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DISCRETE APPROXIMATION OF THE SOLUTION OF THE DIRICHLET PROBLEM BY DISCRETE MEANS

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Dedicated to Professor W. Wade on his 60^{th} birthday

Abstract. In paper [7] using spherical functions we had constructed continuous and discrete approximation processes on the sphere S^2. In this paper we show that these processes give approximations of the solution of three dimensional Dirichlet problem. We give also an estimation for the rate of the convergence.

1. Introduction: The three dimensional Laplace equation and Dirichlet problem

Let consider the three dimensional Laplace equation

$$\frac{\partial^2\Phi}{\partial x_1^2}+\frac{\partial^2\Phi}{\partial x_2^2}+\frac{\partial^2\Phi}{\partial x_3^2}=0,$$

and let $x=(x_1,x_2,x_3)=(\rho\cos\theta,\rho\sin\theta\cos\varphi,\rho\sin\theta\sin\varphi)=\rho v$ and $u=(u_1,u_2,u_3)=(\cos\theta',\sin\theta'\cos\varphi',\sin\theta'\sin\varphi')$.

It is known that the spherical polynomials

$$P_m^\lambda(\xi)=\sum_{0\geq\ell\geq m/2}(-1)^\ell\frac{\Gamma(m-\ell+\lambda)}{\Gamma(\lambda)\ell!(m-2\ell)!}(2\xi)^{m-2\ell}$$

are generated by

$$(1-2\xi\rho+\rho^2)^{-\lambda}=\sum_{m=0}^{\infty}P_m^\lambda(\xi)\rho^m.$$

For $\lambda=1/2$ $P_m^{1/2}(\xi)=P_m(\xi)$ are the Legendre polynomials. Consequently the Poisson kernel of the three dimensional Laplace equation:

$$\Phi(x,u)=\frac{1-xx'}{(1-2ux'+xx')^{3/2}}$$

can be expressed in the following way

$$\Phi(x,u)=(1-\rho^2)\sum_{m=0}^{\infty}P_m^{3/2}(uv')\rho^m=\sum_{\ell=0}^{\infty}(2\ell+1)\rho^\ell P_\ell^{1/2}(uv')=$$

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$$\sum_{\ell=0}^{\infty}(2\ell+1)\rho^{\ell}P_{\ell}(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')).$$

Theorem A (see [6] pg. 20). *If $f\in C(S^2)$ then the function*

$$f(\rho v)=\int_{S^2}\frac{1-\rho^2}{(1-2\rho uv'+\rho^2)^{3/2}}f(\theta',\varphi')\sin\theta' d\theta' d\varphi'$$

$$=\sum_{\ell=0}^{\infty}(2\ell+1)\rho^{\ell}\int_{S^2}f(\theta',\varphi')P_{\ell}(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi'))\sin\theta' d\theta' d\varphi'$$

is solution of the three dimensional Dirichlet problem in the three dimensional unit sphere.

2. Spherical functions

In this section we will summarize some results connected with spherical functions. In the three dimensional case spherical functions can be introduced as matrix elements t^{ℓ}_{jk} of unitary irreducible representations of the matrix group $SU(2)$ ([11, p. 278]), where

$$SU(2)=\{g\in SL(2): g^*=g^{-1}\}$$

is the set of second order unitary matrices.

For $k=0$ we obtain the classical (zonal) spherical functions. The functions $\{\sqrt{2\ell+1}t^{\ell}_{j0} : \ell=0,1,\dots,-\ell\le j\le\ell\}$ constitute an orthonormal system with respect to the invariant measure on three dimensional unit sphere S^2 and the corresponding Fourier series is convergent in $L^2(S^2)$.

Using the irreducible property of the representation we show that the kernel function of Laplace- Fourier series can be expressed by Legendre polynomials P_{ℓ}. Using this property of the kernel function the approximation processes given in paper [7] can be considered as approximations of the Dirichlet-problem on the three dimensional unit sphere.

