Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
20 (2004), 93–104
www.emis.de/journals

A CENTRAL LIMIT THEOREM FOR RANDOM FIELDS

ISTVÁN FAZEKAS AND ALEXEY CHUPRUNOV

Abstract. A central limit theorem is proved for α-mixing random fields. The sets of locations where the random field is observed become more and more dense in an increasing sequence of domains. The central limit theorem concerns these observations. The limit theorem is applied to obtain asymptotic normality of kernel type density estimators. It turns out that in our setting the covariance structure of the limiting normal distribution can be a combination of those of the continuous parameter and the discrete parameter cases.

1. Introduction

In statistics, most asymptotic results concern the increasing domain case, i.e. when the random process (or field) is observed in an increasing sequence of domains T_n, with $|T_n| \to \infty$. However, if we observe a random field in a fixed domain and intend to prove an asymptotic theorem when the observations become dense in that domain, we obtain the so called infill asymptotics (see Cressie [4]). It is known that several estimators being consistent for weakly dependent observations in the increasing domain setup, are not consistent if the infill approach is considered. In this paper we combine the infill and the increasing domain approaches. We call infill-increasing approach if our observations become more and more dense in an increasing sequence of domains. Using this setup, Lahiri [15] and Fazekas [8] studied the asymptotic behaviour of the empirical distribution function. Practical applications of this approach was given in Lahiri, Kaiser, Cressie, and Hsu [17]. Also in the infill-increasing case, consistency and asymptotic normality of the least squares estimator for linear and for linear errors-in-variables models were proved in Fazekas and Kukush [10]. In Putter and Young [22] the kriging was considered using infill-increasing approach. General central limit theorems were obtained in Lahiri [16] for spatial processes under infill-increasing type designs.

The main result of this paper is Theorem 2.1 in Section 2. It is a Bernstein type central limit theorem for α-mixing random fields. It is analogous to Theorem 1.1 in Bosq, Merlevède and Peligrad [2]. The novelties of our theorem are the infill-increasing setting and that it concerns random fields and not only random processes. The detailed proof is given in Section 3. The method of proof is the well-known big-block small-block technique often applied to obtain asymptotic normality of nonparametric statistics (see, e.g., Liebscher [18]). In Section 4 we give an application of Theorem 2.1. Theorem 4.1 states asymptotic normality of the kernel

---

2000 *Mathematics Subject Classification.* 60F05, 62M30.

Key words and phrases. Central limit theorem, random field, α-mixing, infill asymptotics, increasing domain asymptotics, density estimator, asymptotic normality of estimators.

Supported by the Hungarian Foundation of Scientific Researches under Grant No. OTKA T032361/2000 and Grant No. OTKA T032658/2000.

The research was partially realized while the second author was visiting University of Debrecen, Hungary.

type density estimator (4.2) in the infill-increasing case. The underlying random field is α-mixing. The conditions are similar to those of Theorem 2.2 (continuous time process) and Theorem 3.1 (discrete time process) of Bosq, Merlevède and Peligrad [2]. Our result is in some sense between the discrete and the continuous time cases.

Kernel type density estimators are widely studied, see e.g. Prakasa Rao [21], Devroye and Györfi [6], Bosq [1], Kutoyants [14]. Several papers are devoted to the density estimators for weakly dependent stationary sequences (see, e.g., Castellana and Leadbetter [3], Bosq, Merlevède and Peligrad [2], Liebscher [18]). In most of the papers the goal is to find weak dependence conditions of asymptotic normality. A few papers study the relation of the rate of dependence and the asymptotic behaviour (see, e.g., Csörgő and Mielniczuk [5]). The asymptotic normality of the kernel type density estimator is well known for weakly depenedent continuous time processes, too (see, e.g., Bosq, Merlevède and Peligrad [2]). The paper Guillou and Merlevède [12] gives an estimator for the asymptotic variance. However, in the continuous time case, if we calculate numerically the kernel type density estimator, its asymptotic variance can be different from that of the theoretical one. To point out this phenomenon is the goal of Theorem 4.1, and therefore we turn to so called infill-increasing setup.

The results of this paper were announced at conferences, see e.g. Fazekas [9].

2. A Bernstein-type central limit theorem

The following notation is used. $\mathbb{N}$ is the set of positive integers, $\mathbb{Z}$ is the set of all integers, $\mathbb{N}^d$ and $\mathbb{Z}^d$ are d-dimensional lattice points, where d is a fixed positive integer. $\mathbb{R}$ is the real line, $\mathbb{R}^d$ is the d-dimensional space with the usual Euclidean norm $\|\mathbf{x}\|$. In $\mathbb{R}^d$ we shall also consider the distance corresponding to the maximum norm: $\varrho(\mathbf{x},\mathbf{y}) = \max_{1\le i\le d} |x^{(i)} - y^{(i)}|$, where $\mathbf{x} = (x^{(1)},\dots,x^{(d)})$, $\mathbf{y} = (y^{(1)},\dots,y^{(d)})$. The distance of two sets in $\mathbb{R}^d$ corresponding to the maximum norm is also denoted by ϱ: $\varrho(A,B) = \min\{\varrho(a,b) \,:\, a\in A,\, b\in B\}$.

For real valued sequences $\{a_n\}$ and $\{b_n\}$, $a_n = \mathrm{o}(b_n)$ (resp. $a_n = \mathrm{O}(b_n)$) means that the sequence a_n/b_n converges to 0 (resp. is bounded). We shall denote different constants with the same letter c (or C). $|\mathcal{D}|$ denotes the cardinality of the finite set $\mathcal{D}$ and at the same time $|T|$ denotes the volume of the domain T.

We shall suppose the existence of an underlying probability space $(\Omega,\mathcal{F},\mathbb{P})$. The σ-algebra generated by a set of events or by a set of random variables will be denoted by $\sigma\{.\}$. The symbol $\mathbb{E}$ stands for the expectation. The variance and the covariance are denoted by var(.) and cov(.,.), respectively. The L_p-norm of a random (vector) variable η is $\|\eta\|_p = \{\mathbb{E}\|\eta\|^p\}^{1/p}$, $1\le p<\infty$.

The symbol $\Rightarrow$ denotes convergence in distribution. $\mathcal{N}(m,\Sigma)$ stands for the (vector) normal distribution with mean (vector) m and covariance (matrix) Σ.

