point in this grid are in the following relation. $P(x, y, z)$ and $Q(x', y', z') \in \mathbb{Z}^3$ are representing the same grid-point, iff there is an $m \in \mathbb{Z}$, such that $x - x' = y - y' = z - z' = mk = \frac{mn}{3}$. This relation for $k \geq 2$ means that we divide each of the classes of the 3-planes circular grid to k parts.

The other 2 cases are not so natural, but we can use them.

We can suppose that the circular triangular grid with n planes use the points with sum of coordinate-values $0, 1, \ldots, n-1$. Depending on the neighbouring criterion we can define 3 kind of steps and using them several kinds of distances in circular grids.

8. "Triangular grids" in higher dimensions

We can extend our theory to higher dimensions.

Definition 5. Let the triangular grid with n-"plane" in the m dimensional space is that part of $\mathbb{Z}^{m+1}$ in which the sum of the coordinate-values (number m) of each point is $0, 1, \ldots n-1$, respectively.

When we taking a step, we move from an object to a neighbouring point. We have $m+1$ types of neighbouring criteria, so we have $m+1$ types of possible steps in these grids.

Similarly we can define paths and distances depending on steps in triangular grid with n-"plane" in m dimension. We also can use the concept of "lanes" in these grids, where 1 or more coordinate-values are fixed. We can use fixed coordinate values up to number $m-2$. If we fixed $m-2$ coordinates then we have a lane like the traditional triangular grids (in previous sections). If we use smaller number of fixed coordinates then the "lane" is more dimensional.

Note, that these more dimensional grids can be imagine as infinite graphs. Any finite part of them is a graph, therefore it can be drawn to the three dimensional space.

9. Summary

The rectangular and hexagonal grids are widely used in pattern recognition, CNN etc. Her used 3 dependent coordinate values for hexagonal grid. We used 3 coordinate values to represent the points of the so-called triangular grid. We showed that these two grids are the 1- and 2-planes grids in the family of triangular grids. In this article we continued the sequence of hexagonal and triangular planes with arbitrary n-planes triangular grid. We showed a planar representation and we coordinatized the 3-planes triangular grid, so it is easier to make calculations in this grid. We also continued our examination to the n-planes triangular grids. We showed that for $n > 3$ these grids are non-planar. We also defined the circular and the higher dimensional triangular grids. We showed that using the circular 3-planes triangular grid another type of neighbourhood system can be defined on the hexagonal grid. This new type of coordinatization can be a useful tool to get a better approximation of the Euclidean distance and/or circle in the hexagonal grid.

We analysed three types of neighbourhood relations in these triangular grids. We were able to define distance functions on the triangular grids based on these neighbouring relations. (For the rectangular grids and for the so-called triangular grid these definitions are in [1-3] and in [12], respectively.) Since we have two or more types of points on these triangular grids (except the 1-plane grid), we had to face to some additional problems that are not present for widely used rectangular and hexagonal grids. For example the types and locations of the neighbours are different.

The general triangular grids seem to be useful in image processing, in theory of CNN and in some other network related fields.

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