If $g\in SU(2)$, then it can be written in the following form :

$$g=\begin{pmatrix}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{pmatrix},\quad |\alpha|^2+|\beta|^2=1,\alpha,\beta\in\mathbb{C}.$$

Every element from $SU(2)$ can be represented with the so called Euler angles, namely there exist $\theta\in(0,\pi),\varphi\in[0,2\pi),\psi\in[-2\pi,2\pi)$ so that:

$$g=\begin{pmatrix}e^{i\varphi/2} & 0\\ 0 & e^{-i\varphi/2}\end{pmatrix}\begin{pmatrix}\cos(\theta/2) & i\sin(\theta/2)\\ i\sin(\theta/2) & \cos(\theta/2)\end{pmatrix}\begin{pmatrix}e^{i\psi/2} & 0\\ 0 & e^{-i\psi/2}\end{pmatrix}$$

$$:=k(\varphi)a(\theta)k(\psi),$$

where $|\alpha|=\cos(\theta/2),\operatorname{Arg}\alpha=(\varphi+\psi)/2,\operatorname{Arg}\beta=(\varphi-\psi+\pi)/2.$

Denote by

$$[t^{\ell}_{jk}]_{j,k\in I_{\ell}}=T^{\ell}$$

$\ell\in\mathbb{N}$, $j\in I_{\ell}:=\{-\ell,-\ell+1,\dots,\ell\}$ the matrix of this representation regarding to a certain base.

If g has the form $g(\theta)=a(\theta)$ let define

$$P^{\ell}_{jk}(\cos\theta):=t^{\ell}_{jk}(a(\theta))=$$

$$\sqrt{\frac{(\ell-j)!}{(\ell+k)!(\ell-k)!(\ell+j)!}}2^{j-l}i^{j-k}(\cos(\theta/2))^{j+k}(\sin(\theta/2))^{j-k} \tag{2.1}$$

$$\times\frac{d^{\ell+j}}{dy^{\ell+j}}\left[(y-1)^{\ell+k}(y+1)^{\ell-k}\right]|_{y=\cos\theta}.$$

If $g = k(\varphi)a(\theta)k(\psi) \in SU(2)$, then the correspondent t^{ℓ}_{jk} has the following form

$$t^{\ell}_{jk}(g(\theta,\varphi,\psi)) = e^{-i(j\varphi+k\psi)} P^{\ell}_{jk}(\cos\theta), \tag{2.2}$$

where (θ,φ,ψ) are the Euler angles.

For $k = 0$ we obtain $t^{\ell}_{j0}(g) = e^{-ij\varphi} P^{\ell}_{j0}(\cos\theta) := Y_{\ell j}(\varphi,\theta)$, $\ell \in \mathbb{N}$, $j \in I_\ell$ which are called spherical functions.

Let denote by S^2 the three dimensional unit sphere. The normalized spherical functions

$$\sqrt{2\ell+1}\, t^{\ell}_{j0}(\varphi,\theta), \quad \ell \in \mathbb{N}, j \in I_\ell$$

form an orthonormal system regarding to the scalar product generated by the following continuous measure on the unit sphere

$$\int_{S^2} f(x) d\mu(x) := \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(\varphi,\theta) \sin\theta d\theta d\varphi. \tag{2.3}$$

i.e.

$$\sqrt{(2\ell+1)(2\ell'+1)} \int_{S^2} t^{\ell}_{m0}(g) \overline{t^{\ell'}_{m'0}}(g) d\mu(g) = \delta_{mm'} \delta_{\ell\ell'}. \tag{2.4}$$

Moreover, every function f from $L^2(S^2)$ can be represented in the following form

$$f(\varphi,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \sum_{k=-\ell}^{k=\ell} C_{\ell k} t^{\ell}_{k0}(\varphi,\theta), \tag{2.5}$$

where

$$C_{\ell k} = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(\varphi',\theta') \overline{t^{\ell}_{k0}(\varphi',\theta')} \sin\theta' d\theta' d\varphi' \tag{2.6}$$

are the Laplace-Fourier coefficients and the series being convergent in $L^2(S^2)$ with respect to the measure on S^2. Let denote by

$$\chi^{\ell}(\theta,\theta',\varphi,\varphi') := \sum_{k=-\ell}^{\ell} \overline{t^{\ell}_{k0}(\theta',\varphi')} t^{\ell}_{k0}(\theta,\varphi) \tag{2.7}$$

the character of the representation T^{ℓ}.