Now we describe the scheme of observations. For simplicity we restrict ourselves to rectangles as domains of observations. Let $\Lambda>0$ be fixed. By $(\mathbb{Z}/\Lambda)^d$ we denote the Λ-lattice points in $\mathbb{R}^d$, i.e. lattice points with distance $1/\Lambda$:

$$\left(\frac{\mathbb{Z}}{\Lambda}\right)^d = \left\{\left(\frac{k_1}{\Lambda},\dots,\frac{k_d}{\Lambda}\right) \,:\, (k_1,\dots,k_d)\in\mathbb{Z}^d\right\}.$$

T will be a bounded, closed rectangle in $\mathbb{R}^d$ with edges parallel to the axes and $\mathcal{D}$ will denote the Λ-lattice points belonging to T, i.e. $\mathcal{D} = T\cap(\mathbb{Z}/\Lambda)^d$. To describe the limit distribution we consider a sequence of the previous objects. I.e. let $T_1, T_2, \dots$

be bounded, closed rectangles in $\mathbb{R}^d$. Suppose that

$$T_1 \subset T_2 \subset T_3 \subset \dots, \quad \bigcup_{i=1}^{\infty} T_i = T_\infty. \tag{2.1}$$

We assume that the length of each edge of T_n is integer and converges to ∞, as $n \to \infty$. Let $\{\Lambda_n\}$ be an increasing sequence of positive integers (the non-integer case is essentially the same) and $\mathcal{D}_n$ be the Λ_n-lattice points belonging to T_n.

Let $\{\xi_{\mathbf{t}},\ \mathbf{t} \in T_\infty\}$ be a random field. The n-th set of observations involves the values of the random field $\xi_{\mathbf{t}}$ taken at each point $\mathbf{k} \in \mathcal{D}_n$. Actually, each $\mathbf{k} = \mathbf{k}^{(n)} \in \mathcal{D}_n$ depends on n but to avoid complicated notation we often omit superscript (n). By our assumptions, $\lim_{n\to\infty} |\mathcal{D}_n| = \infty$.

Define the discrete parameter (vector valued) random field $Y_n(\mathbf{k})$ as follows. For each $n = 1, 2, \dots,$ and for each $\mathbf{k} \in \mathcal{D}_n$

$$\text{let } \ Y_n(\mathbf{k}) = Y_n(\mathbf{k^{(n)}}) \ \text{ be a Borel measurable function of } \ \xi_{\mathbf{k}^{(n)}}. \tag{2.2}$$

We need the notion of α-mixing (see e.g. Doukhan [7], Guyon [13], Lin and Lu [19]). Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras in $\mathcal{F}$. The α-mixing coefficient of $\mathcal{A}$ and $\mathcal{B}$ is defined as follows.

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(AB)| \ : \ A \in \mathcal{A}, B \in \mathcal{B}\}.$$

The α-mixing coefficients of $\{\xi_{\mathbf{t}} \ : \ \mathbf{t} \in T_\infty\}$ are

$$\alpha(r, u, v) = \sup\{\alpha(\mathcal{F}_{I_1}, \mathcal{F}_{I_2}) \ : \ \varrho(I_1, I_2) \ge r,\ |I_1| \le u,\ |I_2| \le v\},$$
$$\alpha(r) = \sup\{\alpha(\mathcal{F}_{I_1}, \mathcal{F}_{I_2}) \ : \ \varrho(I_1, I_2) \ge r\},$$

where I_i is finite a subset in T_∞ with cardinality $|I_i|$ and $\mathcal{F}_{I_i} = \sigma\{\xi_{\mathbf{t}} \ : \ \mathbf{t} \in I_i\}$, $i = 1, 2$. We shall use the following condition. For some $1 < a < \infty$

$$\int_0^\infty s^{2d-1} \alpha^{\frac{a-1}{a}}(s) ds < \infty\,. \tag{2.3}$$

Now, we turn to the version of the central limit theorem appropriate to our sampling scheme. Our Theorem 2.1 is a modification of Theorem 1.1 of Bosq, Merlevède and Peligrad [2]. The novelties of our theorem are the infill-increasing setting and that it concerns random fields.

We concentrate on the case when $\xi_{\mathbf{t}}$ and $\xi_{\mathbf{s}}$ are dependent if $\mathbf{t}$ and $\mathbf{s}$ are close to each other. Therefore our theorem does not cover the case when $Y_n(\mathbf{k})$'s are independent and identically distributed. On the other hand, if $\xi_{\mathbf{t}}$ is a stationary field with continuous covariance function and positive variance, then the covariance is close to a fixed positive number inside a small hyperrectangle. We intend to cover this case. Recall that $\mathcal{D}_n$ is a sequence of finite sets in $(\mathbb{Z}/\Lambda_n)^d$ with

$$\lim_{n\to\infty} |\mathcal{D}_n| = \infty.$$

Theorem 2.1. *Let $\xi_{\mathbf{t}}$ be a random field and let $Y_n(\mathbf{k}) = (Y_n^{(1)}(\mathbf{k}), \dots, Y_n^{(m)}(\mathbf{k}))$ be an m-dimensional random field defined by (2.2). Let*

$$S_n = \sum_{\mathbf{k}\in\mathcal{D}_n} Y_n(\mathbf{k}), \quad n = 1, 2, \dots.$$

Suppose that for each fixed n, the field $Y_n(\mathbf{k})$, $\mathbf{k} \in \mathcal{D}_n$, is strictly stationary with $\mathbb{E}Y_n(\mathbf{k}) = 0$. Assume that

$$\|Y_n(\mathbf{k})\| \le M_n, \tag{2.4}$$

where M_n depends only on n;

$$\sup_{n,\mathbf{k},r} \mathbb{E}\big(Y_n^{(r)}(\mathbf{k})\big)^2 < \infty; \tag{2.5}$$

for any increasing, unbounded sequence of rectangles G_n with $G_n \subseteq T_n$

$$\lim_{n\to\infty} \frac{1}{\Lambda_n^d |\mathcal{G}_n|} \mathbb{E}\left[\sum_{\mathbf{k}\in\mathcal{G}_n} Y_n^{(r)}(\mathbf{k}) \sum_{\mathbf{l}\in\mathcal{G}_n} Y_n^{(s)}(\mathbf{l})\right] = \sigma_{r,s}, \quad r,s = 1,\dots,m, \tag{2.6}$$

where $\mathcal{G}_n = G_n \cap (\mathbb{Z}/\Lambda_n)^d$; the matrix $\Sigma = (\sigma_{r,s})_{r,s=1}^m$ is positive definite; there exists $1 < a < \infty$ such that (2.3) is satisfied; and

$$M_n \le c|T_n|^{\frac{a^2}{(3a-1)(2a-1)}} \quad \text{for each} \quad n. \tag{2.7}$$

Then

$$\frac{1}{\sqrt{\Lambda_n^d |\mathcal{D}_n|}} S_n \Rightarrow \mathcal{N}(0,\Sigma), \qquad \text{as} \quad n\to\infty. \tag{2.8}$$

3. Proof of the main result

The covariance inequality in the α-mixing case is

$$|\operatorname{cov}(X,Y)| \le C\alpha^{1/t}(X,Y)\, \|X\|_p\, \|Y\|_q\,, \tag{3.1}$$

if $t,p,q > 1$, $1/t + 1/p + 1/q = 1$. We remark that for bounded random variables

$$|\operatorname{cov}(X,Y)| \le C\alpha(X,Y)\, \|X\|_\infty\, \|Y\|_\infty\,,$$

is satisfied, see, e.g., Lin and Lu [19].