Taking into account that the representation T^{ℓ} of $SU(2)$ is unitary and irreducible (see [11] p. 284), we obtain that

$$\begin{aligned} \chi^{\ell}(\theta,\theta',\varphi,\varphi') &= \chi^{\ell}(h^{-1}g) = \operatorname{spur}(T^{\ell}(h^{-1}g)) = t^{l}_{00}(h^{-1}g) \\ &= P^{l}_{00}(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')) \\ &= P_l(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')). \end{aligned} \tag{2.8}$$

Then the Fourier-Laplace series can be written in the following way

$$f(\varphi,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) \int_{S^2} f(\theta',\varphi') P_\ell(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')) \sin\theta' d\theta' d\varphi'$$

Let denote $g = a(\theta)k(\varphi)$, $h = a(\theta')k(\varphi')$ and denote by $(S_n f)(g)$ the partial sum of the series:

$$\begin{aligned} &(S_n f)(g(\theta,\varphi)) \\ &= \sum_{\ell=0}^{n} (2\ell+1) \int_{S^2} f(\theta',\varphi') P_l(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')) d\sin\theta' d\theta' d\varphi'. \end{aligned} \tag{2.9}$$

In what follows we will give the discrete analogies of (2.3), (2.4), (2.6) and (2.9).

3. Discretisation

Let denote by $\lambda_k^N \in (-1,1), k \in \{1,\dots,N\}$ the roots of Legendre polynomials P_N of order N, and for $j = 1,\dots,N$, let

$$\ell_j^N(x) := \frac{(x-\lambda_1^N)\dots(x-\lambda_{j-1}^N)(x-\lambda_{j+1}^N)\dots(x-\lambda_N^N)}{(\lambda_j^N-\lambda_1^N)\dots(\lambda_j^N-\lambda_{j-1}^N)(\lambda_j^N-\lambda_{j+1}^N)\dots(\lambda_j^N-\lambda_N^N)},$$

be the corresponding fundamental polynomials of Lagrange interpolation. Denote by

$$\mathcal{A}_k^N := \int_{-1}^1 \ell_k^N(x)dx, \quad (1 \le k \le N), \tag{3.1}$$

the corresponding Cristoffel-numbers. In paper [7] we gave the set of nodal points in $[0,\pi]\times[0,2\pi]$ and the discrete measure regarding to the orthonormality property of the spherical functions is also valid. In what follows we will summarise the results mentioned before. Let denote by

$$X = \{z_{kj} = (\theta_k,\varphi_j) = (\arccos\lambda_k^N, \frac{2\pi j}{2N+1}) : k = \overline{1,N}, j = \overline{0,2N}\} \tag{3.2}$$

the set of nodal points, and

$$\mu_N(z_{kj}) := \frac{\mathcal{A}_k^N}{2(2N+1)}.$$

Let define the following discrete integral on the set of nodal points X

$$\int_X f d\mu_N := \sum_{k=1}^N \sum_{j=0}^{2N} f(z_{kj})\mu_N(z_{kj}) = \sum_{k=1}^N \sum_{j=0}^{2N} f(\theta_k,\varphi_j)\frac{\mathcal{A}_k^N}{2(2N+1)}. \tag{3.3}$$

Theorem B. *Let $N \in \mathbb{N}$, $N \ge 1$, then the finite collection of normalized spherical functions*

$$\{\sqrt{2\ell+1}t_{m0}^\ell : S^2 \to \mathbb{C} \mid m \in I_\ell, \ell \in \{0,\dots,N-1\}\}$$

form an orthonormal system on the set of nodal points X regarding to the discrete integral defined by (3.3), i.e.

$$\sqrt{2\ell+1}\sqrt{2\ell'+1}\int_X t_{m0}^\ell \overline{t_{p0}^{\ell'}} d\mu_N = \delta_{\ell\ell'}\delta_{mp} \quad (\ell,\ell' < N, m \in I_\ell, p \in I_{\ell'}). \tag{3.4}$$

In paper [7] it was also proved that (3.3) tends to the invariant measure on $SU(2)$ given by (2.3), namely