Lemma 3.1. *Let $\mathcal{D} \subset (\mathbb{Z}/\Lambda)^d$ be a finite set and let $\xi_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{D}$, be a strictly stationary random field with zero mean and with $|\xi_{\mathbf{i}}| \le M < \infty$ and $a > 1$. Then*

$$\mathbb{E}\left(\sum\nolimits_{\mathbf{i}\in\mathcal{D}} \xi_{\mathbf{i}}\right)^4 \le c|\mathcal{D}|^2 \Lambda^{2d} M^{4-\frac{2}{a}} \left(\mathbb{E}\xi_{\mathbf{i}}^2\right)^{\frac{1}{a}}, \tag{3.2}$$

if

$$\int_0^\infty s^{2d-1}\alpha^{\frac{a-1}{a}}(s,u,v)ds < \infty \tag{3.3}$$

for pairs $u = 3$, $v = 1$ and $u = v = 2$, where $\alpha(s,u,v)$ denotes the mixing coefficient of the field $\xi_{\mathbf{i}}$.

Proof of Lemma 3.1. The following calculation is similar to the ones in Lahiri [15] and Maltz [20]. For simplicity, consider the case $\Lambda = 1$ (the other cases can be reduced to this).

$$\mathbb{E}\left\{\sum\nolimits_{\mathbf{i}\in\mathcal{D}} \xi_{\mathbf{i}}\right\}^4 \le \tag{3.4}$$

$$\le C\left\{\sum_{\mathbf{i}\in\mathcal{D}} \mathbb{E}\xi_{\mathbf{i}}^4 + \sum_{\mathbf{i}\ne\mathbf{j}} |\mathbb{E}\xi_{\mathbf{i}}^3\xi_{\mathbf{j}}| + \sum_{\mathbf{i}\ne\mathbf{j}} \mathbb{E}\xi_{\mathbf{i}}^2\xi_{\mathbf{j}}^2 + \sum_{\mathbf{i}\ne\mathbf{j}\ne\mathbf{k}} |\mathbb{E}\xi_{\mathbf{i}}^2\xi_{\mathbf{j}}\xi_{\mathbf{k}}| + \sum_{\mathbf{i}\ne\mathbf{j}\ne\mathbf{k}\ne\mathbf{l}} |\mathbb{E}\xi_{\mathbf{i}}\xi_{\mathbf{j}}\xi_{\mathbf{k}}\xi_{\mathbf{l}}|\right\} =$$
$$= C[J_1 + J_2 + J_3 + J_4 + J_5],$$

where C denotes a finite constant.

$$\begin{aligned}
J_1 &= \sum_{\mathbf{i}\in\mathcal{D}} \mathbb{E}\xi_{\mathbf{i}}^4 \le |\mathcal{D}|M^{4-\frac{2}{a}}\mathbb{E}|\xi_{\mathbf{i}}|^{\frac{2}{a}} \le |\mathcal{D}|M^{4-\frac{2}{a}}\left(\mathbb{E}|\xi_{\mathbf{i}}|^2\right)^{\frac{1}{a}}. \\
J_2 &= \sum_{\mathbf{i}\ne\mathbf{j}} |\operatorname{cov}(\xi_{\mathbf{i}}^3, \xi_{\mathbf{j}})| \le C\sum_{\mathbf{i}\ne\mathbf{j}} \alpha^{\frac{a-1}{a}}(\|\mathbf{i}-\mathbf{j}\|,1,1)\, \|\xi_{\mathbf{i}}^3\|_{2a}\, \|\xi_{\mathbf{j}}\|_{2a} \\
&\le C|\mathcal{D}|\sum_{r=1}^\infty r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)\, M^2\, \|\xi_{\mathbf{i}}\|_{2a}^2 \\
&\le C|\mathcal{D}|\sum_{r=1}^\infty r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1) M^{4-\frac{2}{a}}\left(\mathbb{E}\xi_{\mathbf{i}}^2\right)^{\frac{1}{a}}.
\end{aligned}$$

$$
\begin{aligned}
J_3 &\le \sum_{\mathbf{i}\neq\mathbf{j}} \big(|\operatorname{cov}(\xi_{\mathbf{i}}^2,\xi_{\mathbf{j}}^2)| + \mathbb{E}\xi_{\mathbf{i}}^2\,\mathbb{E}\xi_{\mathbf{j}}^2\big) \\
&\le C\sum_{\mathbf{i}\neq\mathbf{j}} \alpha^{\frac{a-1}{a}}(\|\mathbf{i}-\mathbf{j}\|,1,1)\,\|\xi_{\mathbf{i}}^2\|_{2a}\,\|\xi_{\mathbf{j}}^2\|_{2a} + |\mathcal{D}|^2\big(\mathbb{E}\xi_{\mathbf{i}}^2\big)^2 \\
&\le C|\mathcal{D}|\sum_{r=1}^{\infty} r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)M^{4-\frac{2}{a}}\big(\mathbb{E}\xi_{\mathbf{i}}^2\big)^{\frac{1}{a}} + |\mathcal{D}|^2 M^{4-\frac{4}{a}}\big(\mathbb{E}\xi_{\mathbf{i}}^2\big)^{\frac{2}{a}}\,.
\end{aligned}
$$

For J_4 let r be the greatest distance between subsets of $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$. Then we have two cases. In the first case r is the distance of $\{\mathbf{i}\}$ and $\{\mathbf{j},\mathbf{k}\}$. If $r=\|\mathbf{i}-\mathbf{j}\|$, say, then $\|\mathbf{i}-\mathbf{k}\|\le 2r$. In the second case r is the distance of $\{\mathbf{j}\}$ and $\{\mathbf{i},\mathbf{k}\}$, say. If $r=\|\mathbf{j}-\mathbf{i}\|$, say, then $\|\mathbf{j}-\mathbf{k}\|\le 2r$. Therefore, if we separate terms according to the greatest distance, we obtain the following (the first sum represents the first case while the second sum represents the second case):

$$
\begin{aligned}
J_4 &\le \sum_{\mathbf{i}\neq\mathbf{j}\neq\mathbf{k}} \big(|\operatorname{cov}(\xi_{\mathbf{i}}^2,\xi_{\mathbf{j}}\xi_{\mathbf{k}})| + \mathbb{E}\xi_{\mathbf{i}}^2|\,\mathbb{E}\xi_{\mathbf{j}}\xi_{\mathbf{k}}|\big) + \sum_{\mathbf{i}\neq\mathbf{j}\neq\mathbf{k}} |\operatorname{cov}(\xi_{\mathbf{i}}^2\xi_{\mathbf{k}},\xi_{\mathbf{j}})| \\
&\le C|\mathcal{D}|\sum_{r=1}^{\infty} r^{2d-1}\alpha^{\frac{a-1}{a}}(r,1,2)\,\|\xi_{\mathbf{i}}^2\|_{2a}\,\|\xi_{\mathbf{j}}\xi_{\mathbf{k}}\|_{2a} \\
&+ C|\mathcal{D}|^2\sum_{r=1}^{\infty} r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)\,\mathbb{E}\xi_{\mathbf{i}}^2\,\|\xi_{\mathbf{j}}\|_{2a}\,\|\xi_{\mathbf{k}}\|_{2a} \\
&+ C|\mathcal{D}|\sum_{r=1}^{\infty} r^{2d-1}\alpha^{\frac{a-1}{a}}(r,1,2)\,\|\xi_{\mathbf{i}}^2\xi_{\mathbf{k}}\|_{2a}\,\|\xi_{\mathbf{j}}\|_{2a} \\
&\le C|\mathcal{D}|\sum_{r=1}^{\infty} r^{2d-1}\alpha^{\frac{a-1}{a}}(r,1,2)\,M^{4-\frac{2}{a}}\,\big(\mathbb{E}\xi_{\mathbf{i}}^2\big)^{\frac{1}{a}} \\
&+ C|\mathcal{D}|^2\sum_{r=1}^{\infty} r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)\,M^{4-\frac{4}{a}}\,\big(\mathbb{E}\xi_{\mathbf{i}}^2\big)^{\frac{2}{a}}\,.
\end{aligned}
$$

For J_5 let r be the greatest distance between subsets of $\{\mathbf{i},\mathbf{j},\mathbf{k},\mathbf{l}\}$. Then we have two cases. In the first case r is the distance of a one-point set and a three-point set, $\{\mathbf{i}\}$ and $\{\mathbf{j},\mathbf{k},\mathbf{l}\}$, say. If $r=\|\mathbf{i}-\mathbf{j}\|$, say, then at least one of the remaining points is closer to $\mathbf{j}$ than r: $\|\mathbf{j}-\mathbf{k}\|\le r$. Moreover, the remaining point is closer to $\mathbf{j}$ than $2r$: $\|\mathbf{j}-\mathbf{l}\|\le 2r$. Therefore, for this part of J_5 we have

$$
\begin{aligned}
J_5' &\le \sum_{\mathbf{i}\neq\mathbf{j}\neq\mathbf{k}\neq\mathbf{l}} |\operatorname{cov}(\xi_{\mathbf{i}},\xi_{\mathbf{j}}\xi_{\mathbf{k}}\xi_{\mathbf{l}})| \le C|\mathcal{D}|\sum_{r=1}^{\infty} r^{3d-1}\alpha^{\frac{a-1}{a}}(r,1,3)\,M^2\,\|\xi_{\mathbf{i}}\|_{2a}\,\|\xi_{\mathbf{j}}\|_{2a} \\
&\le C|\mathcal{D}|\sum_{r=1}^{\infty} r^{3d-1}\alpha^{\frac{a-1}{a}}(r,1,3)\,M^{4-\frac{2}{a}}\,\big(\mathbb{E}\xi_{\mathbf{i}}^2\big)^{\frac{1}{a}}\,.
\end{aligned}
$$

In the second case for J_5, r is the distance of two two-point sets. Assume that the sets are $\{\mathbf{i},\mathbf{k}\}$ and $\{\mathbf{j},\mathbf{l}\}$, moreover $\mathbf{i}$ and $\mathbf{j}$ are the closest points of these sets: $r=\|\mathbf{i}-\mathbf{j}\|$. Then the two remaining points are closer to one of them, say, to $\mathbf{i}$, than $2r$: $\|\mathbf{i}-\mathbf{k}\|\le 2r$, $\|\mathbf{i}-\mathbf{l}\|\le 2r$. (Otherwise the distance of $\{\mathbf{i},\mathbf{j}\}$ and $\{\mathbf{k},\mathbf{l}\}$ would be greater than r.) Therefore, for the second part of J_5, we have

$$
J_5'' \le C\sum_{\mathbf{i}}\sum_{\mathbf{j}}\sum_{\substack{\|\mathbf{k}-\mathbf{i}\|\le 2\|\mathbf{i}-\mathbf{j}\| \\ \|\mathbf{l}-\mathbf{i}\|\le 2\|\mathbf{i}-\mathbf{j}\|}} \{|\operatorname{cov}(\xi_{\mathbf{i}}\xi_{\mathbf{k}},\xi_{\mathbf{j}}\xi_{\mathbf{l}})| + |\mathbb{E}\xi_{\mathbf{i}}\xi_{\mathbf{k}}|\,|\mathbb{E}\xi_{\mathbf{j}}\xi_{\mathbf{l}}|\}\,.
$$

Here the second term is bounded by

$$
C\left\{\sum\nolimits_{\mathbf{i}\in\mathcal{D}}\sum\nolimits_{\mathbf{k}\in\mathcal{D}} |\mathbb{E}\xi_{\mathbf{i}}\xi_{\mathbf{k}}|\right\}^2 = C\left\{\sum\nolimits_{\mathbf{i}\in\mathcal{D}}\sum\nolimits_{\mathbf{k}\in\mathcal{D}} |\operatorname{cov}(\xi_{\mathbf{i}},\xi_{\mathbf{k}})|\right\}^2\,.
$$

Therefore

$$\begin{aligned} J_5'' &\le C|\mathcal{D}|\sum_{r=1}^{\infty} r^{3d-1}\alpha^{\frac{a-1}{a}}(r,2,2)\, M^2\, \|\xi_{\mathbf{i}}\|_{2a}\, \|\xi_{\mathbf{j}}\|_{2a} \\ &+ \Big(C|\mathcal{D}|\sum_{r=1}^{\infty} r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)\, \|\xi_{\mathbf{i}}\|_{2a}\, \|\xi_{\mathbf{k}}\|_{2a}\Big)^2 \\ &\le C|\mathcal{D}|\sum_{r=1}^{\infty} r^{3d-1}\alpha^{\frac{a-1}{a}}(r,2,2)\, M^{4-\frac{2}{a}}\left(\mathbb{E}\xi_{\mathbf{i}}^2\right)^{\frac{1}{a}} \\ &+ C|\mathcal{D}|^2\Big(\sum_{r=1}^{\infty} r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)\Big)^2 M^{4-\frac{4}{a}}\left(\mathbb{E}\xi_{\mathbf{i}}^2\right)^{\frac{2}{a}}. \end{aligned}$$

Finally,

$$\begin{aligned} &\mathbb{E}\Big\{\sum_{\mathbf{i}\in\mathcal{D}}\xi_{\mathbf{i}}\Big\}^4 \le C|\mathcal{D}|\Big\{1+\sum_{r=1}^{\infty} r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)+\sum_{r=1}^{\infty} r^{2d-1}\alpha^{\frac{a-1}{a}}(r,1,2) \\ &+\sum_{r=1}^{\infty} r^{3d-1}\alpha^{\frac{a-1}{a}}(r,1,3)+\sum_{r=1}^{\infty} r^{3d-1}\alpha^{\frac{a-1}{a}}(r,2,2)\Big\} M^{4-\frac{2}{a}}\left(\mathbb{E}\xi_{\mathbf{i}}^2\right)^{\frac{1}{a}} \\ &+C|\mathcal{D}|^2\Big\{1+\sum_{r=1}^{\infty} r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)+\Big(\sum_{r=1}^{\infty} r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)\Big)^2\Big\} M^{4-\frac{4}{a}}\left(\mathbb{E}\xi_{\mathbf{i}}^2\right)^{\frac{2}{a}}. \end{aligned}$$