Theorem C. *For all $f \in C(S^2)$,*

$$\lim_{N\to\infty} \int_X f d\mu_N = \int_{S^2} f d\mu.$$

4. (C,α) kernel of Laplace-Fourier series

Let denote $g = a(\theta)k(\varphi)$, $h = a(\theta')k(\varphi')$. Let $n < N$ and denote by

$$(I_{N,n}f)(g) = (I_{N,n}f)(\theta,\varphi) := \sum_{\ell=0}^n (2\ell+1) \sum_{k=-\ell}^{\ell} c_{\ell k}^N t_{k0}^\ell(\theta,\varphi), \tag{4.1}$$

the $n-$th partial sum of discrete Laplace-Fourier series of f, where $c_{\ell k}^N$ is given by

$$c_{\ell k}^N = \int_X \overline{t_{k0}^\ell} f d\mu_N = \sum_{m=1}^N \sum_{j=0}^{2N} f(\theta_m,\varphi_j)\overline{t_{k0}^\ell(\theta_m,\varphi_j)}\frac{\mathcal{A}_m^N}{2(2N+1)}. \tag{4.2}$$

$I_{N,n}f$ is n-th partial sum of the discrete Fourier-Laplace series of the function f defined on the unit sphere S^2. We can observe that

$$(4.3)\qquad (I_{N,n}f)(\theta,\varphi) = \int_X f(\theta',\varphi')\left(\sum_{l=0}^{n}(2\ell+1)\sum_{k=-\ell}^{\ell}\overline{t_{k0}^{\ell}(\theta',\varphi')}t_{k0}^{\ell}(\theta,\varphi)\right)d\mu_N.$$

Then the discrete Fourier-Laplace sum can be expressed in the following way:

$$(4.4)\quad (I_{N,n}f)(g(\theta,\varphi)) = (I_{N,n}f)(\theta,\varphi)$$
$$= \sum_{\ell=0}^{n}(2\ell+1)\int_X f(h(\theta',\varphi'))P_l(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi'))d\mu_N(h(\theta',\varphi')).$$

It can be seen the analogy between the $(I_{N,n}f)$ and the partial sum of Laplace-Fourier series given by (2.9).

Let denote by

$$(4.5)\qquad \begin{aligned} D_n(h^{-1}g) &:= \sum_{\ell=0}^{n}(2\ell+1)\chi^{\ell}(h^{-1}g) \\ &= \sum_{\ell=0}^{n}(2\ell+1)P_\ell(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\varphi-\varphi')) \end{aligned}$$

the kernel function and by

$$(4.6)\qquad K_n^{\alpha} := \frac{1}{A_n^{\alpha}}\sum_{\ell=0}^{n}A_{n-\ell}^{\alpha}(2\ell+1)\chi^{\ell}, \qquad A_n^{\alpha} := \frac{(\alpha+1)(\alpha+2)\dots(\alpha+n)}{n!},$$

the (C,α) kernels of the Laplace-Fourier series. From (1.14) of [3] we get that for $\alpha = 2$

$$(4.7)\qquad K_n^2 := \frac{1}{A_n^2}\sum_{\ell=0}^{n}A_{n-\ell}^2(2\ell+1)\chi^{\ell} \geq 0.$$

Using the orthonormality properties (2.4), (3.4) and the definition of χ^k it is easy to check that

$$(4.8)\qquad \int_X K_n^2(h^{-1}g)d\mu_N(h) = \int_{S^2} K_n^2(h^{-1}g)d\mu(h) = 1.$$

The last two properties show that K_n^2 has the two important properties of Fejér kernel. Let introduce the analogue of de la Valée-Poussin kernel denoted by

$$(4.9)\qquad M_n := \frac{1}{n^2}(A_{3n}^2K_{3n}^2 - 2A_{2n}^2K_{2n}^2 + A_n^2K_n^2).$$

Note that the partial sum of order n of M_n is equal to

$$(4.10)\qquad S_n[M_n] = D_n.$$

Let denote by $\mathcal{T}_n = \operatorname{span}\{t_{k0}^{\ell}, \ell\in\{0,1,\dots,n-1\}, k\in I_\ell\}$. From the orthonormality property of spherical functions and (4.10) follows that

$$(4.11)\qquad \int_{S^2} f(h)M_n(h^{-1}g)d\mu(h) = \int_X f(h)M_n(h^{-1}g)d\mu(h) = f(g),$$

for all $f\in\mathcal{T}_n$. Denote by

$$(4.12)\qquad (V_nf)(g) := \int_{S^2} f(h)M_n(h^{-1}g)d\mu(h)$$

and

$$(4.13)\quad (V_{n,N}f)(g) := \int_{S^2} f(h)M_n(h^{-1}g)d\mu_N(h) \qquad (f\in C(S^2), \quad g\in SU(2)).$$

the continuous and discrete summmation processes corresponding to the M_n kernels. Taking into account that χ_l can be expressed by P_ℓ, (see (2.8)) (4.12) (4.13) can be considered as approximation processes of the three dimensional Dirichlet problem on the unit sphere. In paper [7] it was proved the following theorem

Theorem D. *For all $f \in C(S^2)$,*

1)

$$||V_n f - f|| \to 0 \quad \text{if} \quad n \to \infty \tag{4.14}$$

2)

$$||V_{n,N_n} f - f|| \to 0 \quad \text{if} \quad n \to \infty, \quad \text{so that} \quad 3n < N_n \tag{4.15}$$

where the norm is the maximum norm.

In the rest of the paper we give estimation for the rate of the convergence of the approximation processes defined by (4.12) and (4.13). For this we will use the modulus of continuity and Jackson type theorem for spherical functions.

5. Jackson type inequality for spherical functions

Let denote by

$$E_n f = \inf_{g \in \mathcal{T}_n} ||f - g|| = ||f - g^*||. \tag{5.1}$$

Taking into account that $\mathcal{T}_n$ is a finite dimensional space, the existence of $g^* \in \mathcal{T}_n$ is assured. The generalized translation operator is defined by

$$(T_h f)(x) = \frac{1}{2\pi \sin h} \int_{(x,y)=\cos h} f(y) dt(y), \tag{5.2}$$

where the integral is taken on the circle $(x, y) = \cos h$ of the unit sphere. Let denote by

$$\Omega(f, h) = \sup_{0<t\leq h} ||T_h f - f|| \tag{5.3}$$

the modulus of continuity of the function f. In [8] S. Pawelke proved a Jackson type inequality for spherical functions, namely

Theorem E. *For every function $f \in C(S^2)$ there is a linear combination of spherical functions $G_n f \in \mathcal{T}_n$ so that*

$$||f - G_n f|| \leq K\Omega(f; \frac{1}{n}), \tag{5.4}$$

where K is a constant independent from f.

Consequently $E_n f \leq K\Omega(f; \frac{1}{n})$.

Combining Theorem D. and Theorem E. we can obtain the following Theorem.

6. Main result

Theorem 1. *There exists a positive constant M so that, for all $f \in C(S^2)$,*

1)

$$||V_n f - f|| \leq M\Omega(f; \frac{1}{n})$$

2)

$$||V_{n,N_n} f - f|| \leq M\Omega(f; \frac{1}{n}) \quad \text{if} \quad 3n < N_n$$

where the norm is the maximum norm.

Proof. From (4.7) and (4.8) we obtain that

$$||V_nf|| \leq \frac{1}{n^2}(A_{3n}^2 + 2A_{2n}^2 + A_n^2)||f|| \leq \frac{(3n+2)^2}{n^2}||f|| \leq 25||f||.$$

We obtain in similar way that $||V_{n,N}|| \leq 25||f||$. Consequently the operators $V_n, V_{n,N_n} : C(S^2) \to \mathbb{C}$ are uniformly bounded. From relation (4.1) we obtain that these operators are projection operators on $\mathcal{T}_n$. Let $E_nf = \inf_{g\in\mathcal{T}_n} ||f-g|| = ||f-g^*||$, then $V_ng^* = g^*$. Using Theorem E we obtain that

$$||V_nf - f|| = ||V_nf - g^* + g^* - f|| \leq ||V_nf - V_ng^*|| + ||g^* - f||$$
$$\leq (||V_n|| + 1)||f - g^*|| \leq 26E_nf \leq 26K\Omega(f; \frac{1}{n}).$$

In a similar way it can be obtained the result for V_{n,N_n}. □

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