It is easy to see that we can modify the above argument so that instead of $\sum_{r=1}^{\infty} r^{3d-1}$ we can write $|\mathcal{D}|\sum_{r=1}^{\infty} r^{2d-1}$. Therefore we obtain

$$\begin{aligned} \mathbb{E}\Big\{\sum_{\mathbf{i}\in\mathcal{D}}\xi_{\mathbf{i}}\Big\}^4 &\le C|\mathcal{D}|^2\Big\{1+\sum_{r=1}^{\infty} r^{2d-1}\alpha^{\frac{a-1}{a}}(r,1,3) \\ &+ \sum_{r=1}^{\infty} r^{2d-1}\alpha^{\frac{a-1}{a}}(r,2,2)+\Big(\sum_{r=1}^{\infty} r^{d-1}\alpha^{\frac{a-1}{a}}(r,1,1)\Big)^2\Big\} M^{4-\frac{2}{a}}\left(\mathbb{E}\xi_{\mathbf{i}}^2\right)^{\frac{1}{a}}. \end{aligned}$$

The $\Lambda = 1$ case follows from the above calculation. The infill case follows from the integer lattice case. The field $\xi_{\mathbf{i}}$, $\mathbf{i}\in\mathcal{D}$, where $\mathcal{D}\subset(\mathbb{Z}/\Lambda)^d$, $\Lambda>1$, can be interpreted as a field with integer lattice indices: just multiply the parameter $\mathbf{i}$ by Λ and at the same time use the mixing coefficient $\alpha(r/\Lambda, ., .)$ for parameter subsets of distance r. □

Proof of Theorem 2.1. We use the version of Bernstein's method applied in Bosq, Merlevède and Peligrad [2].

Let $a>1$ be the constant in the theorem and let

$$0<\gamma<\min\left\{\frac{a-1}{3ad}\,,\ \frac{a}{3(a-1)}\right\}. \tag{3.5}$$

Let $A>1$ be fixed. Let $\beta_1=\max\Big\{1\,,\ \max_{k\ge1}\Big\{k\alpha_k^{\frac{a-1}{2da}}\Big\}\Big\}$ and

$$\beta_n=\max\left\{\frac{1}{n^{3/(3\gamma+1)}}\,,\ \max_{k\ge n}\Big\{k\alpha_k^{\frac{a-1}{2da}}\Big\}\,,\ \frac{\beta_{n-1}}{A}\right\}, \tag{3.6}$$

for $n=2,3,\ldots$. Here $\alpha_k=\alpha(k)$, $k=1,2,\ldots$, are the α-mixing coefficients of the underlying random field $\{\xi_{\mathbf{t}}\ :\ \mathbf{t}\in T_\infty\}$. Then

$$\beta_n\ge\frac{1}{n^{3/(3\gamma+1)}}\,,\quad \beta_n\ \text{ is decreasing.} \tag{3.7}$$

We prove that

$$\beta_n^{-d}n^{2d}\alpha_n^{\frac{a-1}{a}}\to0\ \text{ as }\ n\to\infty\,. \tag{3.8}$$

Condition (2.3), i.e. $\int_0^\infty s^{2d-1}\alpha^{\frac{a-1}{a}}(s)ds < \infty$ implies $\sum_{n=1}^\infty n^{2d-1}\alpha_n^{\frac{a-1}{a}} < \infty$. Therefore one can prove that $n^{2d}\alpha_n^{\frac{a-1}{a}} \to 0$. Then $n\alpha_n^{\frac{a-1}{2da}} \to 0$. So we have

$$\beta_n \geq \max_{k\geq n}\left\{k\alpha_k^{\frac{a-1}{2da}}\right\} \to 0\,, \quad \text{as } n\to\infty\,.$$

Furthermore

$$0 < \beta_n^{-d}n^{2d}\alpha_n^{\frac{a-1}{a}} \leq n^d\alpha_n^{\frac{a-1}{2a}} \to 0\,, \quad \text{as } n\to\infty\,.$$

Therefore we have (3.7), (3.8) and $\beta_n \downarrow 0$.

Now $|T_n|$ is the volume of the d-dimensional rectangle T_n. We have

$$|T_n| = |\mathcal{D}_n|/\Lambda_n^d,$$

where $|\mathcal{D}_n|$ is the cardinality of $\mathcal{D}_n$. Denote $[\,.\,]$ the integer part. Let

$$m_n = \left[|T_n|^{\frac{a-1}{d(3a-1)}}\right]\,,\quad p_n = \left[m_n\,\beta^{\gamma}_{[\sqrt{m_n}]}\right]\,,\quad q_n = \left[p_n\,\beta^{1/3}_{[\sqrt{m_n}]}\right]\,. \tag{3.9}$$

Now we prove that

$$p_n \to \infty,\quad q_n\to\infty,\quad \text{and}\ \ p_n/q_n\to\infty. \tag{3.10}$$

Indeed,

$$p_n \geq m_n\,\beta^{\gamma}_{[\sqrt{m_n}]} - 1 \geq m_n\,\left(\sqrt{m_n}\right)^{-\frac{3\gamma}{3\gamma+1}} - 1 = m_n^{1-\frac{3\gamma}{2(3\gamma+1)}} - 1 = m_n^{\frac{3\gamma+2}{2(3\gamma+1)}} - 1 \to \infty,$$

because $m_n\to\infty$. Also

$$\begin{aligned} q_n &> p_n\,\beta^{1/3}_{[\sqrt{m_n}]} - 1 \geq \left(m_n\,\beta^{\gamma}_{[\sqrt{m_n}]} - 1\right)\beta^{1/3}_{[\sqrt{m_n}]} - 1 \\ &= m_n\,\beta^{\gamma+1/3}_{[\sqrt{m_n}]} - \beta^{1/3}_{[\sqrt{m_n}]} - 1 \geq m_n\,\left[\sqrt{m_n}\,\right]^{\frac{-3}{3\gamma+1}\left(\gamma+\frac{1}{3}\right)} - \beta^{1/3}_{[\sqrt{m_n}]} - 1 \\ &= \frac{m_n}{\left[\sqrt{m_n}\,\right]} - \beta^{1/3}_{[\sqrt{m_n}]} - 1 \geq \sqrt{m_n} - o(1) - 1\,. \end{aligned}$$

So we obtained that $q_n > \sqrt{m_n} - o(1) - 1$, i.e. $q_n > \left[\sqrt{m_n}\,\right] - 2$ if n is large enough. Finally,

$$\frac{p_n}{q_n} \geq \beta^{-\frac{1}{3}}_{[\sqrt{m_n}]} \to \infty,$$

because $\beta_n\to 0$.

Now we divide T_n into big and small blocks. First divide T_n into d-dimensional cubes each having size $(p_n+q_n)^d$. Let k_n denote the number of these cubes. Then divide each cube into 2^d d-dimensional rectangles (called a family of rectangles). The largest one of these rectangles is of size p_n^d. This will be a large block. The other ones of sizes $p_n^{d-1}q_n,\dots,p_nq_n^{d-1},q_n^d$ will be the small blocks. However, at the 'border' of T_n we have to make blocks with sizes different from the ones just listed. Namely, we can make big blocks having edge lengths between p_n and $2p_n$. Moreover, the small blocks may have edge lengths between p_n and $2p_n$ but each small block has at least one edge with length between q_n and $2q_n$.

We prove that the contribution of the small blocks converges to 0. We deal with a fixed coordinate of $Y_n(\mathbf{k})$. Recall that k_n is the number of big blocks (it is also the number of 2^d-member families consisting of big and small blocks).

Let $V_{l,j}$ be the sum of random variables having indices in the (l,j)th small block. Here j shows the type of the small block, $j=2,\dots,2^d$, while l is the index of the rectangle family, $l=1,\dots,k_n$.

We show that the L^2-norm of the normed total sum in the small blocks converges to 0. We have

$$L = \frac{1}{\Lambda_n^d|\mathcal{D}_n|}\mathbb{E}\left\{\sum\nolimits_{j=2}^{2^d}\sum\nolimits_{l=1}^{k_n}V_{l,j}\right\}^2 \leq \frac{2^d-1}{\Lambda_n^d|\mathcal{D}_n|}\sum\nolimits_{j=2}^{2^d}\mathbb{E}\left\{\sum\nolimits_{l=1}^{k_n}V_{l,j}\right\}^2\,.$$

Here we used $2\mathbb{E}|XY| \leq \mathbb{E}X^2 + \mathbb{E}Y^2$.

We shall calculate an upper bound for $\mathbb{E}\left\{\sum_{l=1}^{k_n} V_{l,\max}\right\}^2$, where $V_{l,\max}$ denotes the sum in the largest small block (i.e. the block of size $p_n^{d-1}q_n$). It will serve as an upper bound for the other small blocks, therefore we can calculate formally

$$\begin{aligned} L &\le \frac{(2^d-1)^2}{\Lambda_n^d|\mathcal{D}_n|}\mathbb{E}\left\{\sum_{l=1}^{k_n} V_{l,\max}\right\}^2 \\ &\le \frac{(2^d-1)^2}{\Lambda_n^d|\mathcal{D}_n|}\sum_{l=1}^{k_n} \mathbb{E}V_{1,\max}^2 + \frac{c(2^d-1)^2}{\Lambda_n^d|\mathcal{D}_n|}k_n\sum_{l=1}^{\infty} l^{d-1}\alpha_{lp_n}^{1/2}\left\{\mathbb{E}V_{1,\max}^4\right\}^{1/2} \\ &= L_1+L_2\,, \end{aligned}$$

where we used the covariance inequality. We have

$$L_1=\left\{\frac{c\Lambda_n^d k_n p_n^{d-1}q_n}{|\mathcal{D}_n|}\right\}\left\{\frac{1}{\Lambda_n^{2d}p_n^{d-1}q_n}\mathbb{E}V_{1,\max}^2\right\}.$$

Here the first factor is bounded by $cq_n/p_n \to 0$. The second factor, by (2.6), converges to a finite limit. Therefore $L_1\to 0$.

By Lemma 3.1, (2.5), and using that (2.3) implies that $\sum_{l=1}^{\infty} l^{d-1}\alpha_{lp_n}^{1/2}\le cp_n^{-\frac{da}{a-1}}$,

$$\begin{aligned} L_2 &\le \frac{c(2^d-1)^2}{\Lambda_n^d|\mathcal{D}_n|}k_n\sum_{l=1}^{\infty} l^{d-1}\alpha_{lp_n}^{1/2}\left\{\left(p_n^{d-1}q_n\Lambda_n^d\right)^2\Lambda_n^{2d}M_n^{4-\frac{2}{a}}\right\}^{1/2} \\ &\le \frac{ck_np_n^{d-1}q_n\Lambda_n^d\Lambda_n^dM_n^{2-\frac{1}{a}}}{\Lambda_n^d\,|\mathcal{D}_n|\,p_n^{\frac{da}{a-1}}} \le \frac{cq_nM_n^{2-\frac{1}{a}}}{p_np_n^{\frac{da}{a-1}}}\,. \end{aligned}$$

Using the definitions of p_n and q_n,

$$L_2\le c\beta_{[\sqrt{m_n}]}^{\frac{1}{3}}\frac{M_n^{2-\frac{1}{a}}}{\left(m_n\beta_{[\sqrt{m_n}]}^{\gamma}\right)^{\frac{da}{a-1}}}=c\left\{\beta_{[\sqrt{m_n}]}^{\frac{1}{3}-\gamma\frac{da}{a-1}}\right\}\left\{\frac{M_n^{2-\frac{1}{a}}}{m_n^{\frac{da}{a-1}}}\right\}.$$

Here the first factor converges to 0 because $\beta_n\to 0$ and its exponent is positive. The second factor, by the definition of m_n, is smaller than

$$c\frac{M_n^{2-\frac{1}{a}}}{\left(|T_n|^{\frac{a-1}{d(3a-1)}}\right)^{\frac{da}{a-1}}}=c\frac{M_n^{2-\frac{1}{a}}}{|T_n|^{\frac{a}{3a-1}}}\,.$$

By (2.7), this is bounded. Therefore $L_2\to 0$.

We remark that each small block at the 'border' contains at most 2^d times more terms than the corresponding one 'inside' the domain T_n. So their contribution can also be covered by the above calculation.

Now we turn to the big blocks. We use

$$\left|\mathbb{E}e^{it(\eta_1+\cdots+\eta_n)}-\mathbb{E}e^{it(\widetilde{\eta}_1+\cdots+\widetilde{\eta}_n)}\right|\le cn\alpha\,,$$

where $\eta_1,\ldots,\eta_n$ are dependent having maximal α-mixing coefficient α between two disjoint subsets, $\widetilde{\eta}_1,\ldots,\widetilde{\eta}_n$ are independent, moreover $\widetilde{\eta}_l$ has the same distribution as η_l, $l=1,\ldots,n$.

Therefore the difference of the characteristic function of the sum of the big block terms and that of independent blocks is less, than $ck_n\alpha_{q_n}$. Now

$$k_n\alpha_{q_n}\le c\frac{|T_n|}{p_n^d}\left(\frac{\beta_{q_n}^d}{q_n^{2d}}\right)^{\frac{a}{a-1}}$$

because (3.8) implies that $\alpha_{q_n}^{\frac{a-1}{a}} \le c\beta_{q_n}^d / q_n^{2d}$. Therefore

$$\begin{aligned} k_n \alpha_{q_n} &\le c \frac{|T_n|}{p_n^d} \left(\frac{\beta_{q_n}^d}{p_n^{2d} \beta_{[\sqrt{m_n}]}^{\frac{2d}{3}}} \right)^{\frac{a}{a-1}} = c \frac{|T_n|}{p_n^{\frac{d(3a-1)}{a-1}}} \beta_{q_n}^{\frac{da}{a-1}} \beta_{[\sqrt{m_n}]}^{-\frac{2ad}{3(a-1)}} \\ &\le c \left\{ \frac{|T_n|}{m_n^{\frac{d(3a-1)}{a-1}}} \right\} \left\{ \beta_{[\sqrt{m_n}]}^{-\gamma \frac{d(3a-1)}{a-1} - \frac{2ad}{3(a-1)}} \beta_{q_n}^{\frac{da}{a-1}} \right\}. \end{aligned}$$

The limit of the first factor is 1. The second factor is less than $c\beta_{[\sqrt{m_n}]}^{\frac{ad}{3(a-1)} - \gamma\frac{d(3a-1)}{a-1}} A^{\frac{2da}{a-1}}$. This expression converges to 0 because $\beta_n \to 0$ and the exponent is positive.

Therefore, we can consider independent big blocks. We shall apply Lyapunov's theorem. Let $b \in \mathbb{R}^d$ be an arbitrary nonzero vector, then we use $b^\top Y_n(\mathbf{k})$, i.e. a linear combination of the coordinates of $Y_n(\mathbf{k})$. Let U_i denote the sum of these linear combinations in the i-th big block. Using that $U_1, \dots, U_{k_n}$ are independent,

$$\operatorname{var}\left\{ \frac{\sum_{i=1}^{k_n} U_i}{\sqrt{\Lambda_n^d |\mathcal{D}_n|}} \right\} = \left\{ \frac{k_n p_n^d \Lambda_n^d}{|\mathcal{D}_n|} \right\} \operatorname{var}\left\{ \frac{U_i}{\sqrt{p_n^d \Lambda_n^{2d}}} \right\}.$$

Here the first factor converges to 1. The second factor, by (2.6), converges to $b^\top \Sigma b > 0$. Therefore, Lyapunov's condition is

$$U = \sum_{i=1}^{k_n} \mathbb{E} \left\{ \frac{U_i}{\sqrt{\Lambda_n^d |\mathcal{D}_n|}} \right\}^4 \to 0\,.$$

But this is true since, by Lemma 3.1,

$$\begin{aligned} U &\le k_n c \frac{(p_n^d \Lambda_n^d)^2 \Lambda_n^{2d} M_n^{4-\frac{2}{a}}}{\Lambda_n^{2d} |\mathcal{D}_n|^2} \le k_n c \frac{p_n^{2d} \Lambda_n^{2d} M_n^{4-\frac{2}{a}}}{(|T_n|\Lambda_n^d)(k_n p_n^d \Lambda_n^d)} \\ &= \frac{c p_n^d M_n^{4-\frac{2}{a}}}{|T_n|} \le c\beta_{[\sqrt{m_n}]}^{\gamma d} \frac{m_n^d}{|T_n|} M_n^{4-\frac{2}{a}} \le c \left\{ \beta_{[\sqrt{m_n}]}^{\gamma d} \right\} \left\{ \frac{M_n^{\frac{2(2a-1)}{a}}}{|T_n|^{\frac{2a}{3a-1}}} \right\}. \end{aligned}$$

Here the first factor converges to 0. By (2.7), the second factor is bounded. So $U \to 0$. Therefore Lyapunov's condition is satisfied. The theorem is proved. □

4. Application: asymptotic normality of kernel-type density estimators

Now we apply Theorem 2.1 to kernel-type density estimators. We obtain the asymptotic normality of the kernel type density estimator when the sets of locations of observations become more and more dense in an increasing sequence of domains. It turns out that the covariance structure of the limiting normal distribution depends on the ratio of the bandwidth of the kernel estimator and the diameter of the subdivision. This is an important issue when we approximate the integral in the estimator $f_{T_n}(x) = \frac{1}{|T_n|} \frac{1}{h_n} \int_{T_n} K\left(\frac{x - \xi_{\mathbf{t}}}{h_n}\right) d\mathbf{t}$ by a sum, i.e. in practical applications we use an estimator of the form $f_{\mathcal{D}_n}(x) = \frac{1}{|\mathcal{D}_n|} \frac{1}{h_n} \sum_{\mathbf{i} \in \mathcal{D}_n} K\left(\frac{x - \xi_{\mathbf{i}}}{h_n}\right)$.

Let $\xi_{\mathbf{t}}$, $\mathbf{t} \in T_\infty$, be a strictly stationary random field with unknown continuous marginal density function f. We shall estimate f from the data $\xi_{\mathbf{i}}$, $\mathbf{i} \in \mathcal{D}_n$.

A function $K\colon \mathbb{R} \to \mathbb{R}$ will be called a kernel if K is a bounded continuous symmetric density function (with respect to the Lebesgue measure),

$$(4.1) \qquad \lim_{|u| \to \infty} |u| K(u) = 0, \qquad \int_{-\infty}^{+\infty} u^2 K(u)\, du < \infty\,.$$

If K is a kernel and $h_n > 0$, then the kernel-type density estimator is

$$f_n(x) = \frac{1}{|\mathcal{D}_n|} \frac{1}{h_n} \sum\nolimits_{\mathbf{i} \in \mathcal{D}_n} K\left(\frac{x - \xi_{\mathbf{i}}}{h_n}\right), \qquad x \in \mathbb{R}. \tag{4.2}$$

Let $f_{\mathbf{u}}(x, y)$ be the joint density function of $\xi_{\mathbf{0}}$ and $\xi_{\mathbf{u}}$, $\mathbf{u} \neq \mathbf{0}$. Denote by $\mathbb{R}^d_{\mathbf{0}}$ the set $\mathbb{R}^d \setminus \{\mathbf{0}\}$. Let

$$g_{\mathbf{u}}(x, y) = f_{\mathbf{u}}(x, y) - f(x)f(y), \quad \mathbf{u} \in \mathbb{R}^d_{\mathbf{0}},\ x, y \in \mathbb{R}. \tag{4.3}$$

We assume that $g_{\mathbf{u}}(x, y)$ is continuous in x and y for each fixed $\mathbf{u}$. Let $g_{\mathbf{u}}$ denote $g_{\mathbf{u}}(x, y)$ as a function $g\colon \mathbb{R}^d_{\mathbf{0}} \to C(\mathbb{R}^2)$, i.e. a function with values in $C(\mathbb{R}^2)$, the space of continuous real-valued functions over $\mathbb{R}^2$. Let $\|g_{\mathbf{u}}\| = \sup_{(x,y)\in\mathbb{R}^2} |g_{\mathbf{u}}(x, y)|$ be the norm of $g_{\mathbf{u}}$.

For a fixed positive integer m and fixed distinct real numbers $x_1, \dots, x_m$, introduce the notation

$$\sigma(x_i, x_j) = \int_{\mathbb{R}^d_{\mathbf{0}}} g_{\mathbf{u}}(x_i, x_j)\, d\mathbf{u}, \quad i, j = 1, \dots, m, \tag{4.4}$$

$$\Sigma^{(m)} = \big(\sigma(x_i, x_j)\big)_{1\le i,j\le m}. \tag{4.5}$$

Theorem 4.1. *Assume that $g_{\mathbf{u}}$ is Riemann integrable (as a function $g\colon \mathbb{R}^d_{\mathbf{0}} \to C(\mathbb{R}^2)$) on each bounded closed d-dimensional rectangle $R \subset \mathbb{R}^d_{\mathbf{0}}$, moreover $\|g_{\mathbf{u}}\|$ is directly Riemann integrable (as a function $\|g\|\colon \mathbb{R}^d_{\mathbf{0}} \to \mathbb{R}$). Let $x_1, \dots, x_m$ be distinct real numbers and assume that $\Sigma^{(m)}$ in (4.5) is positive definite. Suppose that there exists $1 < a < \infty$ such that (2.3) is satisfied and*

$$(h_n)^{-1} \le c|T_n|^{\frac{a^2}{(3a-1)(2a-1)}} \quad \text{for each} \quad n. \tag{4.6}$$

Assume that $\lim_{n\to\infty} \Lambda_n = \infty$ and $\lim_{n\to\infty} h_n = 0$.

If

$$\lim_{n\to\infty} \frac{1}{\Lambda_n^d} \frac{1}{h_n} = 0, \tag{4.7}$$

then

$$\sqrt{\frac{|\mathcal{D}_n|}{\Lambda_n^d}} \Big\{ \big(f_n(x_i) - \mathbb{E} f_n(x_i)\big),\, i = 1, \dots, m \Big\} \Rightarrow \mathcal{N}(0, \Sigma^{(m)}) \quad \text{as} \quad n \to \infty. \tag{4.8}$$

If, instead of (4.7),

$$\lim_{n\to\infty} \frac{1}{\Lambda_n^d} \frac{1}{h_n} = L > 0 \tag{4.9}$$

is satisfied, then (4.8) remains valid if $\Sigma^{(m)}$ is replaced by

$$\Sigma'^{(m)} = \Sigma^{(m)} + D, \tag{4.10}$$

where D is a diagonal matrix with diagonal elements

$$L f(x_i) \int_{-\infty}^{+\infty} K^2(u)\, du, \quad i = 1, \dots, m.$$

If $f(x)$ has bounded second derivative and $\lim_{n\to\infty} |T_n| h_n^4 = 0$, then in (4.8) $\mathbb{E} f_n(x_i)$ can be replaced by $f(x_i)$, $i = 1, \dots, m$, and both of the above statements remain valid.

The proof of Theorem 4.1 is given in Fazekas and Chuprunov [11]. In that paper numerical evidence is also given for the interesting and important phenomenon following from the form of the limiting covariance matrix.

Acknowledgement. We are grateful to the referee for careful reading the manuscript and for helpful comments.

References

[1] Bosq, D.: *Nonparametric Statistics for Stochastic Processes.* Springer, New York–Berlin–Heidelberg, 1998.

[2] Bosq, D., Merlevède, F. and Peligrad, M.: Asymptotic normality for density kernel estimators in discrete and continuous time. *J. Multivariate Analysis*, **68** (1999), 78–95.

[3] Castellana, J. V. and Leadbetter, M. R.: On smoothed probability density estimation for stationary processes. *Stochastic Process. Appl.*, **21** (1986), no.2, 179–193.

[4] Cressie, N. A. C.: *Statistics for Spatial Data.* Wiley, New York, 1991.

[5] Csörgő S. and Mielniczuk, J.: Density estimation under long-range dependence. *Ann. Statist.* **23** (1995), 990–999.

[6] Devroye, L. and Györfi, L.: *Nonparametric Density Estimation. The L_1 View.* Wiley, New York, 1985.

[7] Doukhan, P.: *Mixing. Properties and Examples.* Lecture Notes in Statistics **85**, Springer, New York, 1994.

[8] Fazekas, I.: Limit theorems for the empirical distribution function in the spatial case. *Statistics & Probability Letters*, **62** (2003), 251–262.

[9] Fazekas, I.: A central limit theorem for kernel density estimators. In *Book of Abstracts of 24th European Meeting of Statisticians*, p. 186, Prague, 2002.

[10] Fazekas, I. and Kukush, A. G.: Infill asymptotics inside increasing domains for the least squares estimator in linear models. *Stat. Inf. Stoch. Proc.* **3** (2000), 199–223.

[11] Fazekas, I. and Chuprunov, A.: Asymptotic normality of kernel type density estimators for random fields. *Manuscript* (submitted for publication), Debrecen, 2003.

[12] Guillou A. and Merlevède, F.: Estimation of the asymptotic variance of kernel density estimators for continuous time processes. *J. Multivariate Analysis*, **79** (2001), 114–137.

[13] Guyon, X.: *Random Fields on a Network. Modeling, Statistics, and Applications.* Springer, New York, 1995.

[14] Kutoyants, Y. A.: Some problems of nonparametric estimation by observations of ergodic diffusion processes. *Statistics & Probability Letters*, **32** (1997), 311–320.

[15] Lahiri, S. N.: Asymptotic distribution of the empirical spatial cumulative distribution function predictor and prediction bands based on a subsampling method. *Probab. Theory Relat. Fields*, **114** (1999), 55–84.

[16] Lahiri, S. N.: Central limit theorems for weighted sums of a spatial process under a class of stochastic and fixed designs. *Sankhya*, **65** (2003), 356–388.

[17] Lahiri, S. N., Kaiser, M. S., Cressie, N., and Hsu, N-J.: Prediction of spatial cumulative distribution functions using subsampling. *J. Amer. Statist. Assoc.* **94** (1999), no. 445, 86–110.

[18] Liebscher E.: Central limit theorems for α-mixing triangular arrays with applications to nonparametric statistics. *Math. Methods Statist.* **10** (2001), 194–214.

[19] Lin, Z. and Lu, C.: *Limit Theory for Mixing Dependent Random Variables.* Science Press, New York–Beijing, and Kluwer, Dordrecht–Boston–London, 1996.

[20] Maltz, A. L.: On the central limit theorem for nonuniform φ-mixing random fields. *J. Theoretical Probability*, **12** (1999), 643–660.

[21] Prakasa Rao, B. L. S.: *Nonparametric Functional Estimation.* Academic Press, New York, 1983.

[22] Putter, H. and Young, G. A.: On the effect of covariance function estimation on the accuracy of kriging predictors. *Bernoulli* **7** (2001), no. 3, 421–438.

Received February 10, 2004.

István Fazekas,
Institute of Informatics,
University of Debrecen,
4010 Debrecen, P.O. Box 12.
Hungary
E-mail address: `fazekasi@inf.unideb.hu`

Alexey Chuprunov,
Department of Mathematical Statistics and Probability,
Research Institute of Mathematics and Mechanics,
Kazan State University
Universitetskaya 17, 420008 Kazan,
Russia
E-mail address: `alexey.chuprunov@ksu.ru`